Are "Cubic" Linear Algebra Algorithms Really Cubic?

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Running Time - Is n^3 the Limit?

- How fast is matrix multiplication?

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} \end{aligned}$$

- Looks like 8 multiplies, 4 adds, right?
  (In general n^3 multiplies and $n^2(n-1)$ adds...)

Nothing Up My Sleeve…

- Strassen's method [1969]

Volker Strassen

$$
\begin{aligned}
M_1 &= (a_{11} + a_{22})(b_{11} + b_{22}) \\
M_2 &= (a_{21} + a_{22})b_{11} \\
M_3 &= a_{11}(b_{12} - b_{22}) \\
M_4 &= a_{22}(b_{21} - b_{11}) \\
M_5 &= (a_{11} + a_{12})b_{22} \\
M_6 &= (a_{21} - a_{11})(b_{11} + b_{12}) \\
M_7 &= (a_{12} - a_{22})(b_{21} + b_{22}) \\
c_{11} &= M_1 + M_4 - M_5 + M_7 \\
c_{12} &= M_3 + M_5 \\
c_{21} &= M_2 + M_4 \\
c_{22} &= M_1 - M_2 + M_3 + M_6
\end{aligned}
$$

Strassen's Method

- Requires only 7 multiplies (and a whole bunch of adds)
- Can be applied recursively!
  - Recursive application for 4 half-size submatrices needs 7 half-size matrix multiplies
  - Asymptotic running time proportional to $n^{\log_2 7} \approx n^{2.8}$
  - Only worth it for large n, because of all those additions...
  - Still, practically useful for $n >$ hundreds or thousands
- Current best algorithm achieves $n^{2.3728639}$
  - Not useful in practice...

Running Time - Is n^3 the Limit?

- Similar sub-cubic algorithms for inverse, LU, etc.
  - Most "cubic" linear-algebra problems aren't!
- What is the ultimate limit? We …ummm …don't know.
  - Is it n^2? $n^2 \log n$? $n^{\text{some ugly constant}}$?
  - Major open question, for such a fundamental problem!