Finite Difference Approximations For Derivatives

Taylor Series

- Goal: given smooth function $f : \Re \rightarrow \Re$,
  find approximate derivatives at some point x
- Consider Taylor series expansions around x:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{2}h^3 + \ldots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{2}h^3 + \ldots$$

Forward Difference

- Starting from first equation,

$$f(x+h) \approx f(x) + f'(x)h + O(h^2)$$

$$\boxed{f'(x) \approx \frac{f(x+h) - f(x)}{h}} + O(h)$$

- This is the *forward-difference* approximation to the first derivative: first-order accurate

Backward Difference

- Similarly, starting from second equation,

$$f(x-h) \approx f(x) - f'(x)h + O(h^2)$$

$$\boxed{f'(x) \approx \frac{f(x) - f(x-h)}{h}} + O(h)$$

- This is the *backward-difference* approximation to the first derivative: also first-order accurate

Centered Difference

- Subtract the two Taylor-series expansions:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{2}h^3 + \dots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{2}h^3 + \dots$$

$$f(x+h) - f(x-h) \approx 2f'(x)h + O(h^3)$$

$$\boxed{f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}} + O(h^2)$$

- This is the *centered-difference* approximation to the first derivative: second-order accurate

Second Derivative

- Now add the two Taylor-series expansions:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{2}h^3 + \ldots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{2}h^3 + \ldots$$

$$f(x+h) + f(x-h) \approx 2f(x) + f''(x)h^2 + O(h^4)$$

$$\boxed{f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}} + O(h^2)$$

- This is the centered-difference approximation to the *second* derivative: second-order accurate