



# Scene Graphs & Modeling Transformations

COS 426, Spring 2017  
Princeton University

# 3D Object Representations

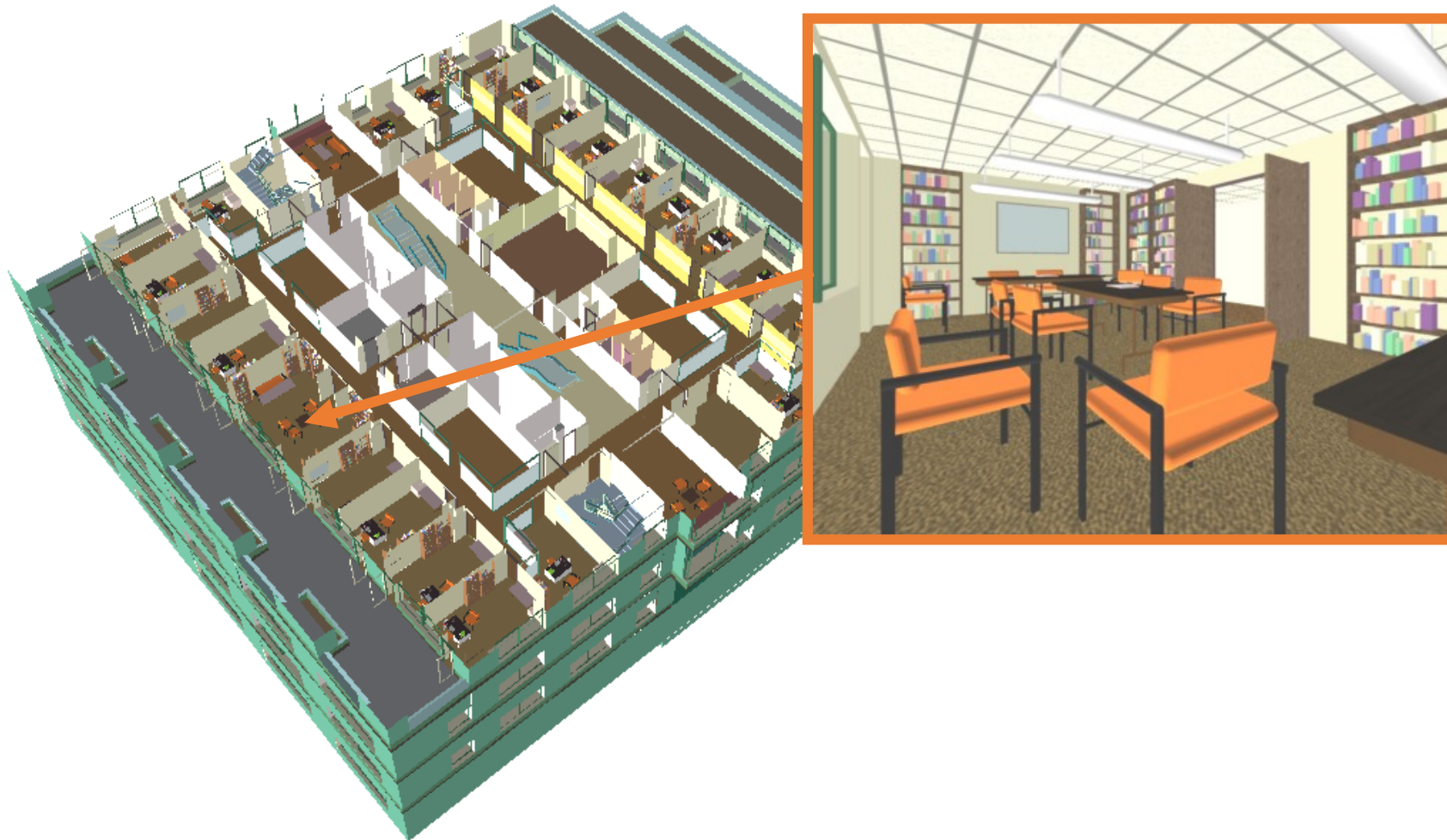


- Points
  - Range image
  - Point cloud
- Surfaces
  - Polygonal mesh
  - Subdivision
  - Parametric
  - Implicit
- Solids
  - Voxels
  - BSP tree
  - CSG
  - Sweep
- High-level structures
  - Scene graph
  - Application specific

# 3D Object Representations



- What object representation is best for this?

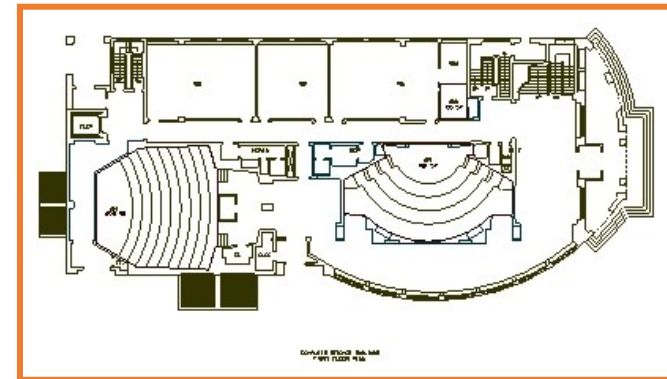
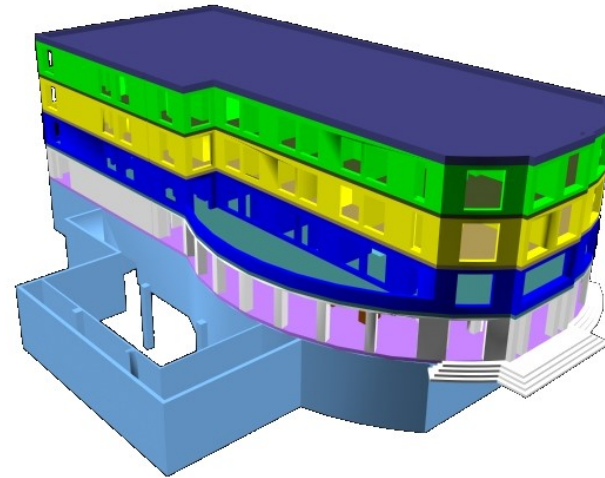


# 3D Object Representations



- Desirable properties of an object representation

- Easy to acquire
- Accurate
- Concise
- Intuitive editing
- Efficient editing
- Efficient display
- Efficient intersections
- Guaranteed validity
- Guaranteed smoothness
- etc.



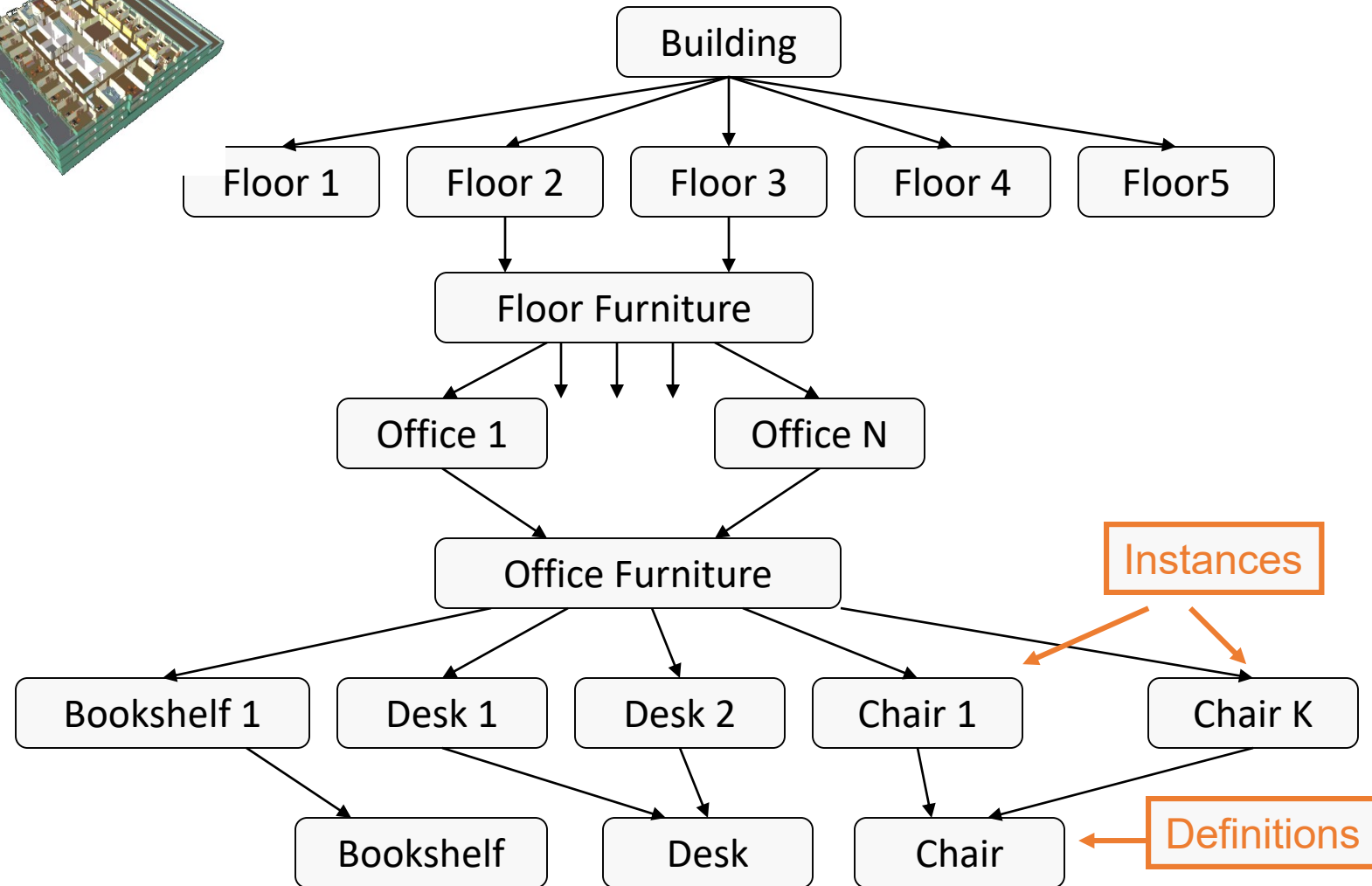
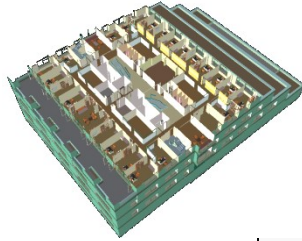
*(CS Building, Princeton University)*

# Overview



- Scene graphs
  - Geometry & attributes
  - Transformations
  - Bounding volumes
- Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
  - 3D transformations

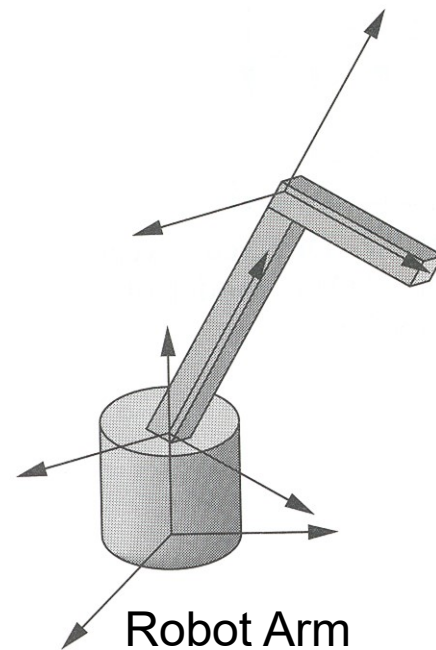
# Scene Graphs



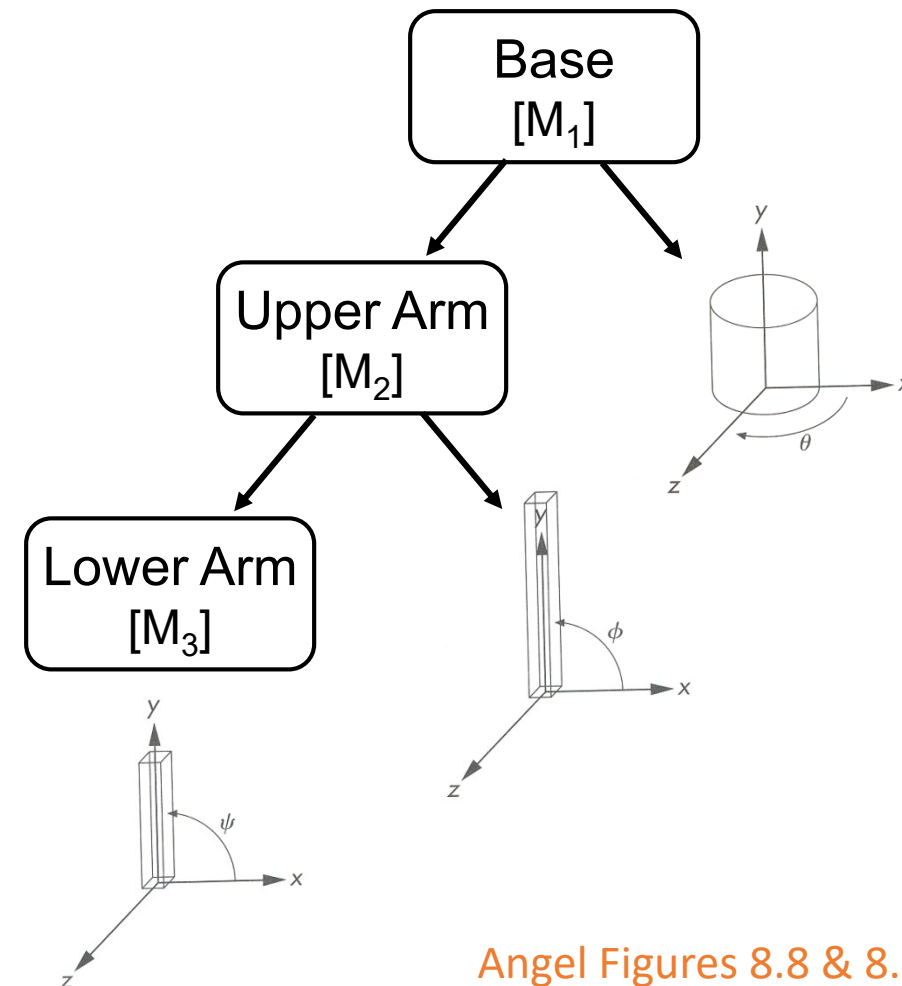
# Scene Graphs



- Hierarchy (DAG) of nodes, where each may have:
  - Geometry representation
  - Modeling transformation
  - Parents and/or children
  - Bounding volume



Robot Arm

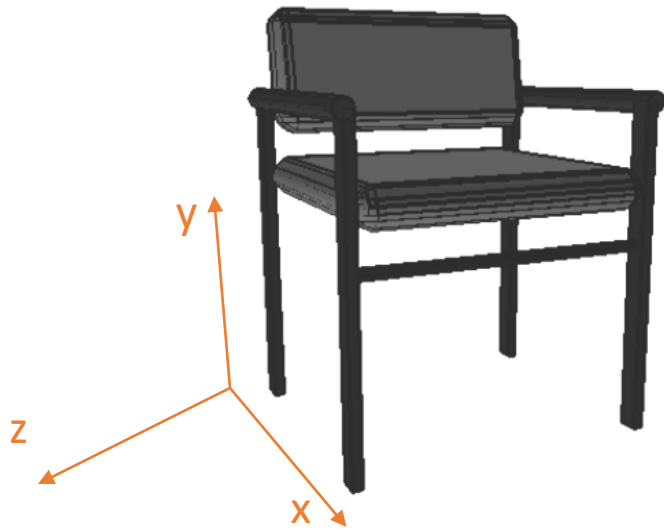


Angel Figures 8.8 & 8.9

# Scene Graphs



- Advantages
  - Allows definitions of objects in own coordinate systems



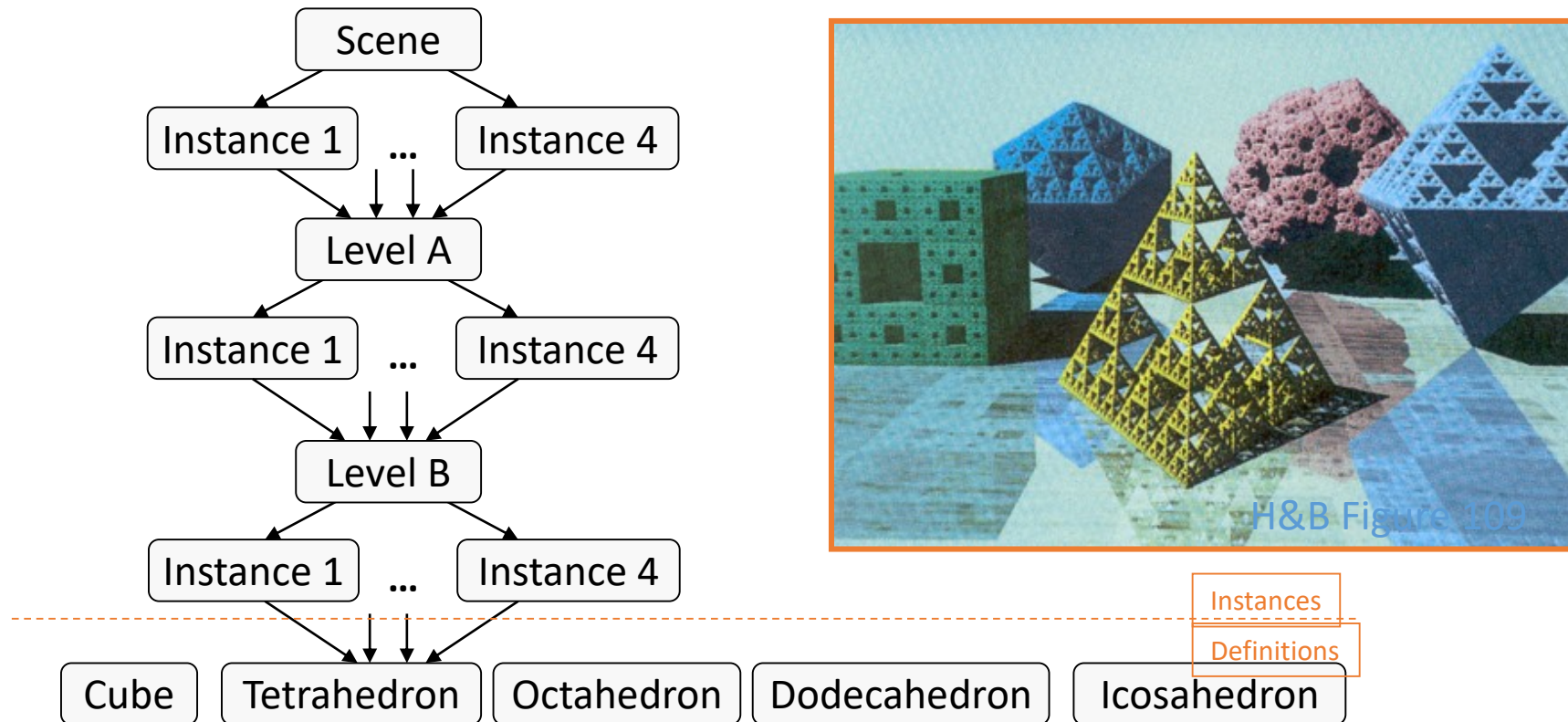


# Scene Graphs



- Advantages

- Allows definitions of objects in own coordinate systems
- Allows use of object definition multiple times in a scene

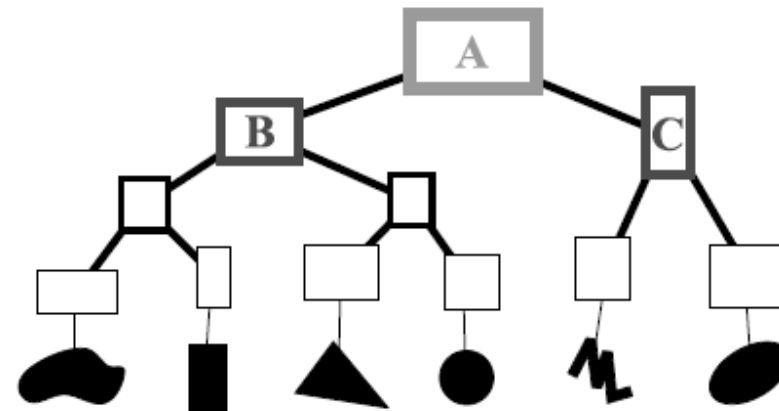
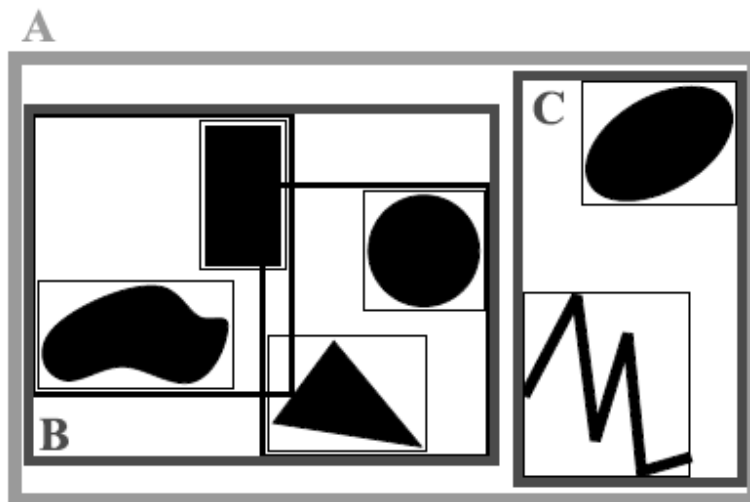


# Scene Graphs



- Advantages

- Allows definitions of objects in own coordinate systems
- Allows use of object definition multiple times in a scene
- Allows hierarchical processing (e.g., intersections)

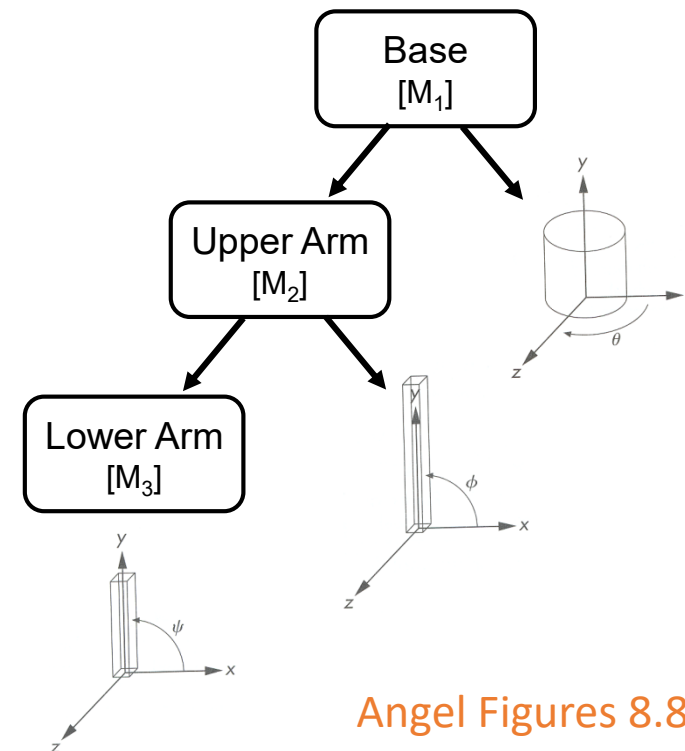
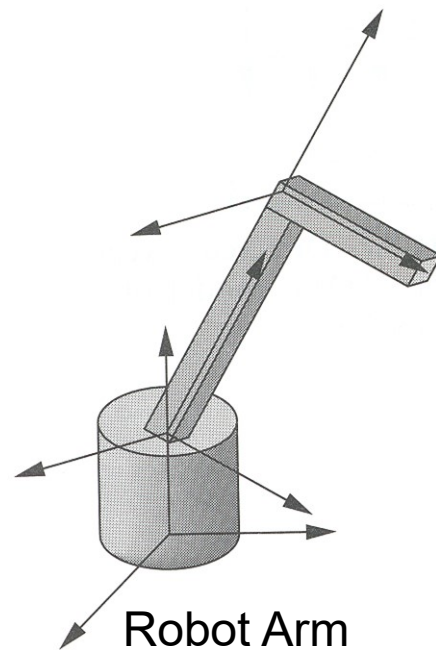


# Scene Graphs



- Advantages

- Allows definitions of objects in own coordinate systems
- Allows use of object definition multiple times in a scene
- Allows hierarchical processing (e.g., intersections)
- Allows articulated animation



Angel Figures 8.8 & 8.9

# Scene Graph Example

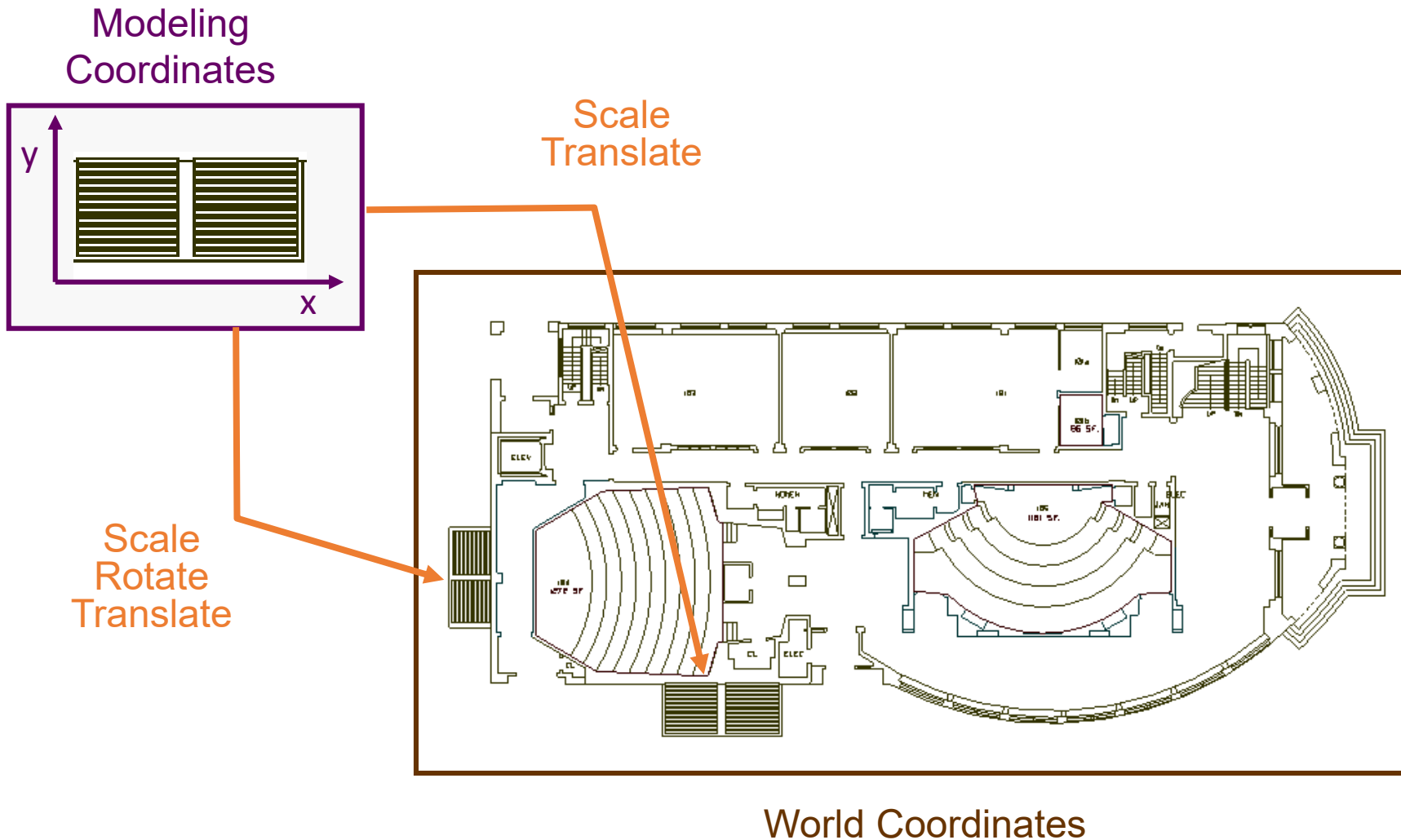


# Overview



- Scene graphs
  - Geometry & attributes
  - Transformations
  - Bounding volumes
- Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
  - 3D transformations

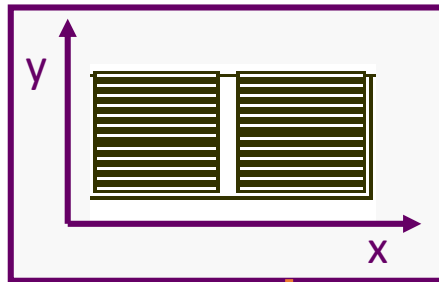
# 2D Modeling Transformations



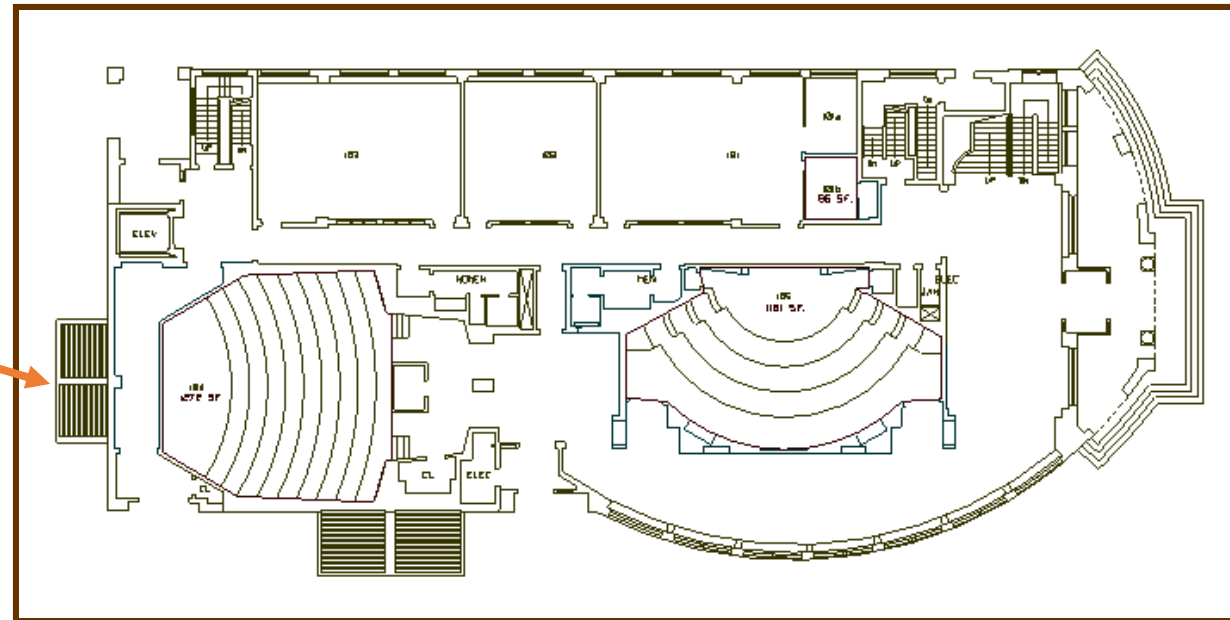
# 2D Modeling Transformations



Modeling  
Coordinates



Let's look  
at this in  
detail...

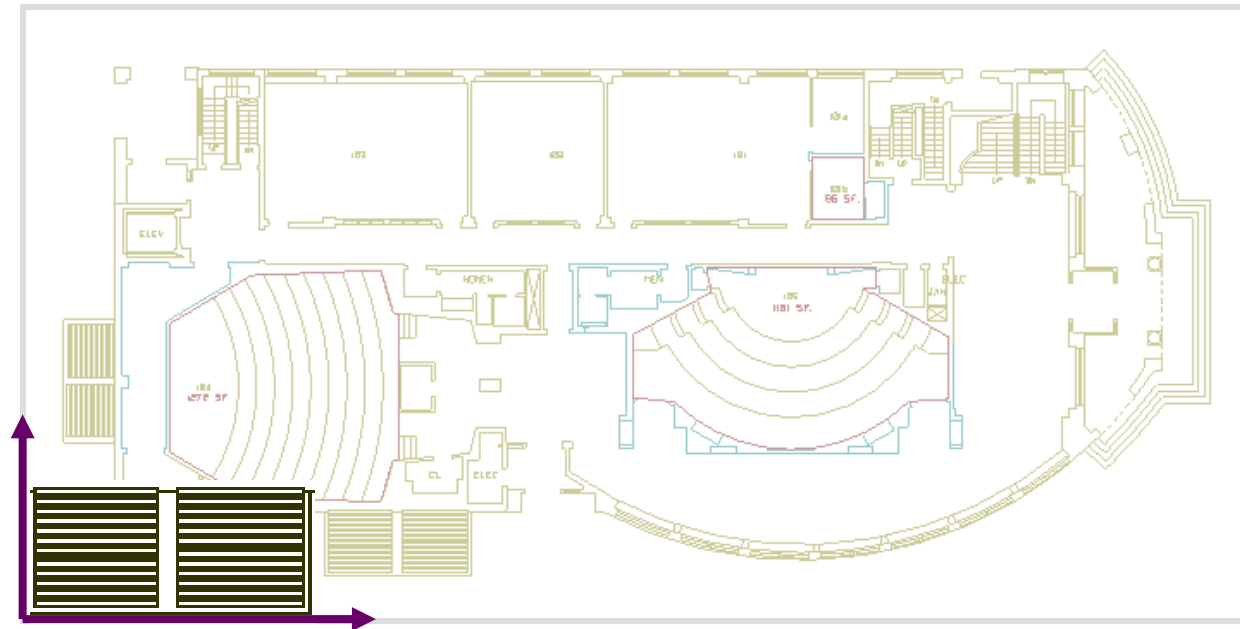
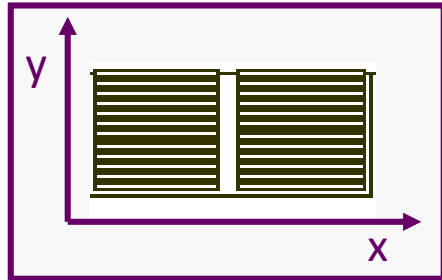


World Coordinates

# 2D Modeling Transformations



Modeling  
Coordinates

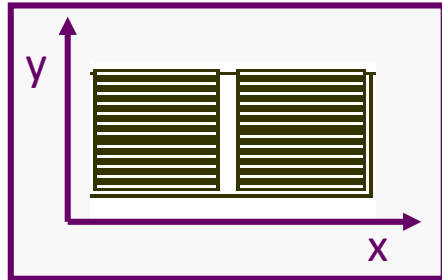




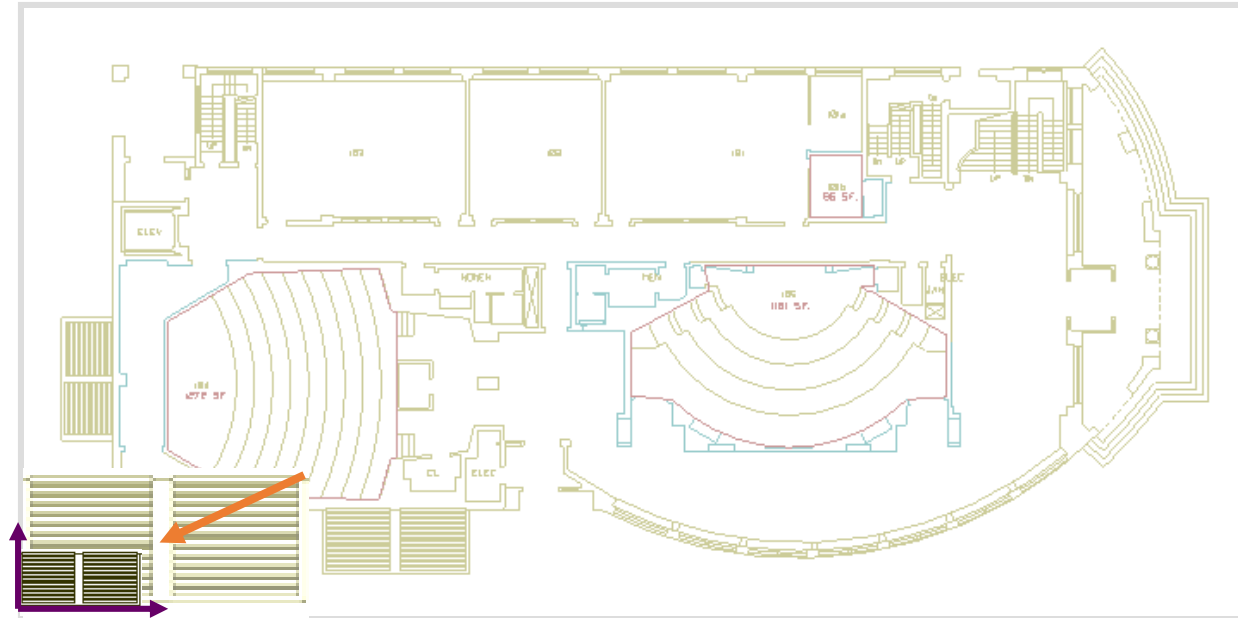
# 2D Modeling Transformations



Modeling  
Coordinates



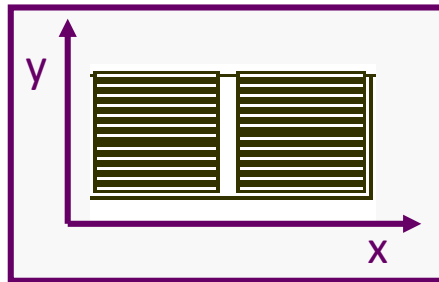
Scale .3, .3



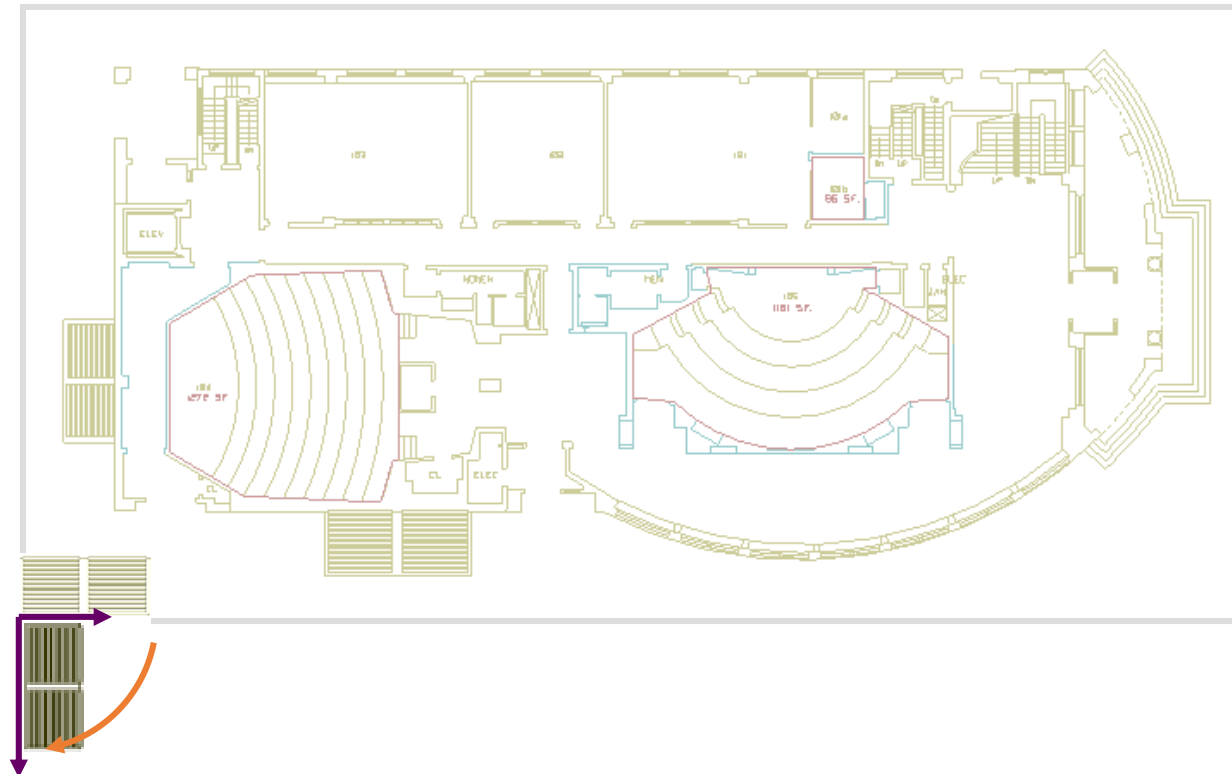
# 2D Modeling Transformations



Modeling  
Coordinates



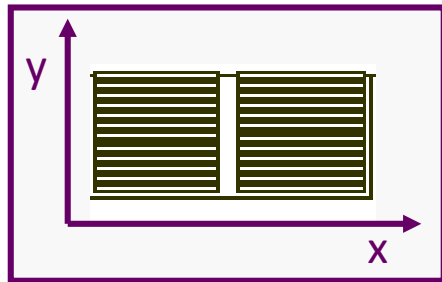
Scale .3, .3  
Rotate -90



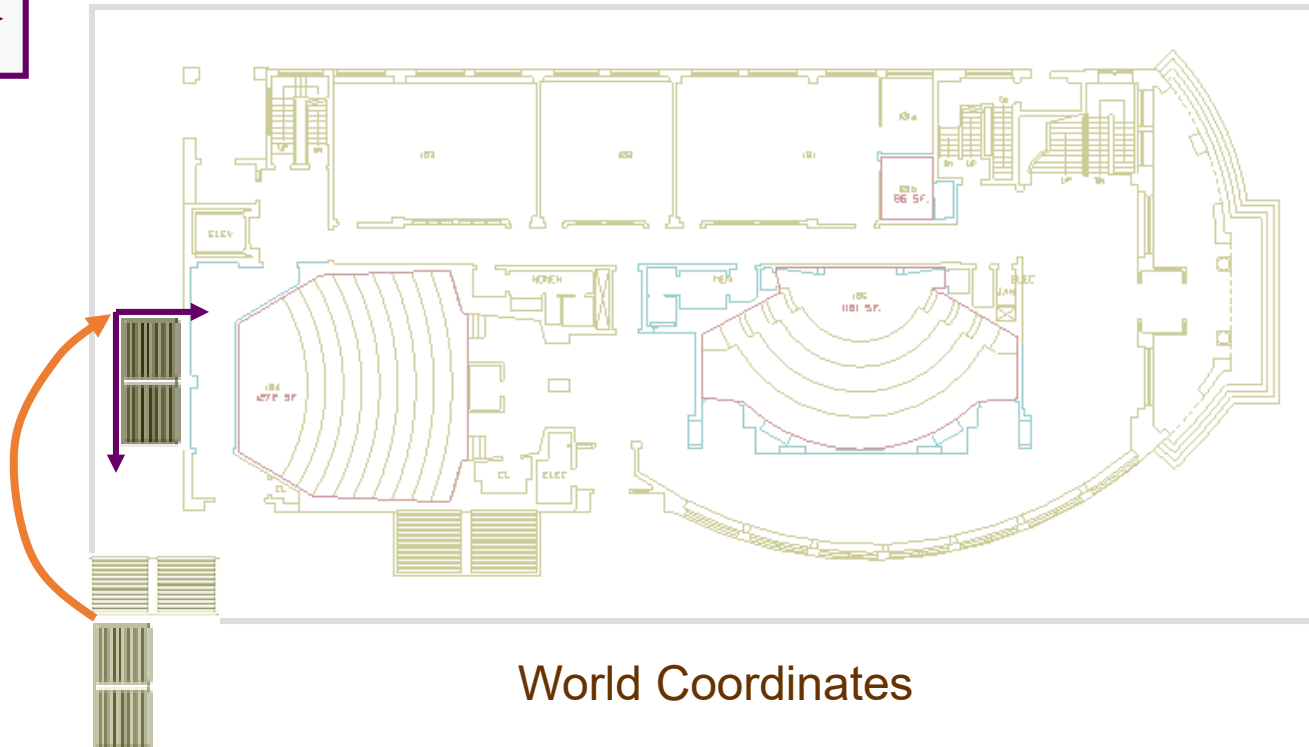
# 2D Modeling Transformations



Modeling  
Coordinates



Scale .3, .3  
Rotate -90  
Translate 5, 3



World Coordinates

# Basic 2D Transformations



- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

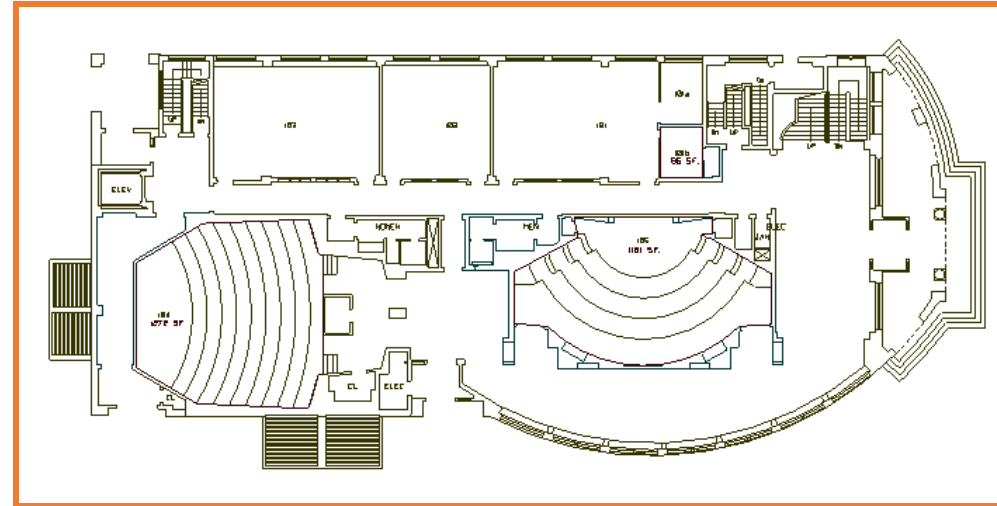
- $x' = x * sx$
- $y' = y * sy$

- Shear:

- $x' = x + hx*y$
- $y' = y + hy*x$

- Rotation:

- $x' = x*\cos\Theta - y*\sin\Theta$
- $y' = x*\sin\Theta + y*\cos\Theta$



Transformations  
can be combined  
(with simple algebra)

# Basic 2D Transformations



- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

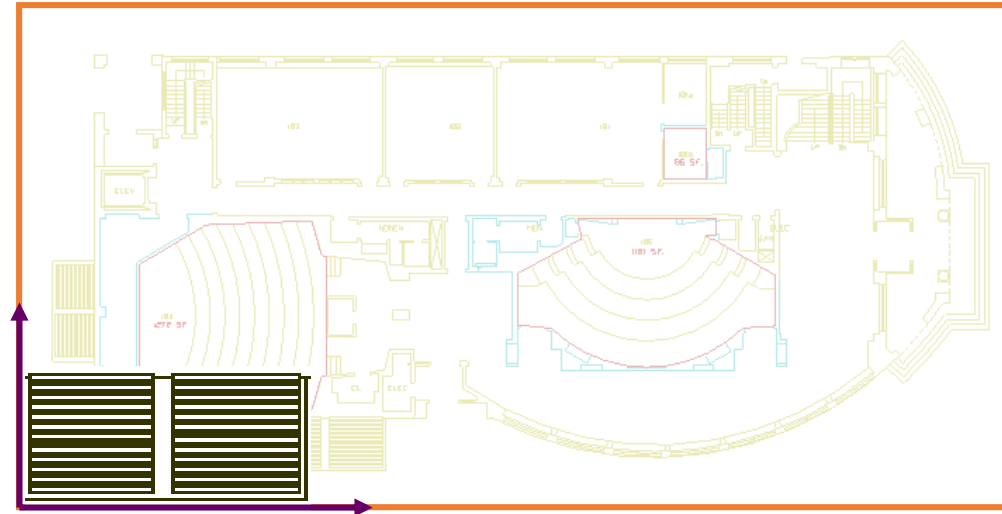
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# Basic 2D Transformations



- Translation:

- $x' = x + tx$
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- Scale:

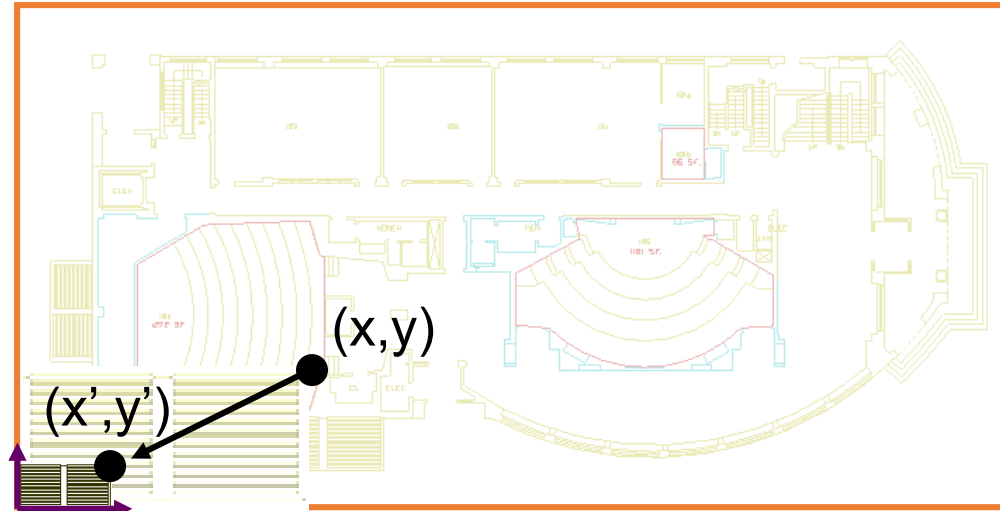
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- Rotation:

- $x' = x*cos\Theta - y*sin\Theta$
- $y' = x*sin\Theta + y*cos\Theta$



$$\begin{aligned} x' &= x * sx \\ y' &= y * sy \end{aligned}$$

# Basic 2D Transformations



- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

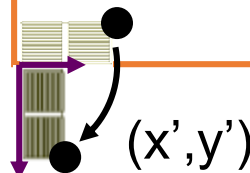
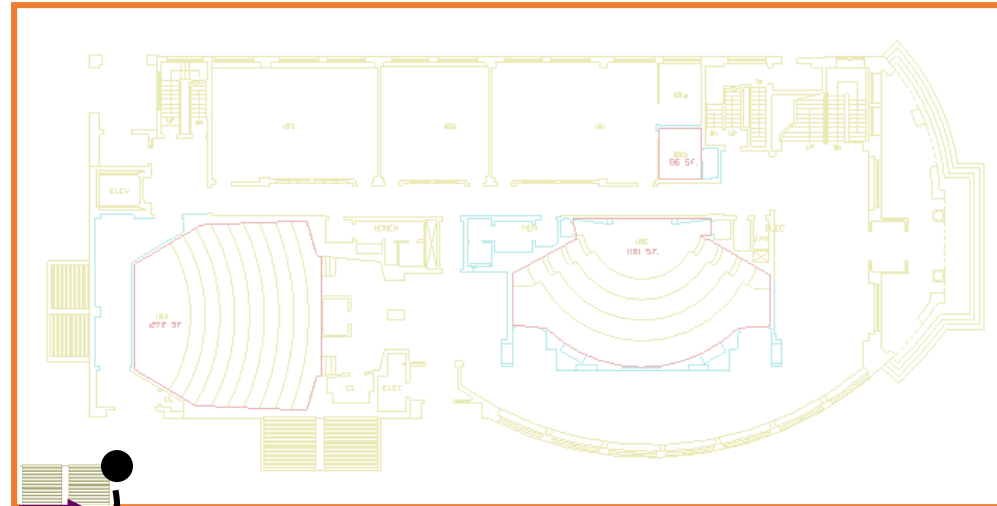
- $x' = x * sx$
- $y' = y * sy$

- Shear:

- $x' = x + hx*y$
- $y' = y + hy*x$

- Rotation:

- $x' = x*cos\Theta - y*sin\Theta$
- $y' = x*sin\Theta + y*cos\Theta$



$$\begin{aligned}x' &= (x * sx) * \cos\Theta - (y * sy) * \sin\Theta \\y' &= (x * sx) * \sin\Theta + (y * sy) * \cos\Theta\end{aligned}$$

# Basic 2D Transformations



- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

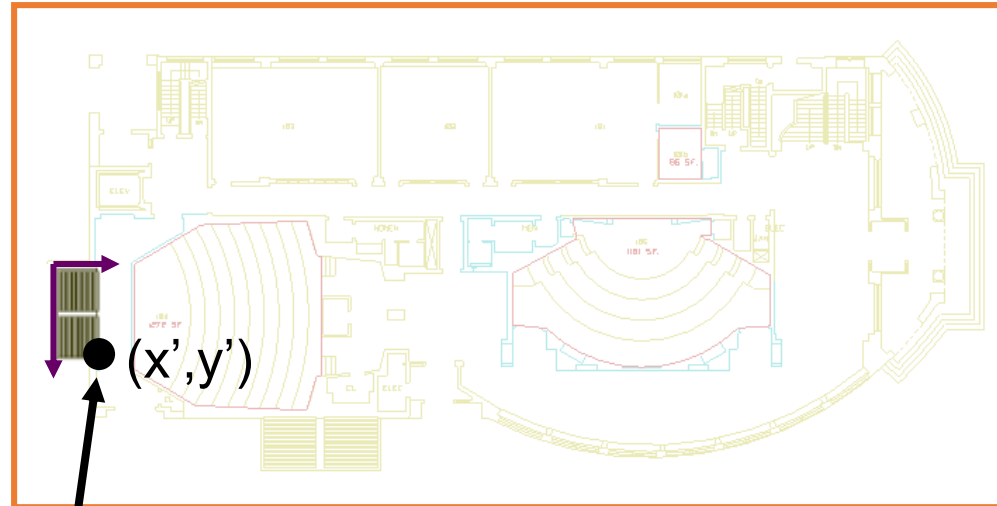
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- Rotation:

- $x' = x*cos\Theta - y*sin\Theta$
- $y' = x*sin\Theta + y*cos\Theta$



$$x' = ((x*sx)*cos\Theta - (y*sy)*sin\Theta) + tx$$
$$y' = ((x*sx)*sin\Theta + (y*sy)*cos\Theta) + ty$$



# Basic 2D Transformations



- Translation:

- $x' = x + tx$
- $y' = y + ty$

- Scale:

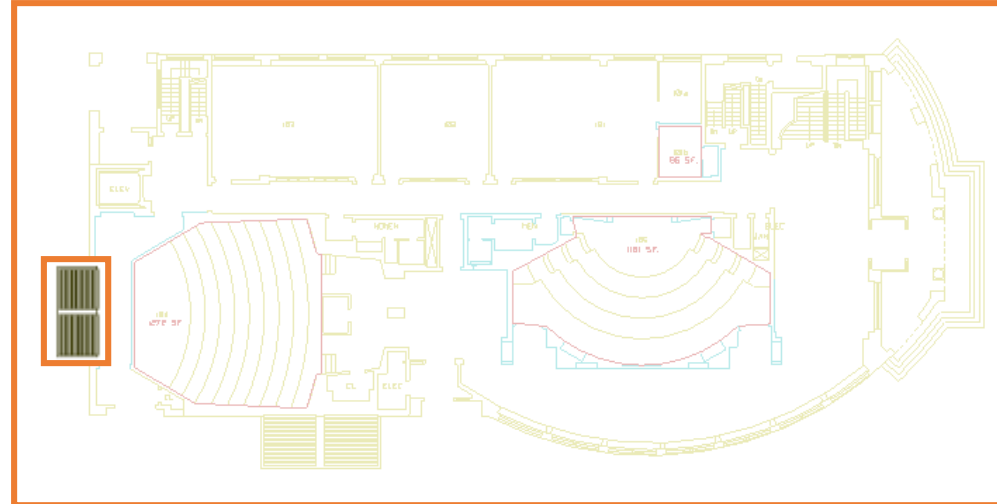
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$$x' = ((x*sx)*cos\Theta - (y*sy)*sin\Theta) + tx$$
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# Overview



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  - Matrix composition
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# Matrix Representation



- Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Multiply matrix by column vector  
⇔ apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = ax + by$$

$$y' = cx + dy$$

# Matrix Representation



- Transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices are a convenient and efficient way to represent a sequence of transformations

# 2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

## 2D Identity?

$$\begin{aligned}x' &= x \\ y' &= y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

## 2D Scale around (0,0)?

$$\begin{aligned}x' &= sx * x \\ y' &= sy * y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

# 2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

## 2D Rotate around (0,0)?

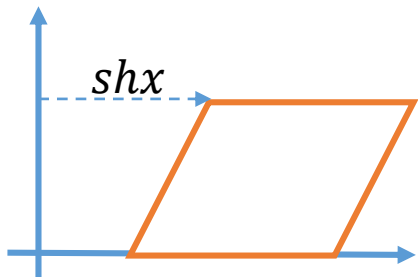
$$\begin{aligned}x' &= \cos \Theta * x - \sin \Theta * y \\y' &= \sin \Theta * x + \cos \Theta * y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

## 2D Shear?

$$\begin{aligned}x' &= x + shx * y \\y' &= shy * x + y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & shx \\ shy & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



# 2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

## 2D Mirror over Y axis?

$$\begin{aligned}x' &= -x \\ y' &= y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

## 2D Mirror over (0,0)?

$$\begin{aligned}x' &= -x \\ y' &= -y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

# 2x2 Matrices



- What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + tx$$

$$y' = y + ty$$

NO.

Only *linear* 2D transformations  
can be represented with a 2×2 matrix



# Linear Transformations



- 2D linear transformations are combinations of ...

- Scale,
- Rotation,
- Shear, and
- Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Properties of linear transformations:


- Satisfies:  $T(s_1\mathbf{p}_1 + s_2\mathbf{p}_2) = s_1T(\mathbf{p}_1) + s_2T(\mathbf{p}_2)$
- Origin maps to origin
- Points at infinity stay at infinity
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

# 2D Translation



- 2D translation represented by a 3x3 matrix
  - Point represented with *homogeneous coordinates*

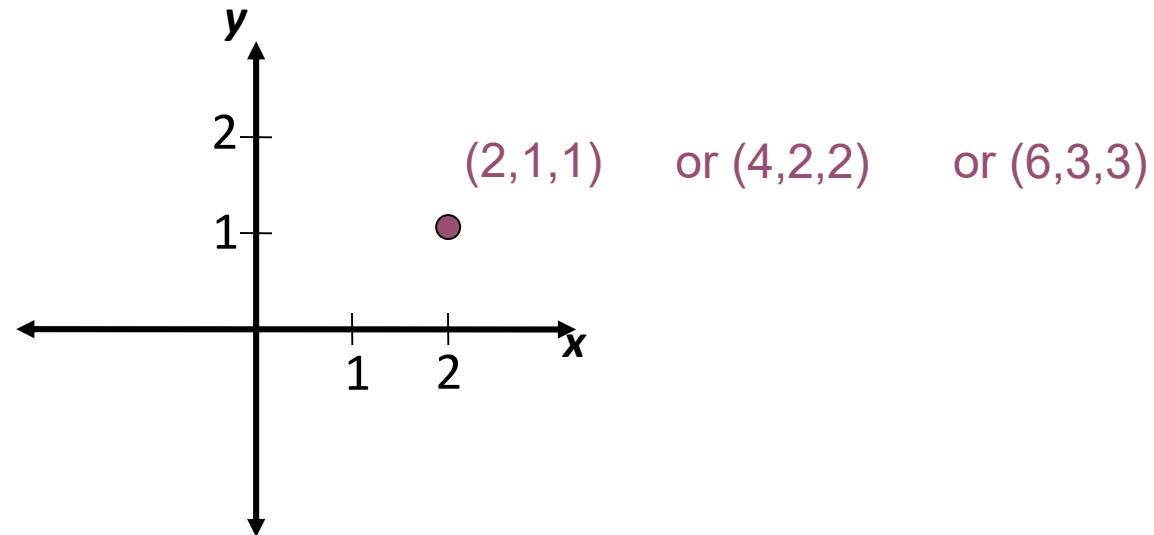
$$\begin{aligned}x' &= x + tx \\ y' &= y + ty\end{aligned}$$


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Homogeneous Coordinates



- Add a 3rd coordinate to every 2D point
  - $(x, y, w)$  represents a point at location  $(x/w, y/w)$
  - $(x, y, 0)$  represents a point at infinity
  - $(0, 0, 0)$  is not allowed



Convenient coordinate system to represent many useful transformations

# Basic 2D Transformations



- Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & shx & 0 \\ shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

# Affine Transformations



- Affine transformations are combinations of ...
  - Linear transformations, and
  - Translations

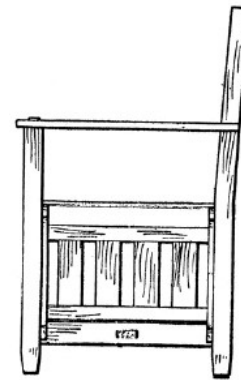
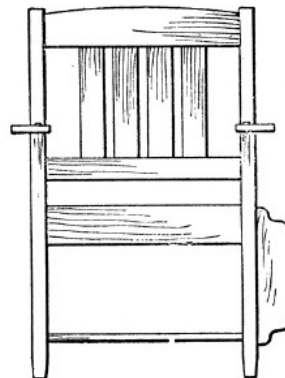
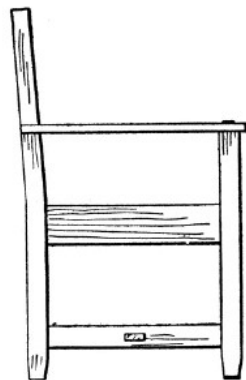
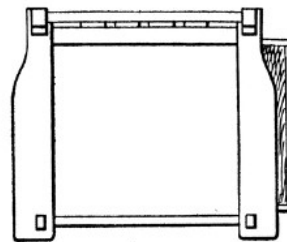
$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of affine transformations:
  - Origin does not necessarily map to origin
  - Points at infinity remain at infinity
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

# Projective Transformations



- The world is in 3D, the screen is flat. How to *Project*?



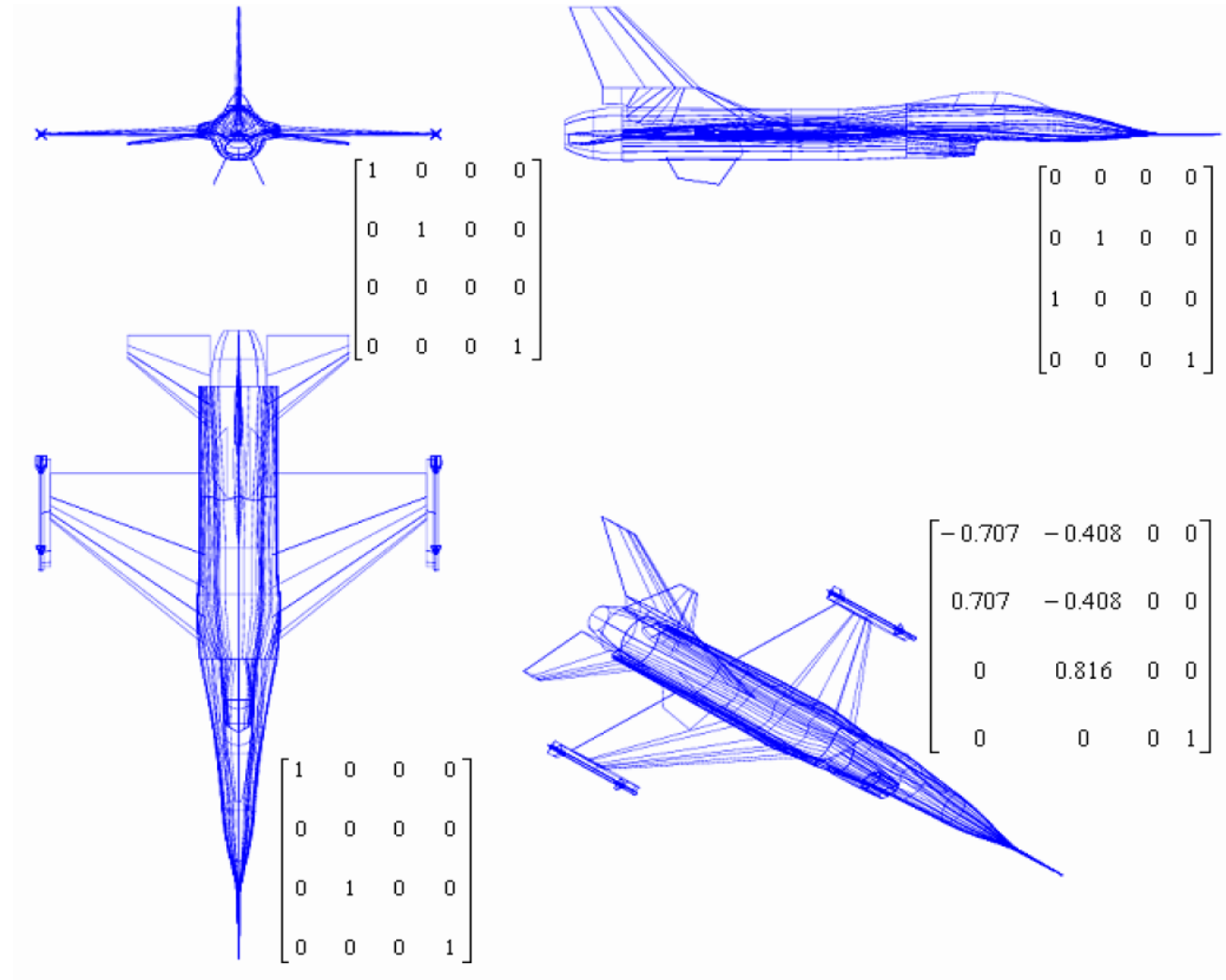
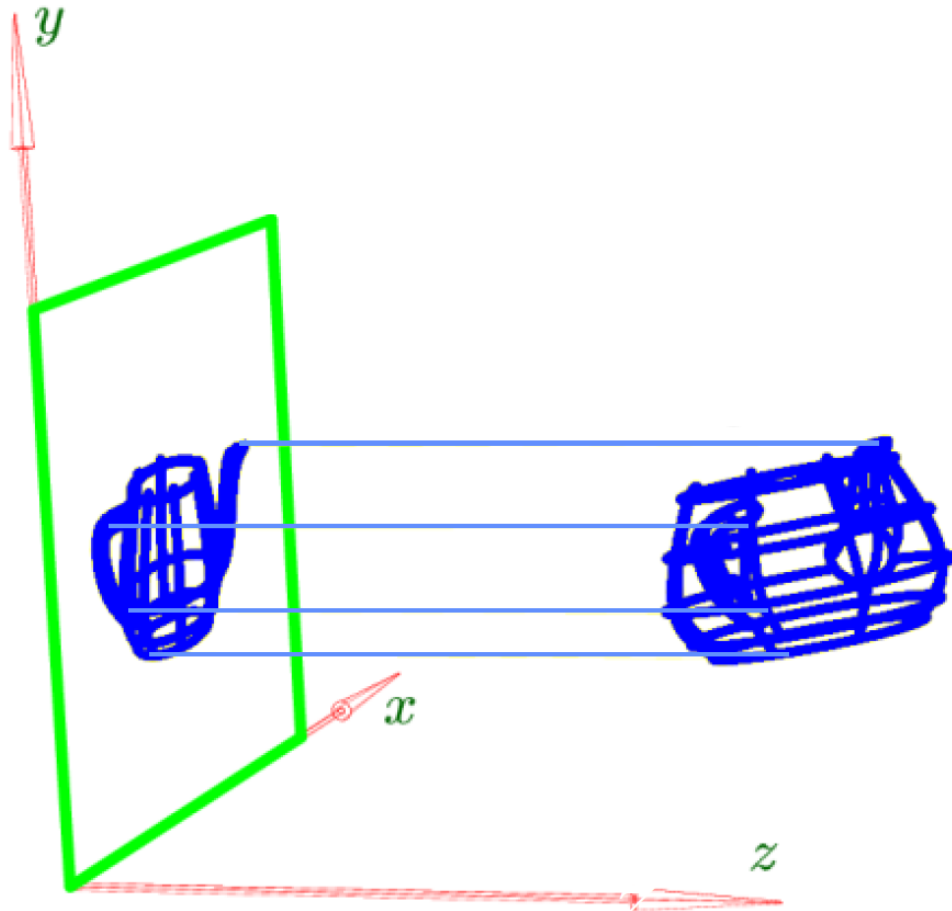
<http://www.nikonweb.com/fisheye/>  
[http://etc.usf.edu/clipart/52100/52103/52103\\_chair\\_o-p.htm](http://etc.usf.edu/clipart/52100/52103/52103_chair_o-p.htm)  
[http://en.wikipedia.org/wiki/File:One\\_point\\_perspective.jpg](http://en.wikipedia.org/wiki/File:One_point_perspective.jpg)

(Thanks Justin the almighty)

# Projective Transformations



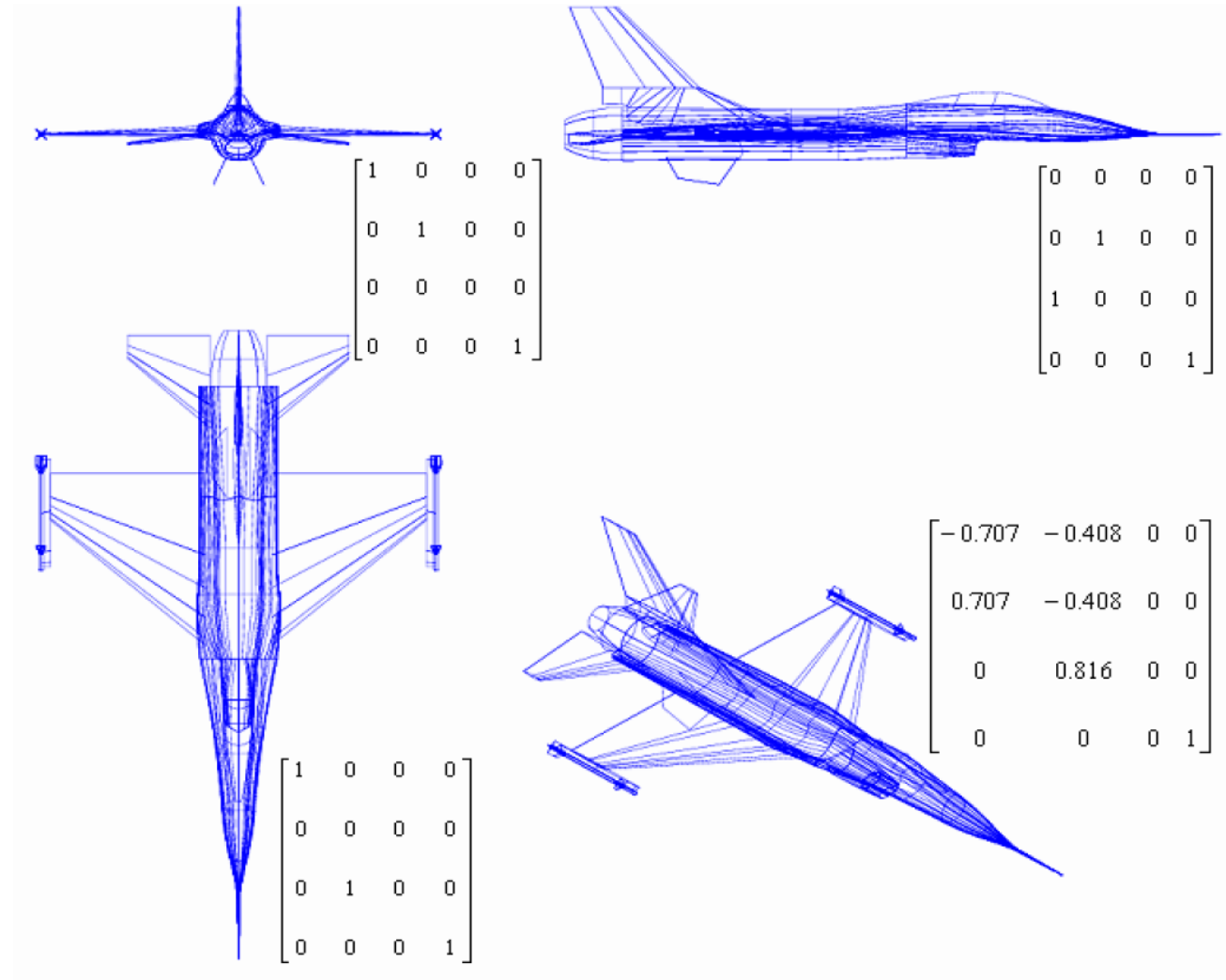
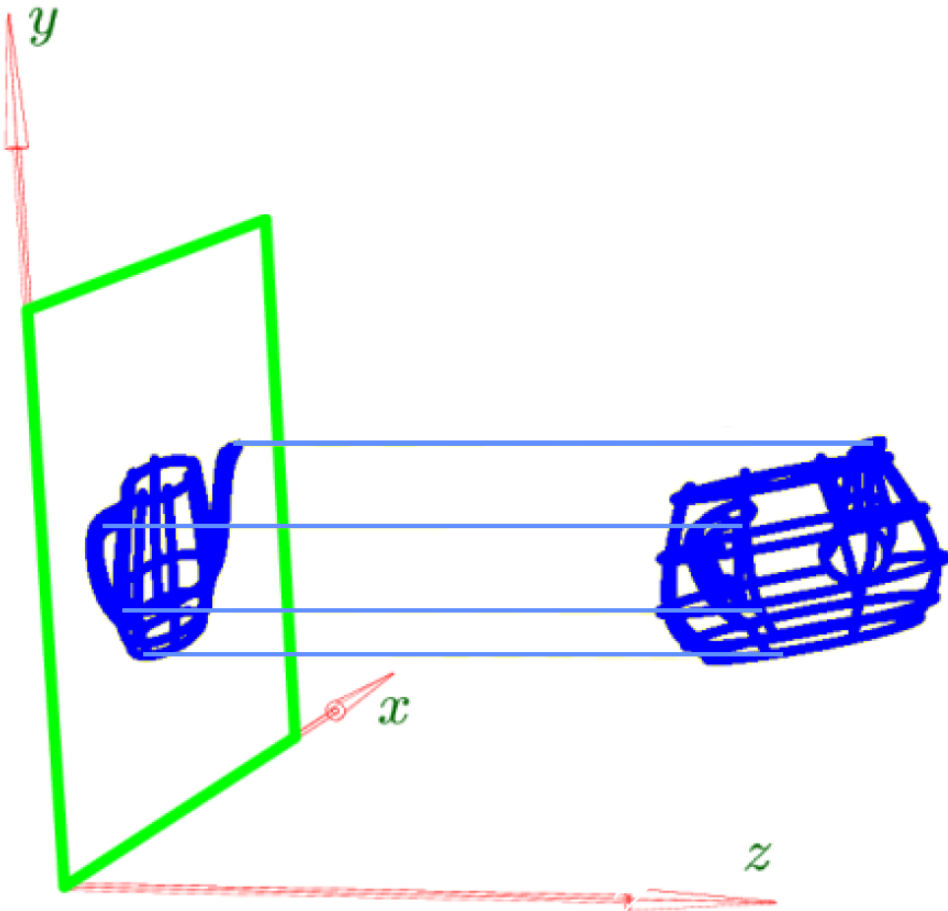
- Drop one axis?



# Projective Transformations



- Drop one axis?

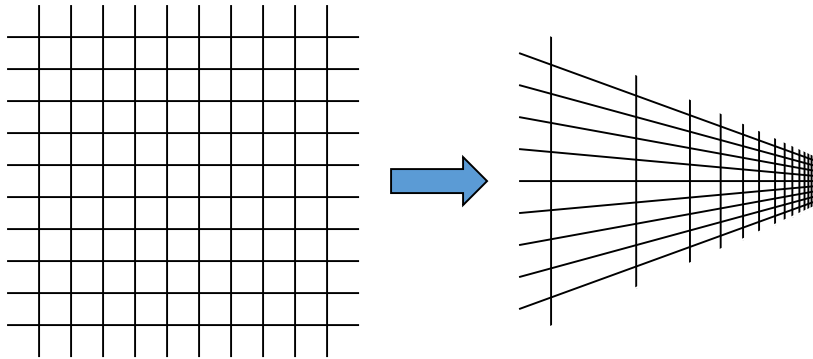




# Projective Transformations



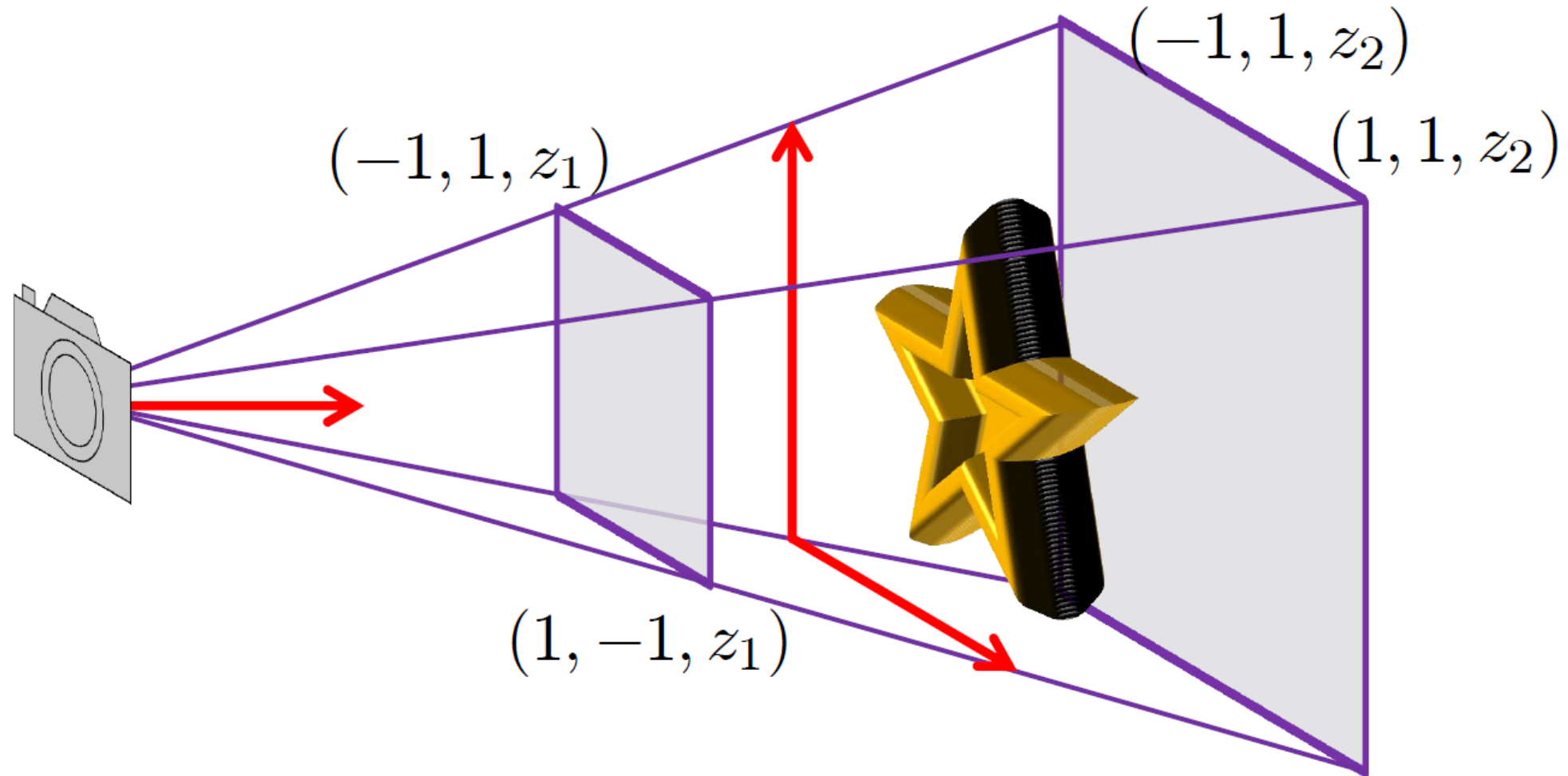
- What is wrong with this picture?



# Projective Transformations



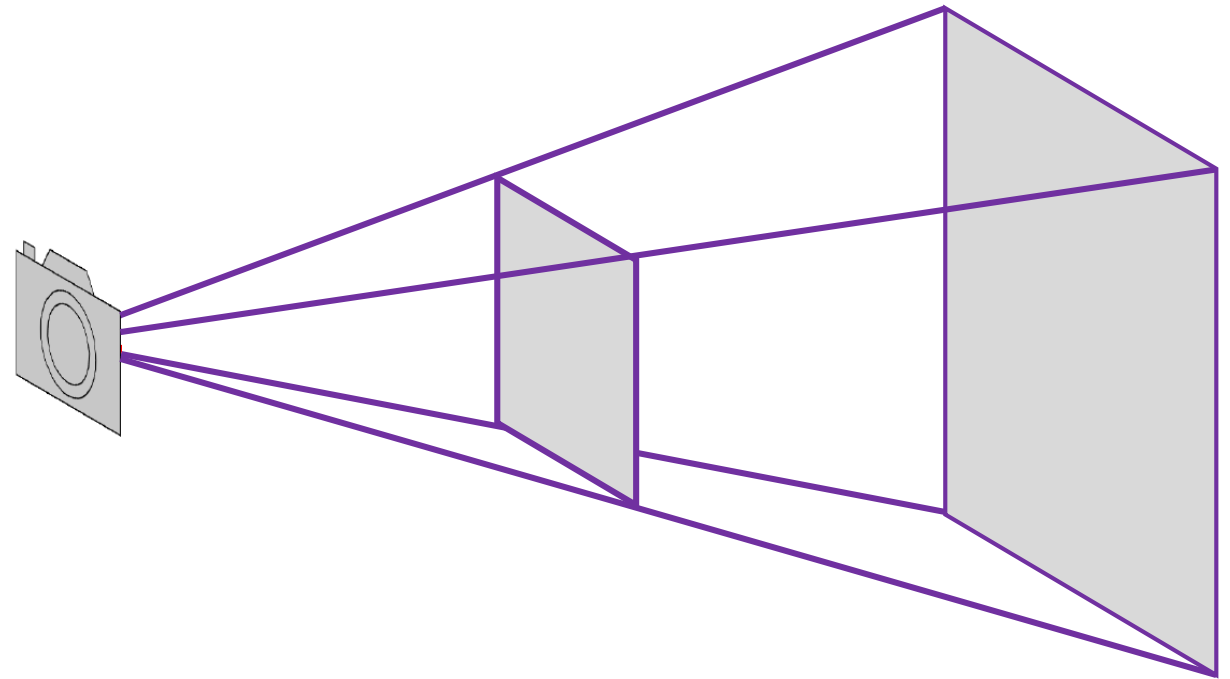
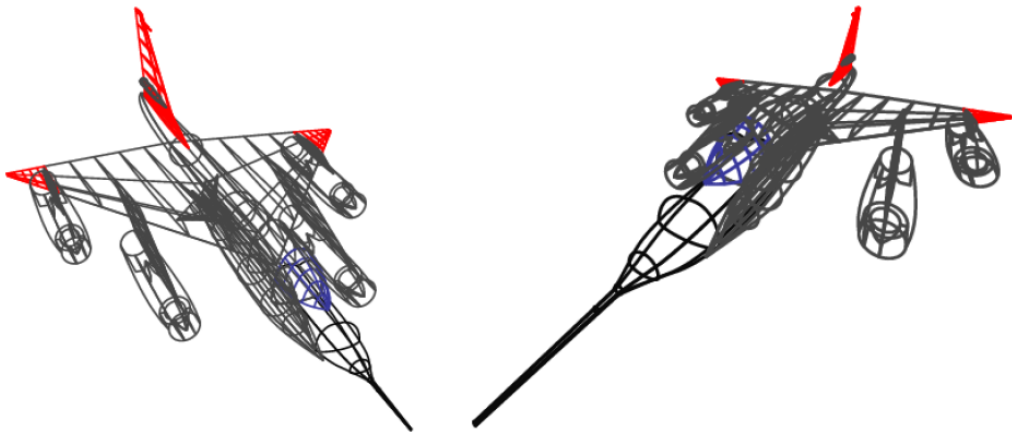
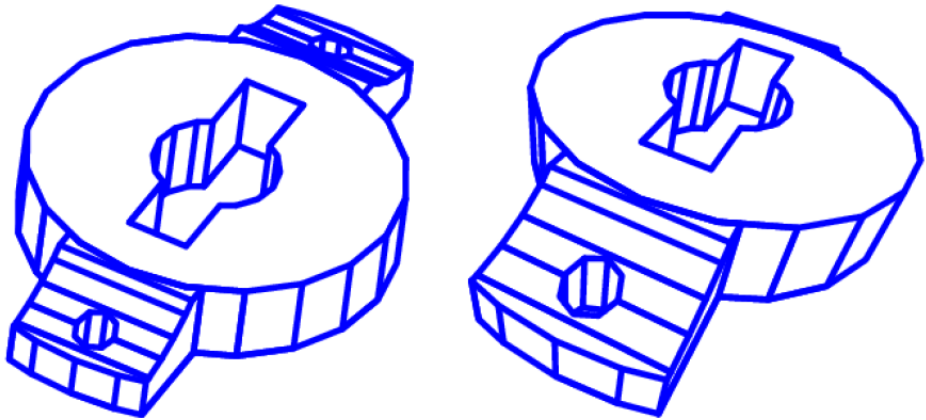
- Pinhole camera model



# Projective Transformations



- Perspective Warp!

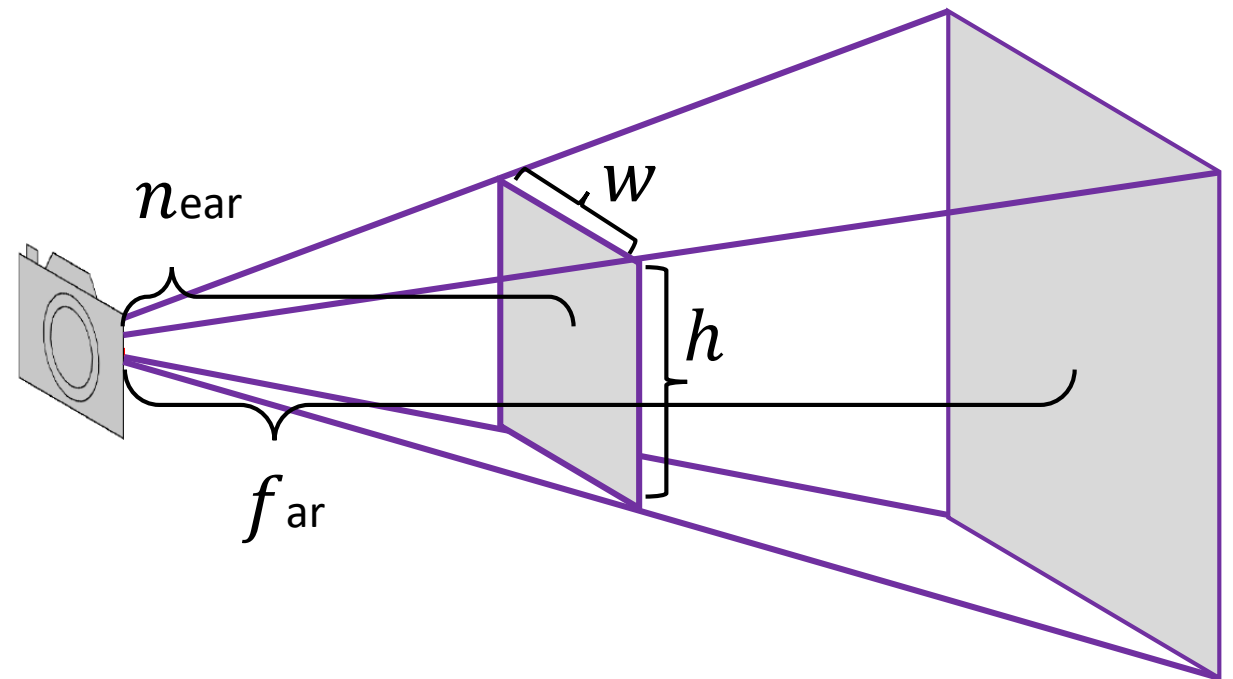


# Projective Transformations



- OpenGL's version

$$\begin{pmatrix} \frac{2n}{w} & 0 & 0 & 0 \\ 0 & \frac{2n}{h} & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$



Demo: [http://www.songho.ca/opengl/gl\\_transform.html](http://www.songho.ca/opengl/gl_transform.html)



# Projective Transformations



- Projective transformations (homographies):

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of projective transformations:

- Origin does not necessarily map to origin
- Point at infinity may map to finite point
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Closed under composition



# Projective Transformations



- Perspective warp in art
  - Julian Beever





# Projective Transformations



- Perspective warp in art
  - Julian Beever



# Overview



- Scene graphs
  - Geometry & attributes
  - Transformations
  - Bounding volumes
- Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
  - 3D transformations



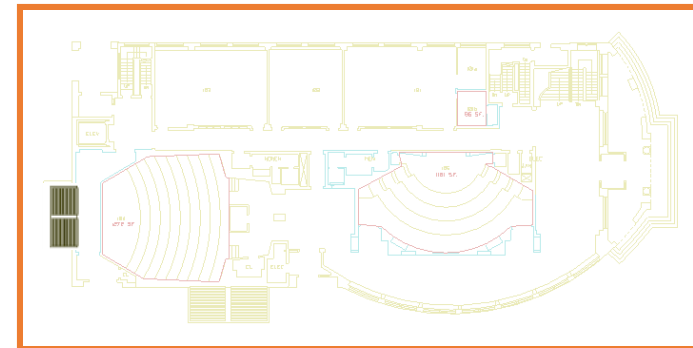
# Matrix Composition



- Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left( \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$\mathbf{p}' = \mathbf{T}(tx,ty) \mathbf{R}(\Theta) \mathbf{S}(sx,sy) \mathbf{p}$

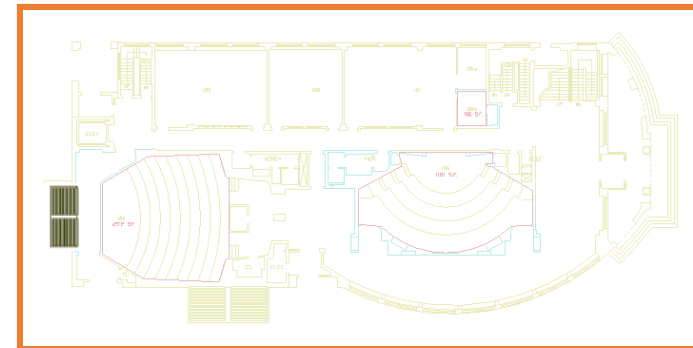


# Matrix Composition



- Matrices are a convenient and efficient way to represent a sequence of transformations
  - General purpose representation
  - Hardware matrix multiply
  - Efficiency with premultiplication
    - Matrix multiplication is associative

$$\mathbf{p}' = (T * (R * (S * \mathbf{p})))$$
$$\mathbf{p}' = (T * R * S) * \mathbf{p}$$



# Matrix Composition

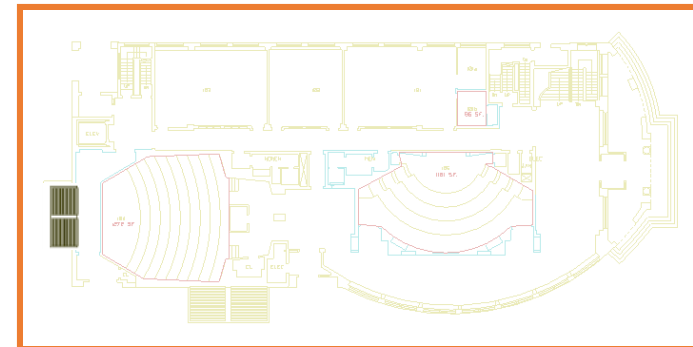


- Be aware: order of transformations matters
  - Matrix multiplication is **not** commutative

$$\mathbf{p}' = \mathbf{T} * \mathbf{R} * \mathbf{S} * \mathbf{p}$$

←—————→

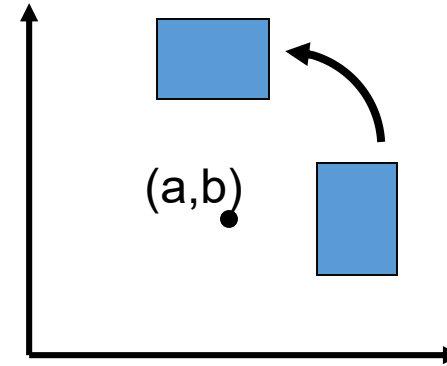
“Global”                      “Local”



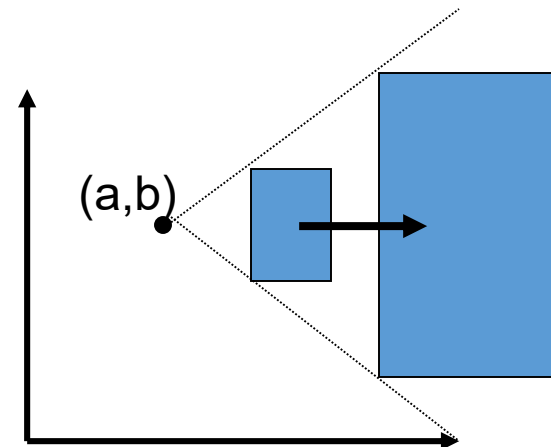
# Matrix Composition



- Rotate by  $\Theta$  around arbitrary point  $(a,b)$



- Scale by  $s_x, s_y$  around arbitrary point  $(a,b)$



# Overview



- Scene graphs
  - Geometry & attributes
  - Transformations
  - Bounding volumes
- Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
  - 3D transformations

# 3D Transformations



- Same idea as 2D transformations
  - Homogeneous coordinates:  $(x,y,z,w)$
  - 4x4 transformation matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

# Basic 3D Transformations



$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Identity

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Mirror over X axis

# Basic 3D Transformations



Rotate around Z axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 & 0 \\ \sin \Theta & \cos \Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around Y axis:

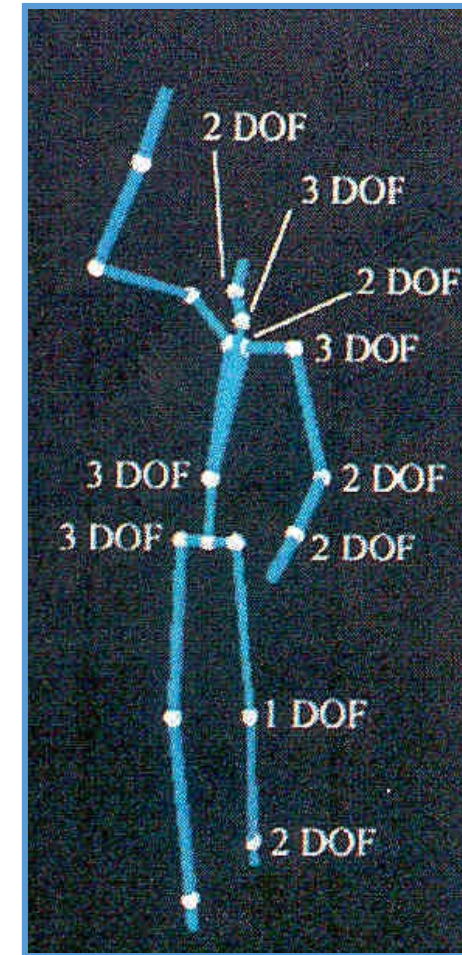
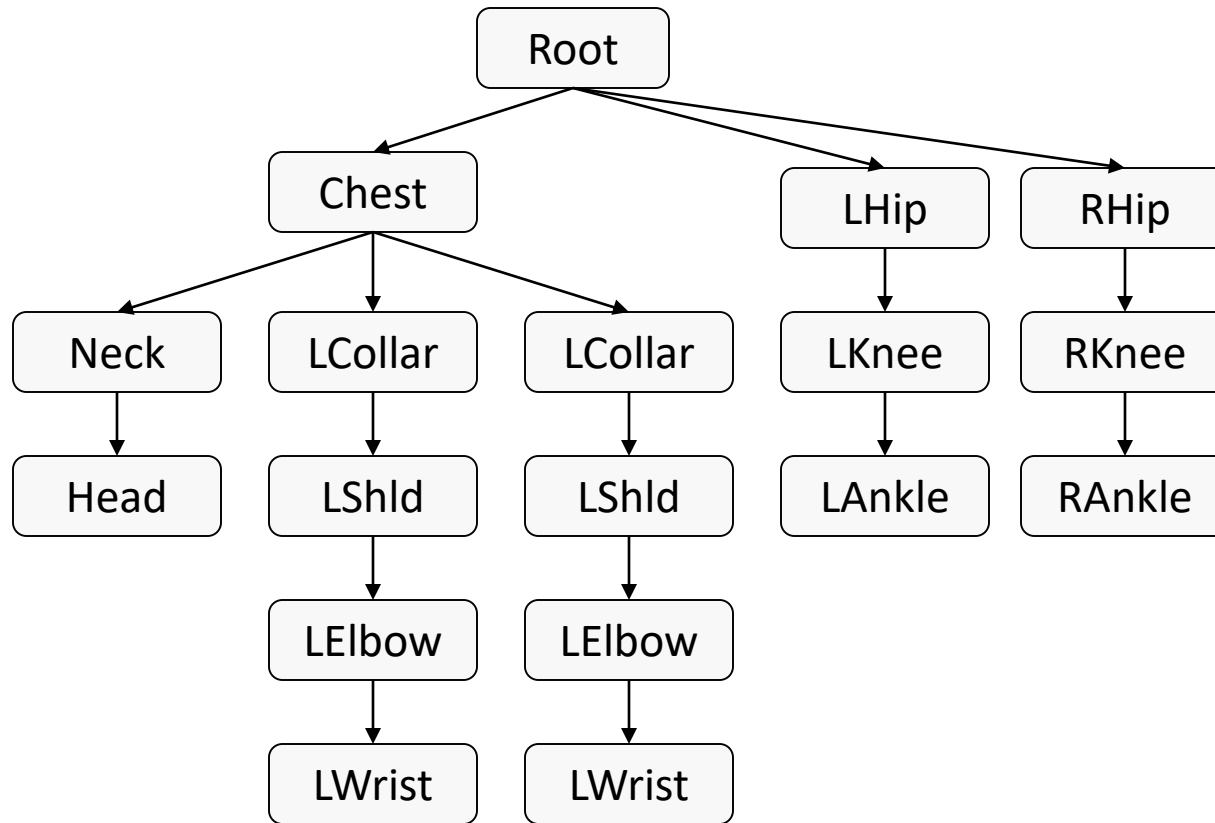
$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos \Theta & 0 & -\sin \Theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \Theta & 0 & \cos \Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around X axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \Theta & -\sin \Theta & 0 \\ 0 & \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$



# Transformations in Scene Graphs



Rose et al. '96

# Summary



- Scene graphs
  - Hierarchical
  - Modeling transformations
  - Bounding volumes
- Coordinate systems
  - World coordinates
  - Modeling coordinates
- 3D modeling transformations
  - Represent most transformations by 4x4 matrices
  - Composite with matrix multiplication (order matters)