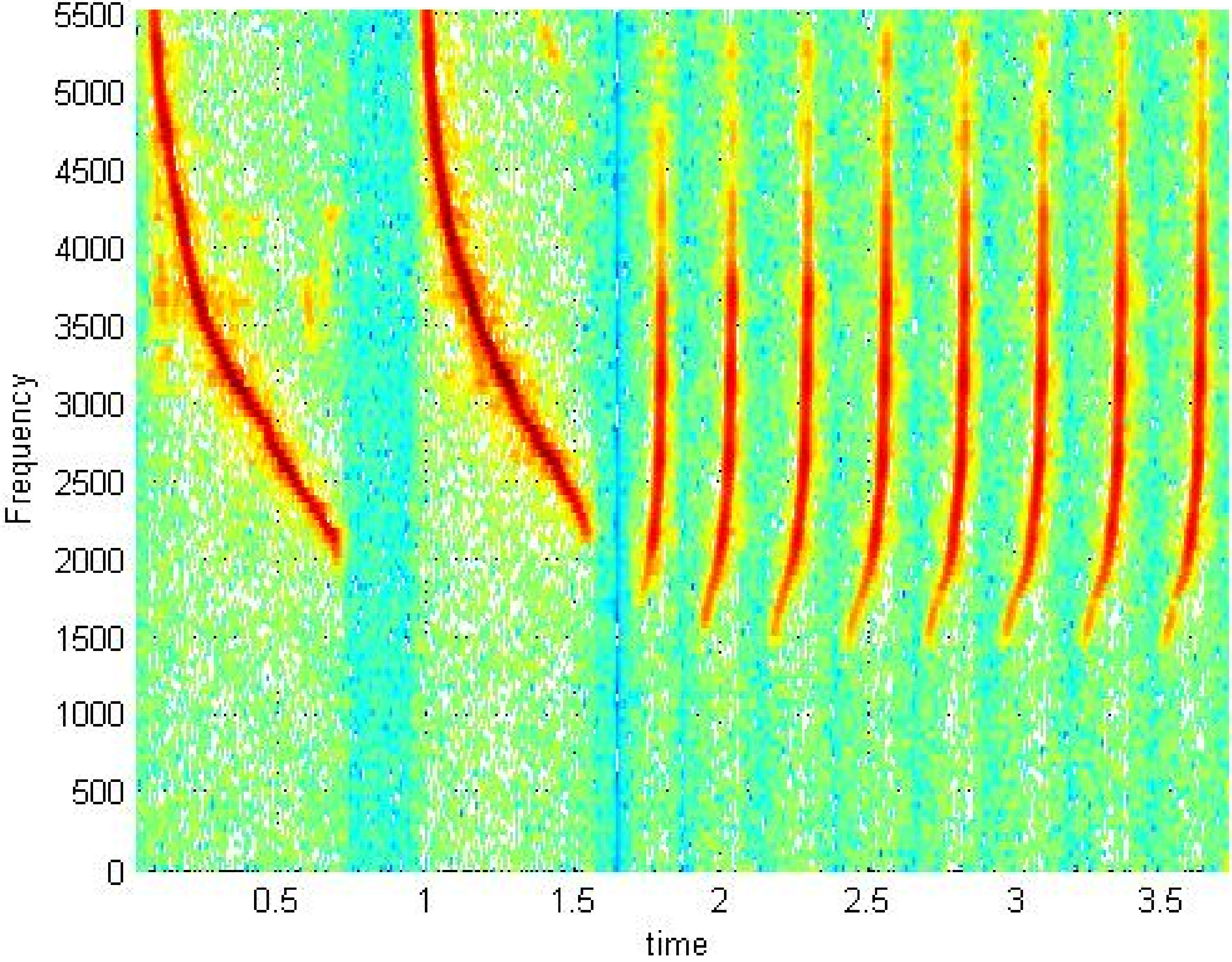
Fourier Transforms

COS 323

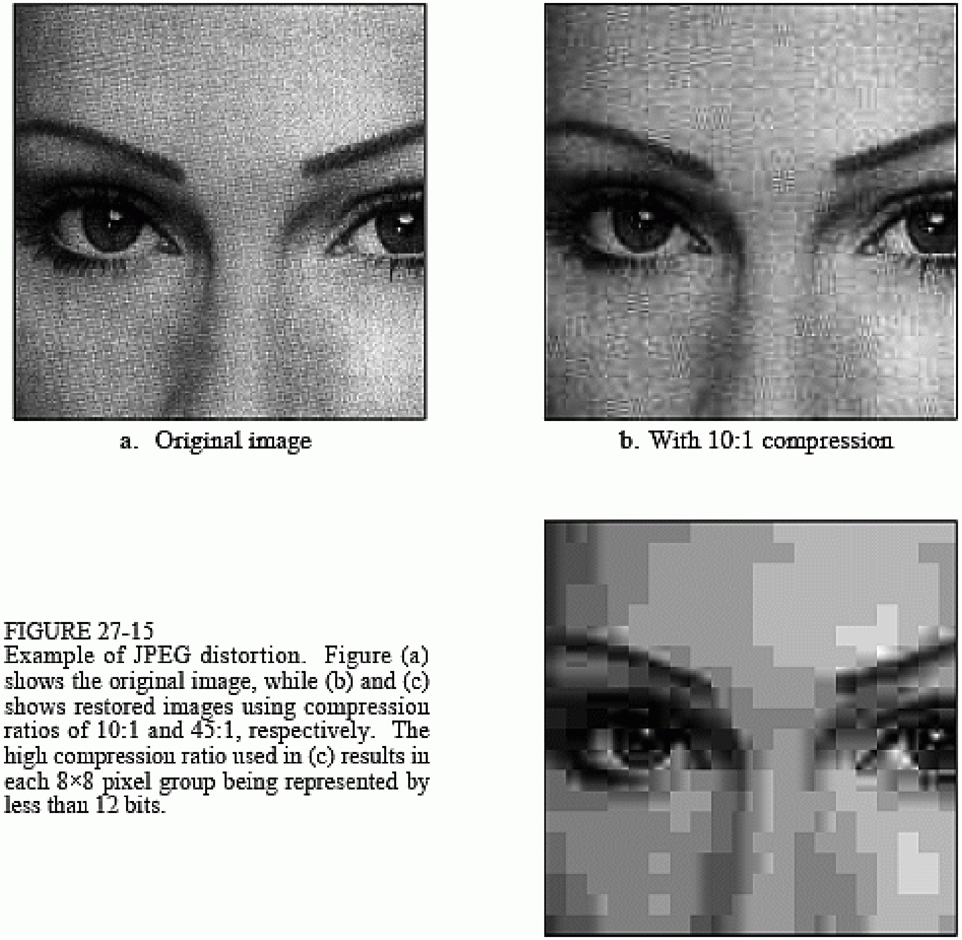
Life in the Frequency Domain

Jean Baptiste Joseph
Fourier (1768-1830)

Spectrogram, Northern Cardinal



JPEG Image Compression



a. Original image

b. With 10:1 compression

c. With 45:1 compression

FIGURE 27-15
Example of JPEG distortion. Figure (a) shows the original image, while (b) and (c) shows restored images using compression ratios of 10:1 and 45:1, respectively. The high compression ratio used in (c) results in each 8×8 pixel group being represented by less than 12 bits.

Discrete
Cosine
Transform
(DCT)

[Steven W. Smith 1997]

The Convolution Theorem

- Fourier transform turns convolution into multiplication:

$$\mathcal{F}\big(f(x) * g(x)\big) = \mathcal{F}\big(f(x)\big)\ \mathcal{F}\big(g(x)\big)$$

(and vice versa):

$$\mathcal{F}\big(f(x)\, g(x)\big) = \mathcal{F}\big(f(x)\big) * \mathcal{F}\big(g(x)\big)$$

Discrete Fourier Transform (DFT)

- *f* is a discrete signal: samples $f_0, f_1, f_2, \ldots, f_{n-1}$
- *f* can be built up out of sinusoids (or complex exponentials) of frequencies 0 through *n*-1:

$$f_x = \frac{1}{n} \sum_{k=0}^{n-1} F_k e^{2\pi i \frac{k}{n} x}$$

- *F* is a function of frequency – describes "how much" *f* contains of sinusoids at frequency *k*
- Computing *F* – the Discrete Fourier Transform:

$$F_k = \sum_{x=0}^{n-1} f_x e^{-2\pi i \frac{k}{n} x}$$

DFT and Inverse DFT (IDFT)

$$F_k = \sum_{x=0}^{n-1} f_x \, e^{-2\pi i \frac{k}{n} x}$$

$$\updownarrow$$

$$f_x = \frac{1}{n} \sum_{k=0}^{n-1} F_k \, e^{2\pi i \frac{k}{n} x}$$

Computing Discrete Fourier Transform

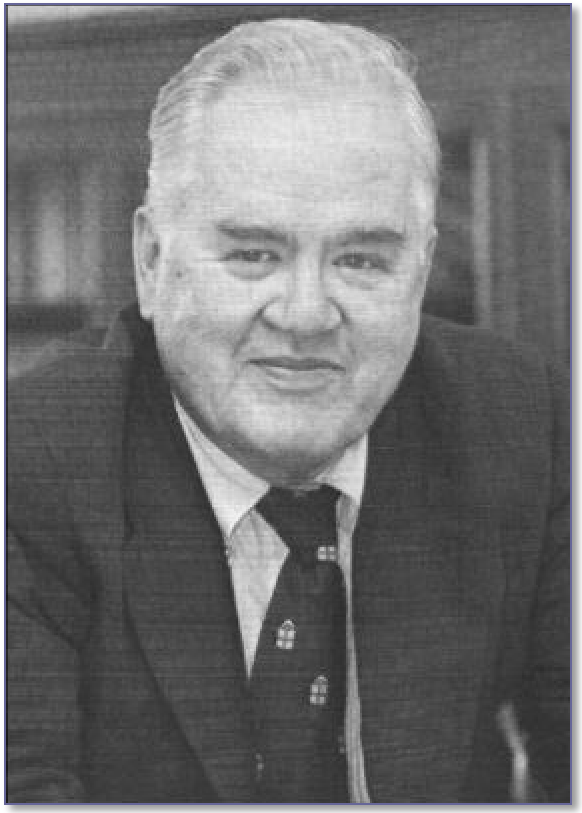
$$F_k = \sum_{x=0}^{n-1} f_x \, e^{-2\pi i \frac{k}{n} x}$$

- Straightforward computation: for each of *n* DFT values, loop over *n* input samples. Total: $O(n^2)$
- Fast Fourier Transform (FFT): $O(n \log_2 n)$ time
  - Revolutionized signal processing, filtering, compression, etc.
  - Also turns out to have less roundoff error

The FFT

Discovered by Johann Carl Friedrich Gauss (1777-1855)

The FFT



Rediscovered and popularized in 1965 by
J. W. Cooley and John Tukey (Princeton alum and faculty)

The FFT

$$F_k = \sum_{x=0}^{n-1} f_x \, e^{-2\pi i \frac{k}{n} x}$$

Let $\omega_n = e^{-2\pi i / n} = \cos(2\pi / n) - i \sin(2\pi / n)$

Then $F_k = \sum_{x=0}^{n-1} f_x \omega_n^{xk}$

$$= \sum_{x=0}^{n/2-1} f_{2x} \omega_n^{2xk} + \sum_{x=0}^{n/2-1} f_{2x+1} \omega_n^{(2x+1)k}$$

The FFT

Key idea: divide and conquer

- Separate computation on even and odd elements

$$
\begin{aligned}
F_k &= \sum_{x=0}^{n/2-1} f_{2x}\,\omega_n^{2xk} + \sum_{x=0}^{n/2-1} f_{2x+1}\,\omega_n^{(2x+1)k} \\
&= \underbrace{\sum_{x=0}^{n/2-1} f_{2x}\,\omega_{n/2}^{xk}}_{\text{Half-size FFT on even elements}} + \omega_n^{k} \underbrace{\sum_{x=0}^{n/2-1} f_{2x+1}\,\omega_{n/2}^{xk}}_{\text{Half-size FFT on odd elements}}
\end{aligned}
$$

Example (n = 4)

- From the definition:

$$F_0 = f_0\omega_n^{0\cdot 0} + f_1\omega_n^{1\cdot 0} + f_2\omega_n^{2\cdot 0} + f_3\omega_n^{3\cdot 0}$$

$$F_1 = f_0\omega_n^{0\cdot 1} + f_1\omega_n^{1\cdot 1} + f_2\omega_n^{2\cdot 1} + f_3\omega_n^{3\cdot 1}$$

$$F_2 = f_0\omega_n^{0\cdot 2} + f_1\omega_n^{1\cdot 2} + f_2\omega_n^{2\cdot 2} + f_3\omega_n^{3\cdot 2}$$

$$F_3 = f_0\omega_n^{0\cdot 3} + f_1\omega_n^{1\cdot 3} + f_2\omega_n^{2\cdot 3} + f_3\omega_n^{3\cdot 3}$$

Example (n = 4)

- Using the fact that $\omega_n^4 = 1$

$$F_0 = f_0\omega_n^0 + f_1\omega_n^0 + f_2\omega_n^0 + f_3\omega_n^0$$

$$F_1 = f_0\omega_n^0 + f_1\omega_n^1 + f_2\omega_n^2 + f_3\omega_n^3$$

$$F_2 = f_0\omega_n^0 + f_1\omega_n^2 + f_2\omega_n^0 + f_3\omega_n^2$$

$$F_3 = f_0\omega_n^0 + f_1\omega_n^3 + f_2\omega_n^2 + f_3\omega_n^5$$

Example (n = 4)

- Group even and odd terms, factor:

$$
\begin{aligned}
F_0 &= \left(f_0\omega_n{}^0 + f_2\omega_n{}^0\right) + \omega_n{}^0\left(f_1\omega_n{}^0 + f_3\omega_n{}^0\right)\\
F_1 &= \left(f_0\omega_n{}^0 + f_2\omega_n{}^2\right) + \omega_n{}^1\left(f_1\omega_n{}^0 + f_3\omega_n{}^2\right)\\
F_2 &= \left(f_0\omega_n{}^0 + f_2\omega_n{}^0\right) + \omega_n{}^2\left(f_1\omega_n{}^0 + f_3\omega_n{}^0\right)\\
F_3 &= \left(f_0\omega_n{}^0 + f_2\omega_n{}^2\right) + \omega_n{}^3\left(f_1\omega_n{}^0 + f_3\omega_n{}^2\right)
\end{aligned}
$$

Example (n = 4)

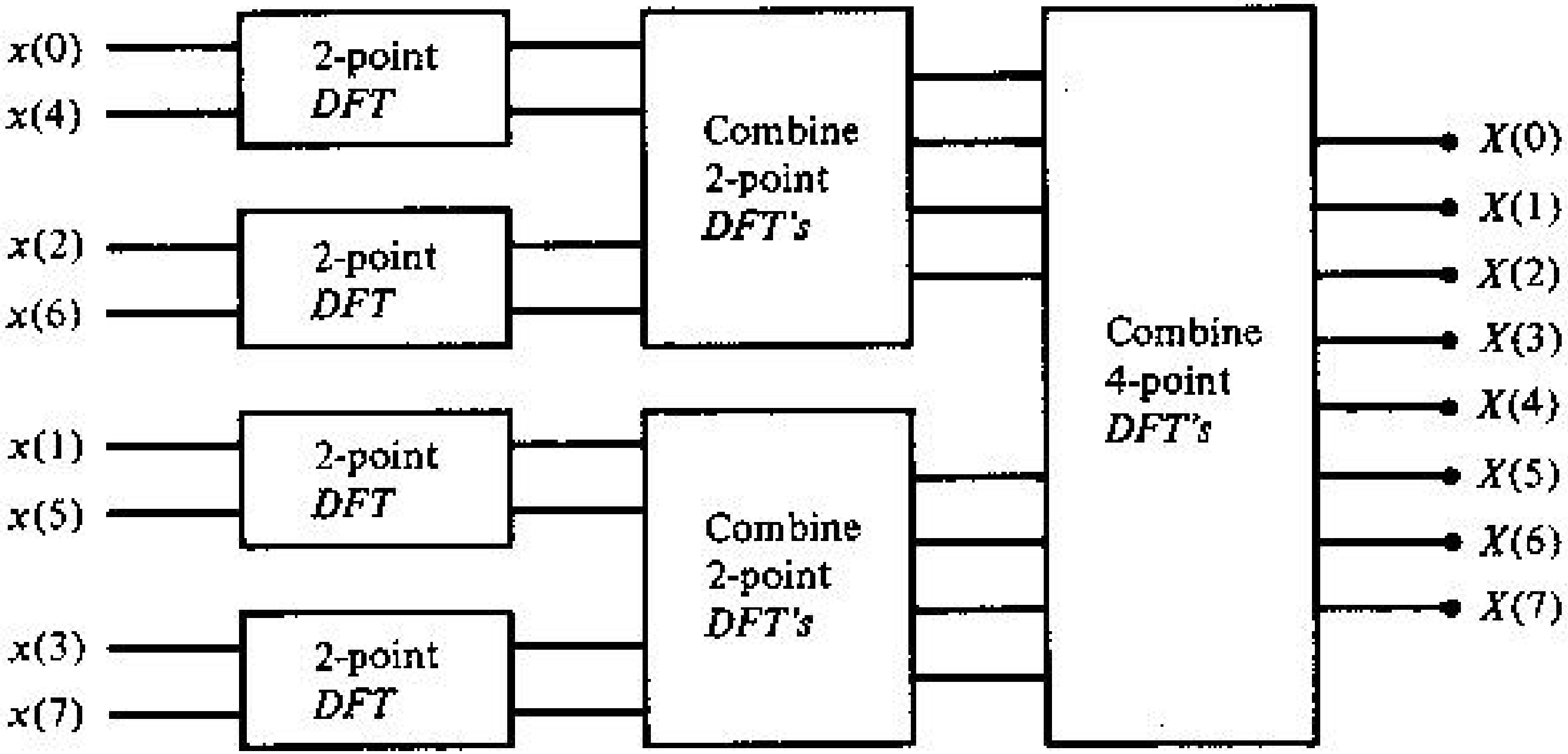
- This can be computed from two length-2 DFTs, with some “twiddle factors”

$$
\begin{aligned}
F_0 &= \left(f_0\omega_n{}^0 + f_2\omega_n{}^0\right) + \omega_n{}^0\left(f_1\omega_n{}^0 + f_3\omega_n{}^0\right)\\
F_1 &= \left(f_0\omega_n{}^0 + f_2\omega_n{}^2\right) + \omega_n{}^1\left(f_1\omega_n{}^0 + f_3\omega_n{}^2\right)\\
F_2 &= \left(f_0\omega_n{}^0 + f_2\omega_n{}^0\right) + \omega_n{}^2\left(f_1\omega_n{}^0 + f_3\omega_n{}^0\right)\\
F_3 &= \left(f_0\omega_n{}^0 + f_2\omega_n{}^2\right) + \omega_n{}^3\left(f_1\omega_n{}^0 + f_3\omega_n{}^2\right)
\end{aligned}
$$

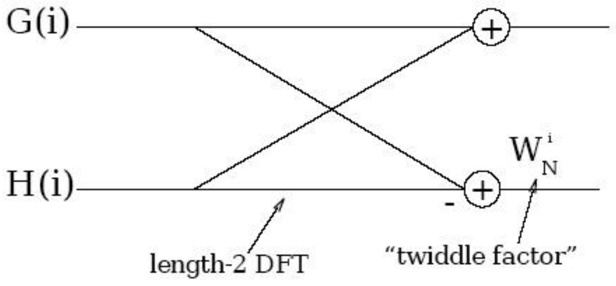
$$
\begin{matrix} f_0 \\ f_2 \end{matrix} \xrightarrow{DFT} \begin{matrix} f_0\omega_{n/2}{}^0 + f_2\omega_{n/2}{}^0 \\ f_0\omega_{n/2}{}^0 + f_2\omega_{n/2}{}^1 \end{matrix}
\qquad
\begin{matrix} f_1 \\ f_3 \end{matrix} \xrightarrow{DFT} \begin{matrix} f_1\omega_{n/2}{}^0 + f_3\omega_{n/2}{}^0 \\ f_1\omega_{n/2}{}^0 + f_3\omega_{n/2}{}^1 \end{matrix}
$$

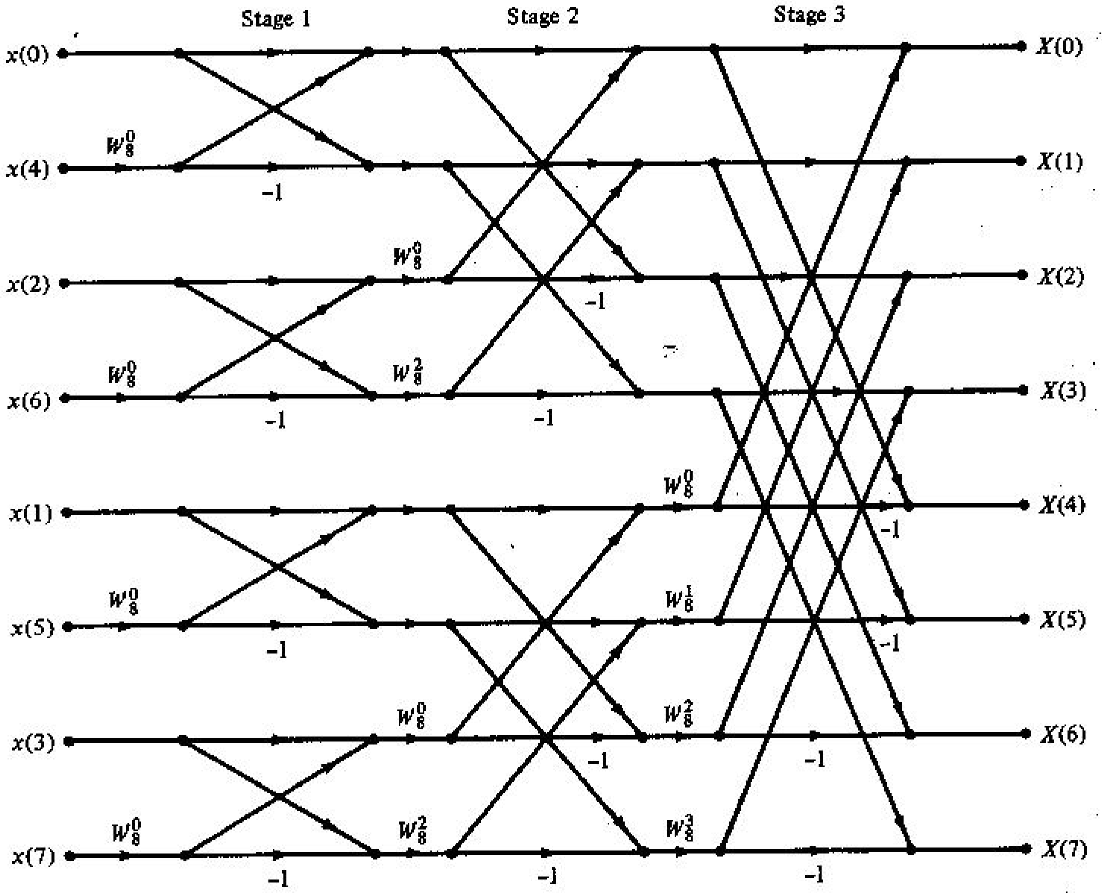
The FFT

- Now apply algorithm recursively!



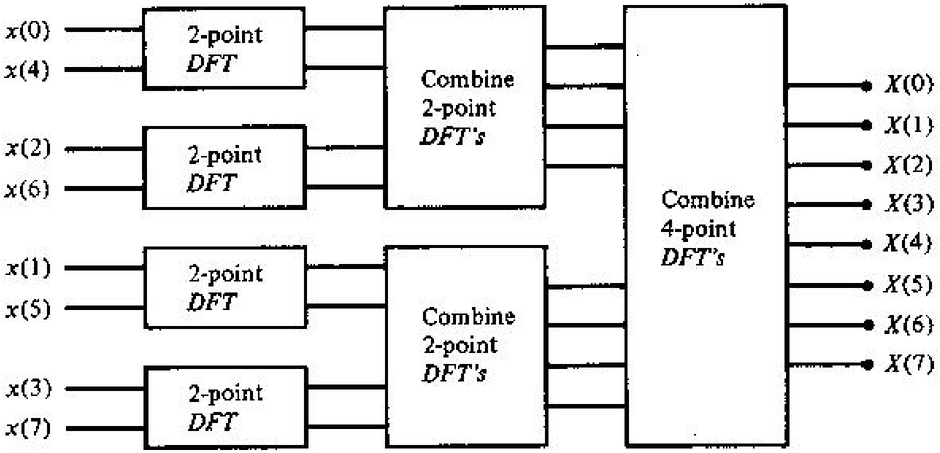
FFT Butterfly





The FFT

- Final detail: how to find elements involved in initial size-2 FFTs?



- Bit reversal!

$0 \to 000 \to 000 \to 0$
$1 \to 001 \to 100 \to 4$
$2 \to 010 \to 010 \to 2$
$3 \to 011 \to 110 \to 6$
$4 \to 100 \to 001 \to 1$
$5 \to 101 \to 101 \to 5$
$6 \to 110 \to 011 \to 3$
$7 \to 111 \to 111 \to 7$

FFT Running Time

- Time to compute FFT of length *n*:
  - Solve two subproblems of length *n*/2
  - Additional processing proportional to *n*

$$T(n) = 2T(n/2) + cn$$

- Recurrence relation with solution

$$T(n) = c\,n\log_2 n$$

FFT Running Time

- Proof:

$$T(n) = 2T(n/2) + cn$$

$$c\,n \log_2 n \overset{?}{=} 2\left(c\,{}^{n}\!/_{2} \log_2 {}^{n}\!/_{2}\right) + cn$$

$$c\,n \log_2 n \overset{?}{=} c\,n\left((\log_2 n) - 1\right) + cn$$

$$c\,n \log_2 n \overset{\checkmark}{=} c\,n \log_2 n - cn + cn$$

DFT of Real Signals

- Standard FFT is complex → complex
  - n real numbers as input yields n complex numbers
  - But: symmetry relation for real inputs $F_{n-k} = (F_k)^*$
  - Variants of FFT to compute this efficiently
- Discrete Cosine Transform (DCT)
  - Reflect real input to get signal of length $2n$
  - Resulting FFT real and symmetric
  - n real numbers as input, n real numbers as output

Application: JPEG Image Compression

- Perceptually-based lossy compression of images
- Algorithm
  - Transform colors
  - Divide into 8×8 blocks
  - 2-dimensional DCT on each block
  - Perceptually-guided quantization
  - Lossless run-length and Huffman encoding

Application: JPEG Image Compression

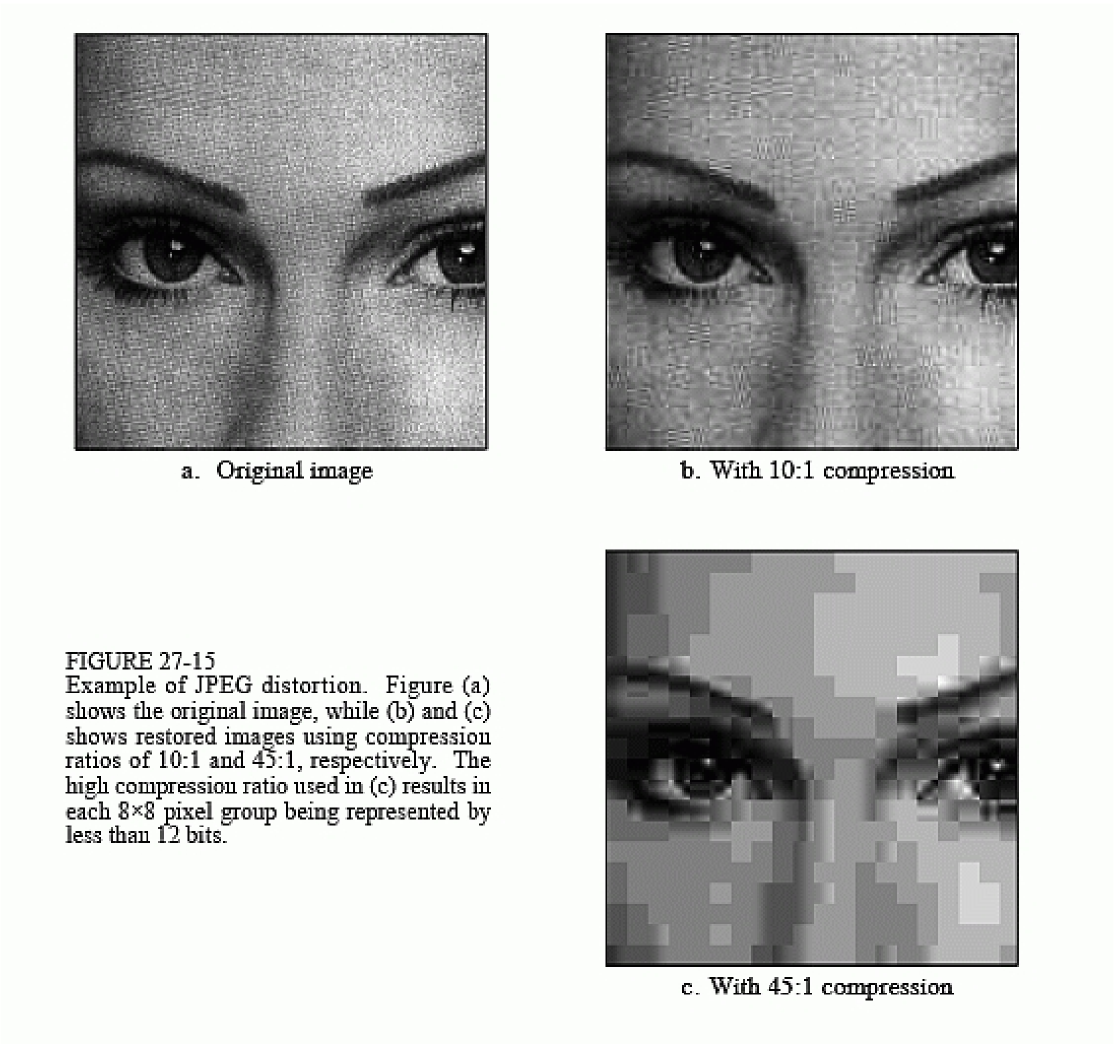


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FIGURE 27-15
Example of JPEG distortion. Figure (a) shows the original image, while (b) and (c) shows restored images using compression ratios of 10:1 and 45:1, respectively. The high compression ratio used in (c) results in each 8×8 pixel group being represented by less than 12 bits.

Discrete
Cosine
Transform
(DCT)

[Steven W. Smith 1997]

Application: Polynomial Multiplication

- Usual algorithm for multiplying two polynomials of degree n is $O(n^2)$
- Observation: can use DFT to efficiently go between polynomial coefficients f_x

  $$f(t) = \sum_{x=0}^{n-1} f_x t^x$$

  and polynomial evaluated at ω_n^k

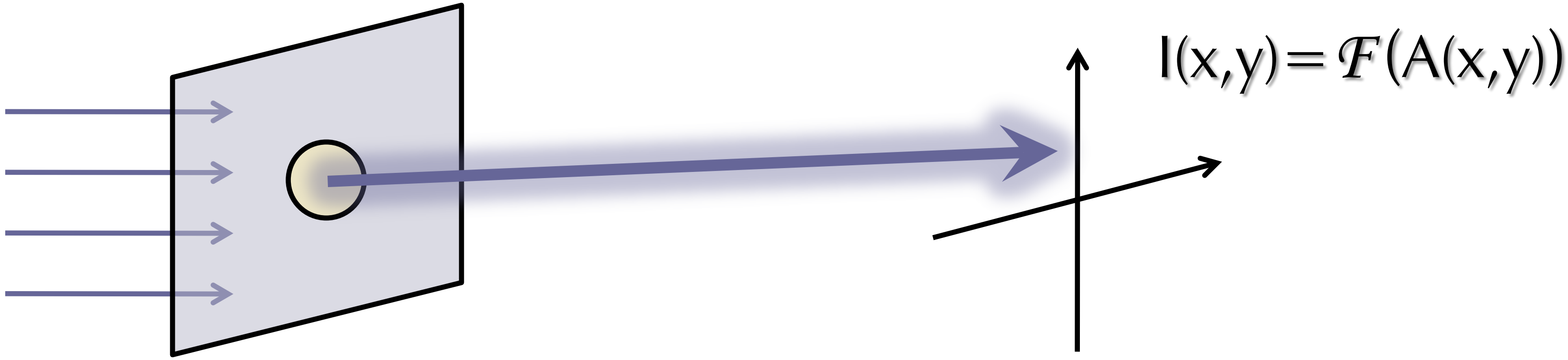
  $$f(\omega_n^k) = F_k = \sum_{x=0}^{n-1} f_x \omega_n^{kx}$$

Application: Polynomial Multiplication

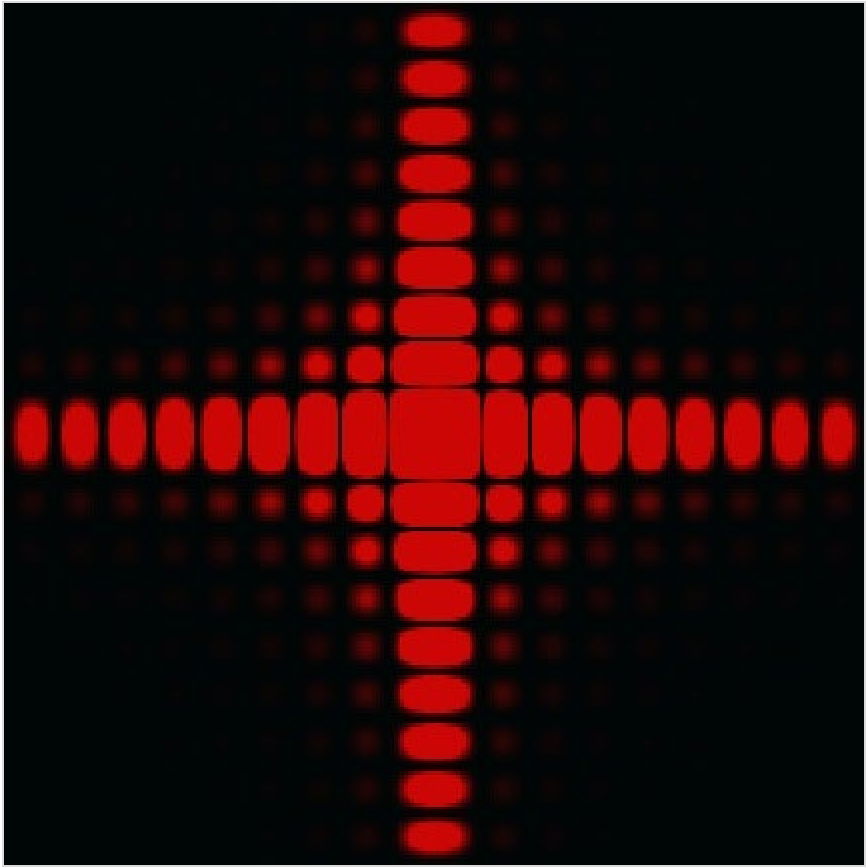
- So, we have an O(*n* log *n*) algorithm for multiplying two degree-*n* polynomials:
  - DFT on coefficients
  - Multiply
  - Inverse DFT
- Polynomial multiplication is convolution!

Application: Diffraction

- (Far-field) diffraction pattern of parallel light passing through an aperture is Fourier transform of aperture

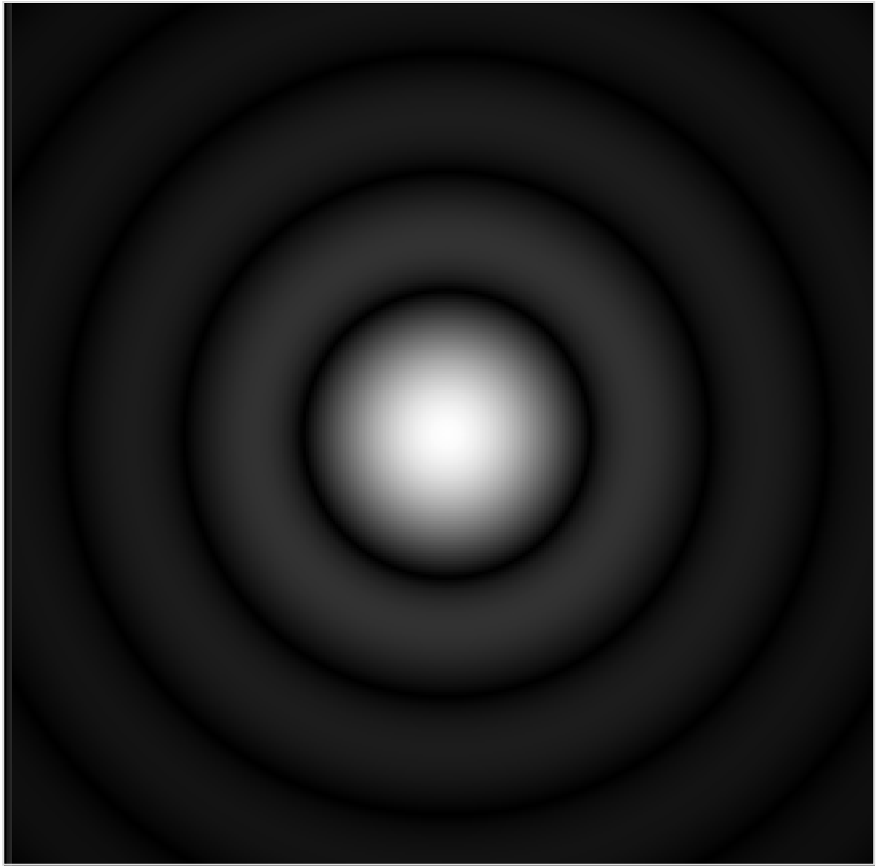


Application: Diffraction



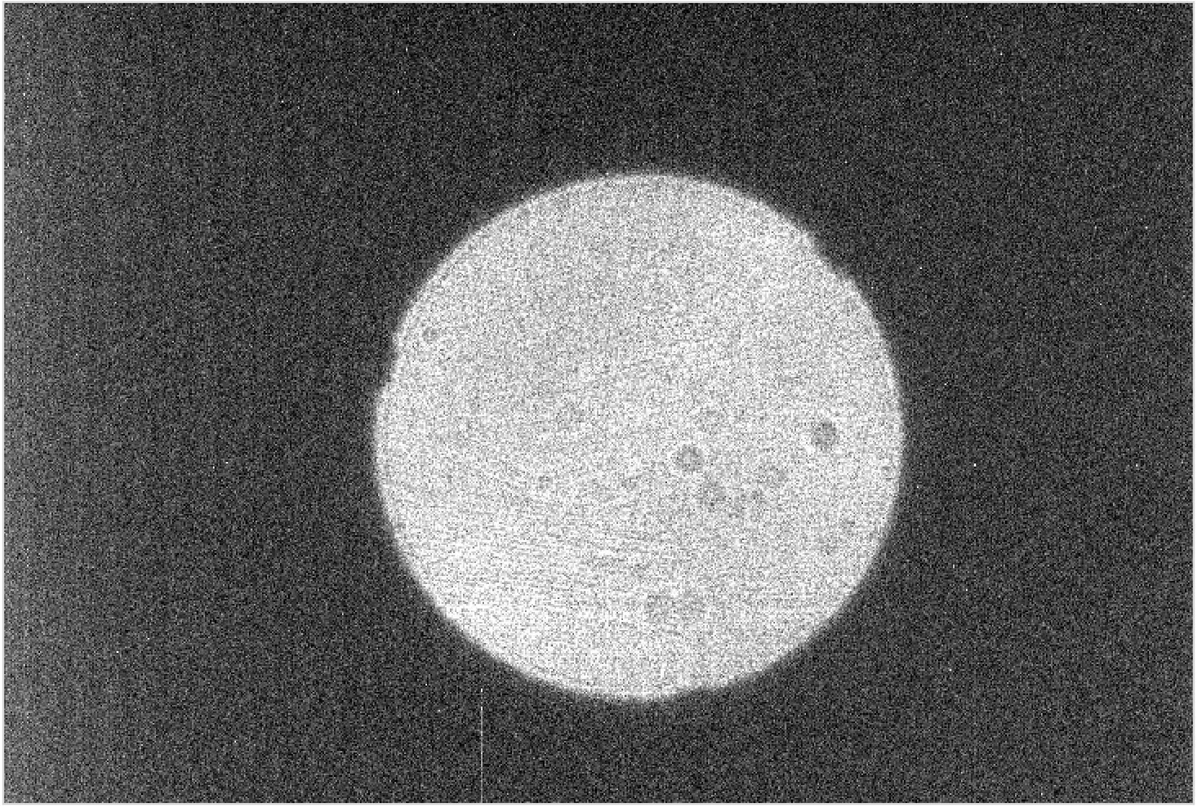
Square aperture

Application: Diffraction



Circular aperture: Airy disk

Application: Diffraction



Diffraction + defocus in telescope image