Chaos

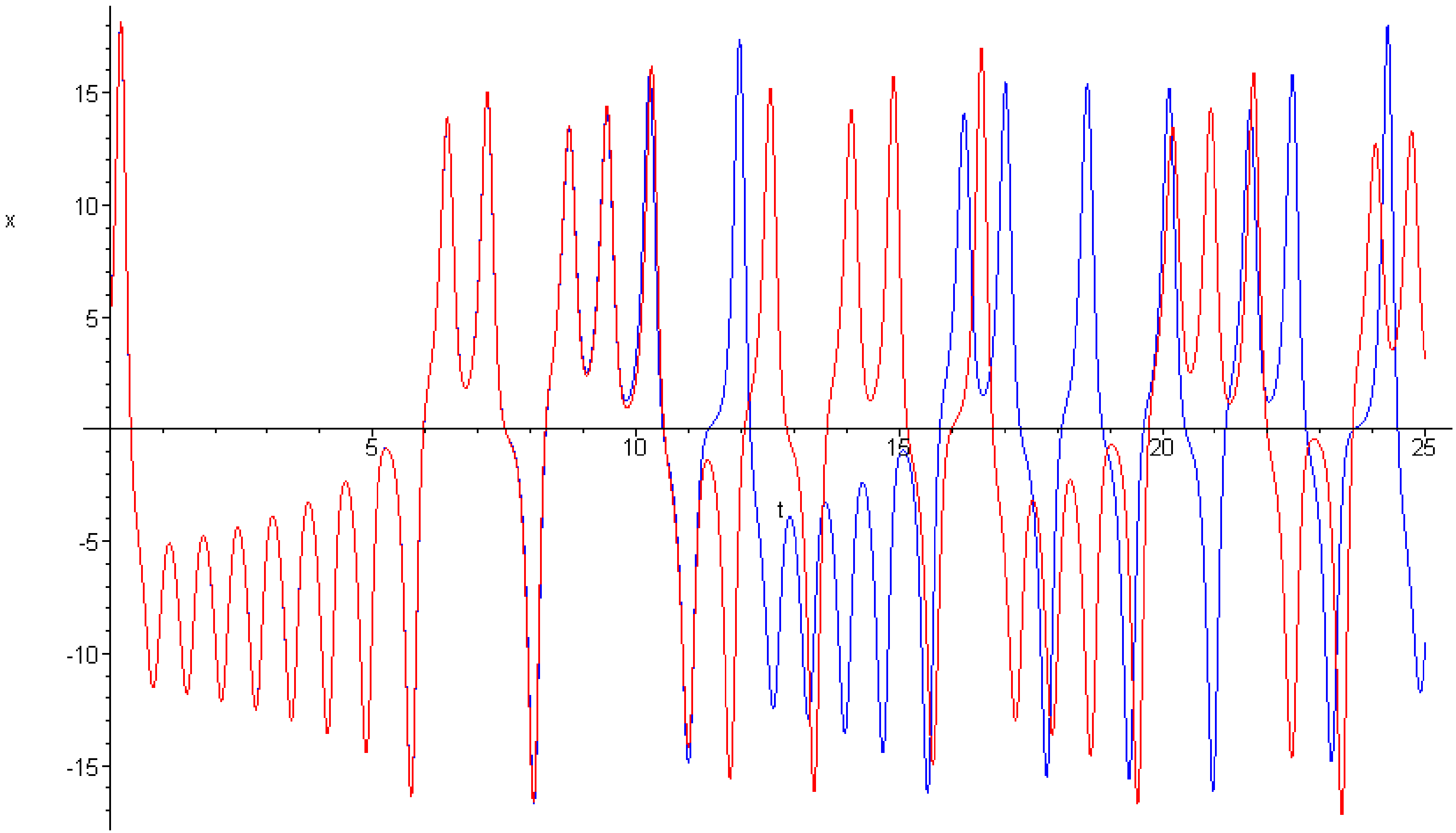
Lorenz Equations

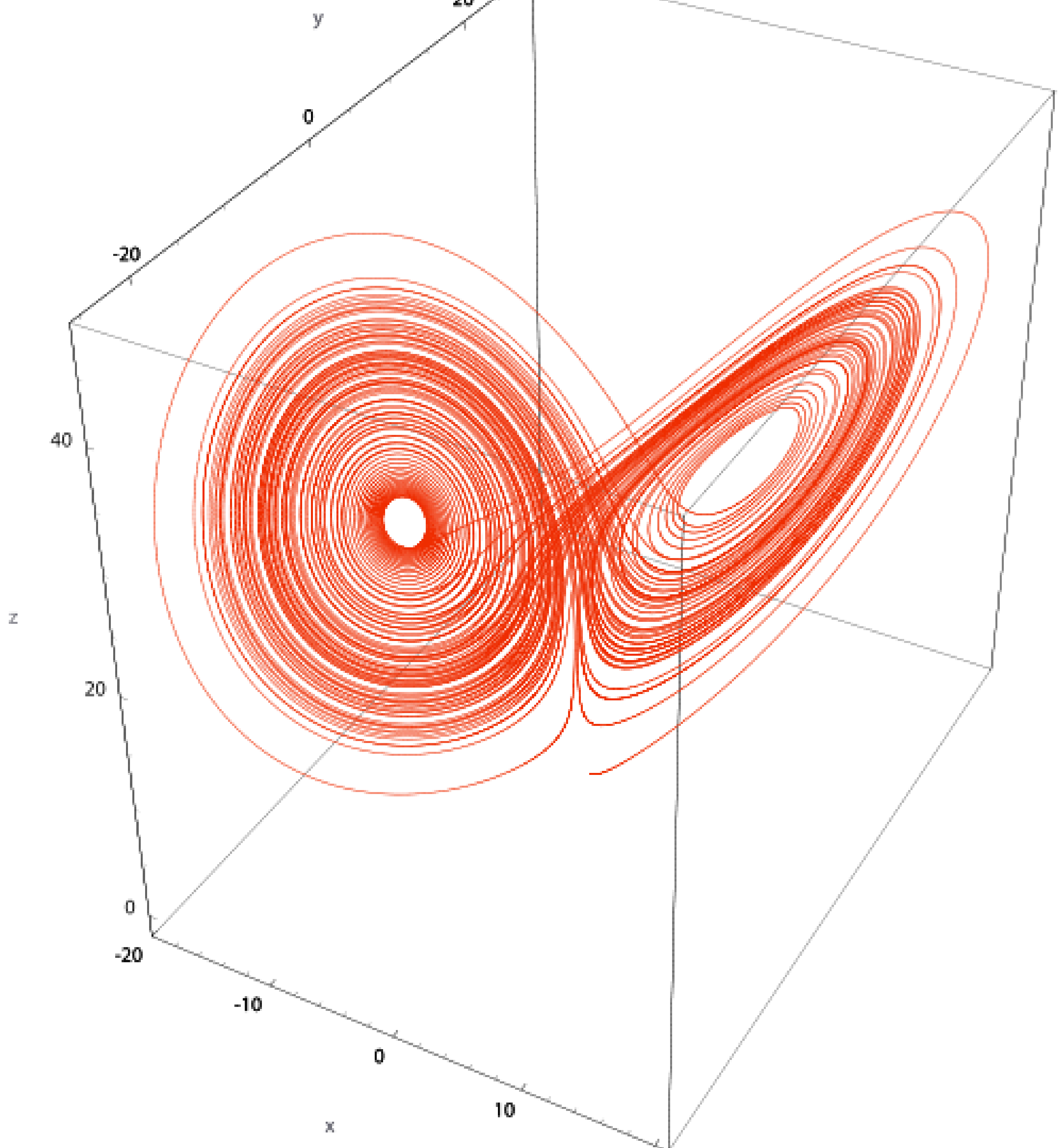
$$\frac{dx}{dt} = -\sigma x + \sigma y$$

$$\frac{dy}{dt} = rx - y - xz$$

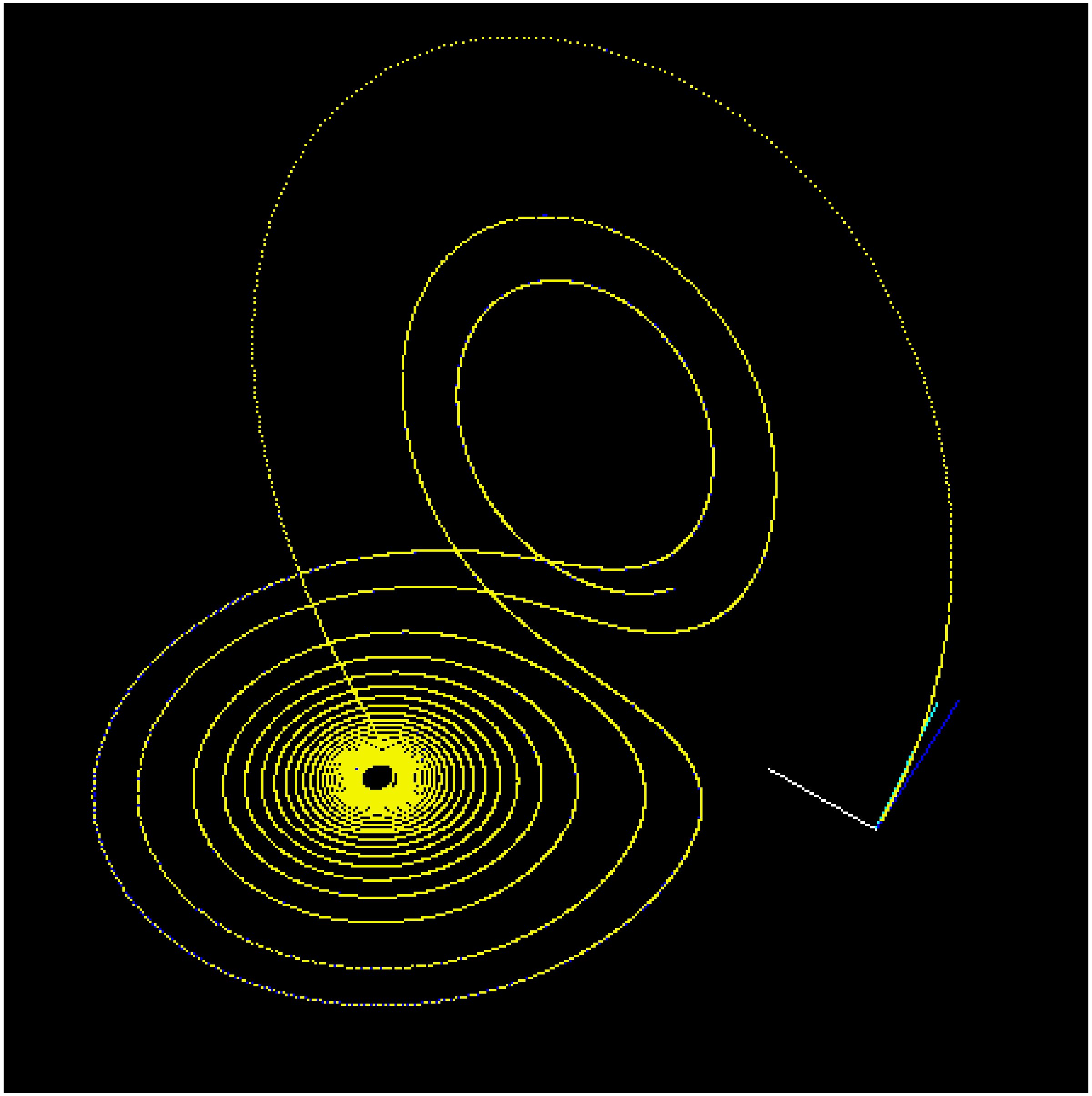
$$\frac{dz}{dt} = -bz + xy$$

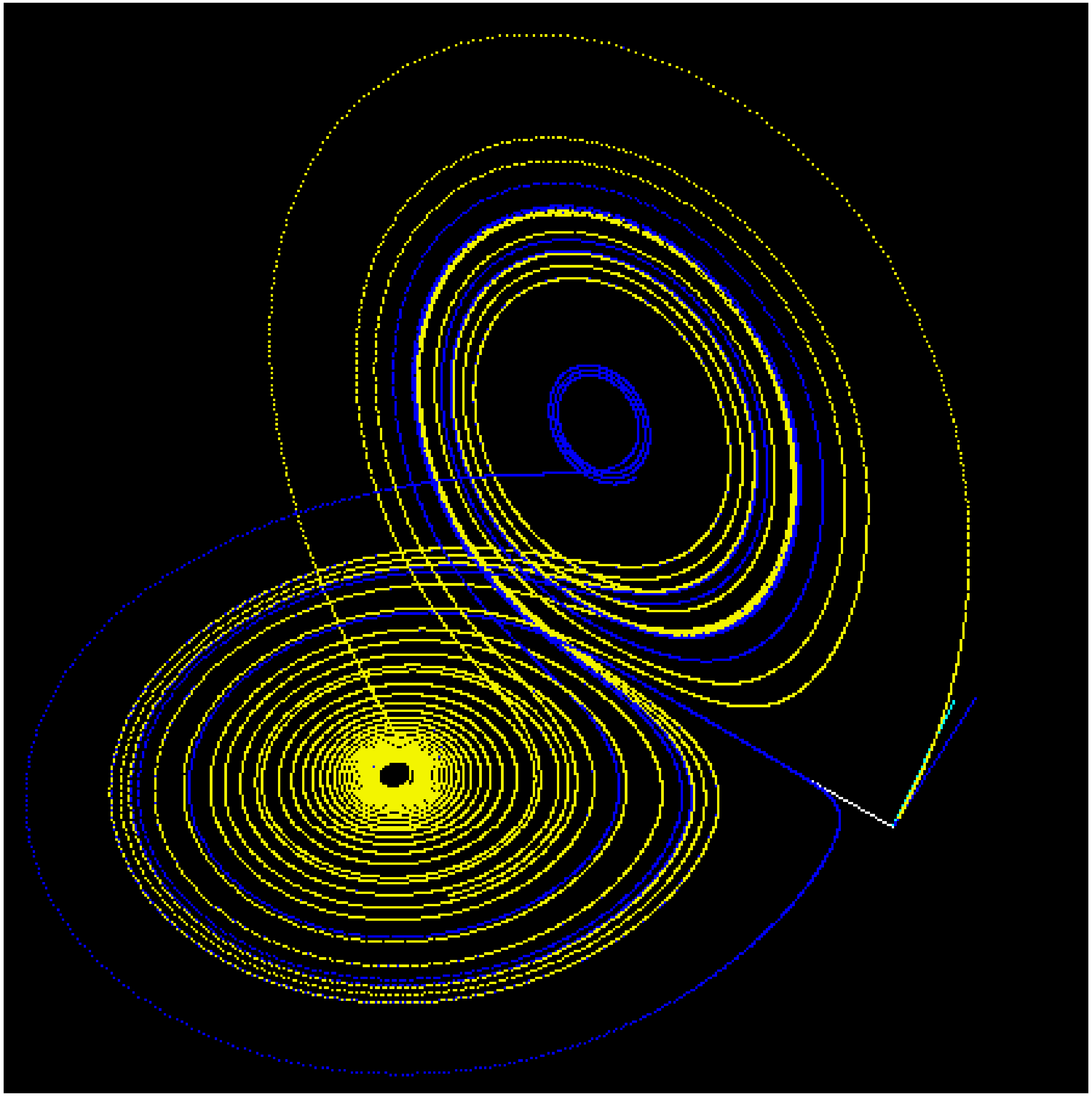
x over time for 2 initial conditions

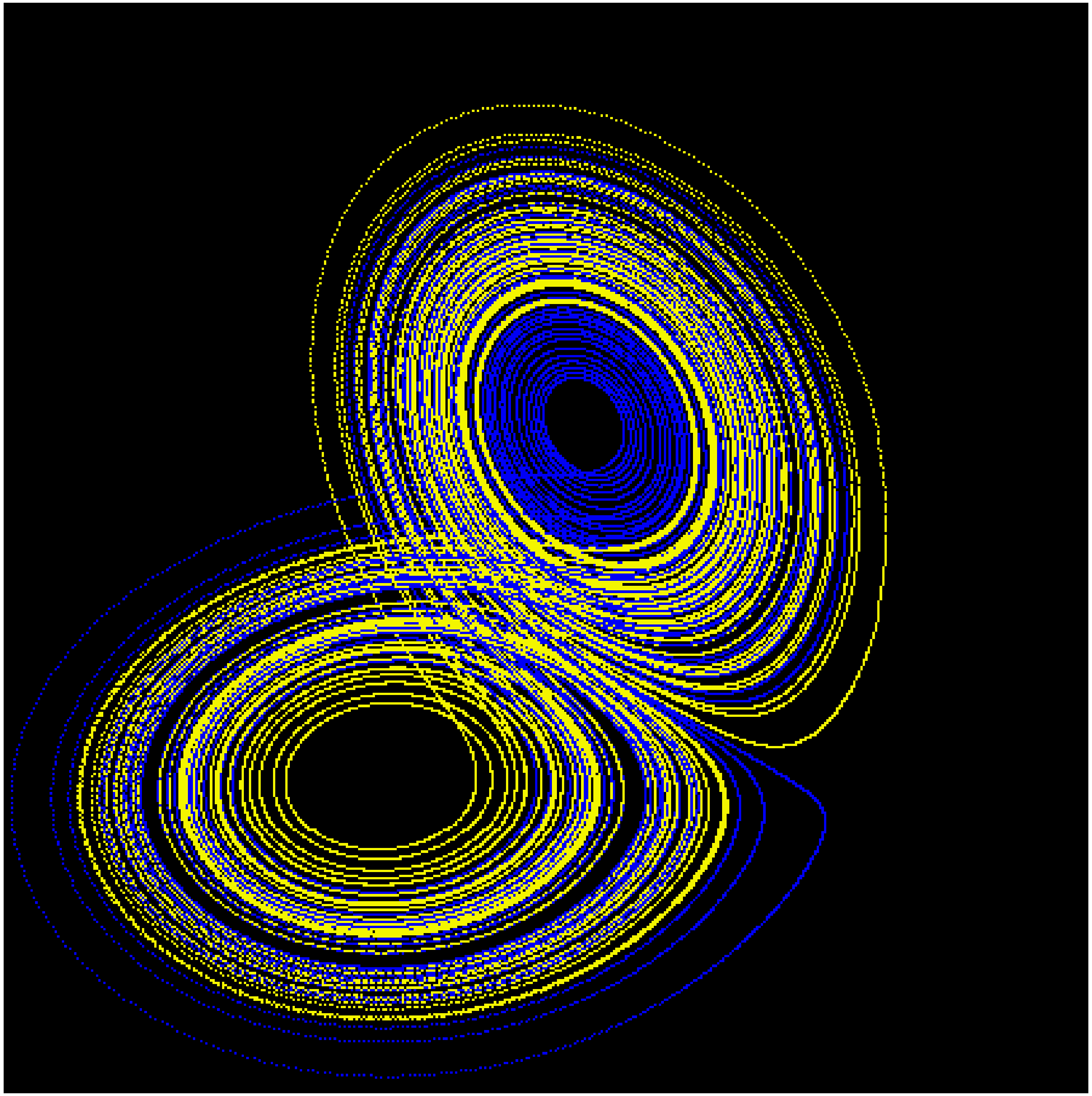




y
20
0
-20
40
z
20
0
-20
-10
0
10
x







Logistic Map

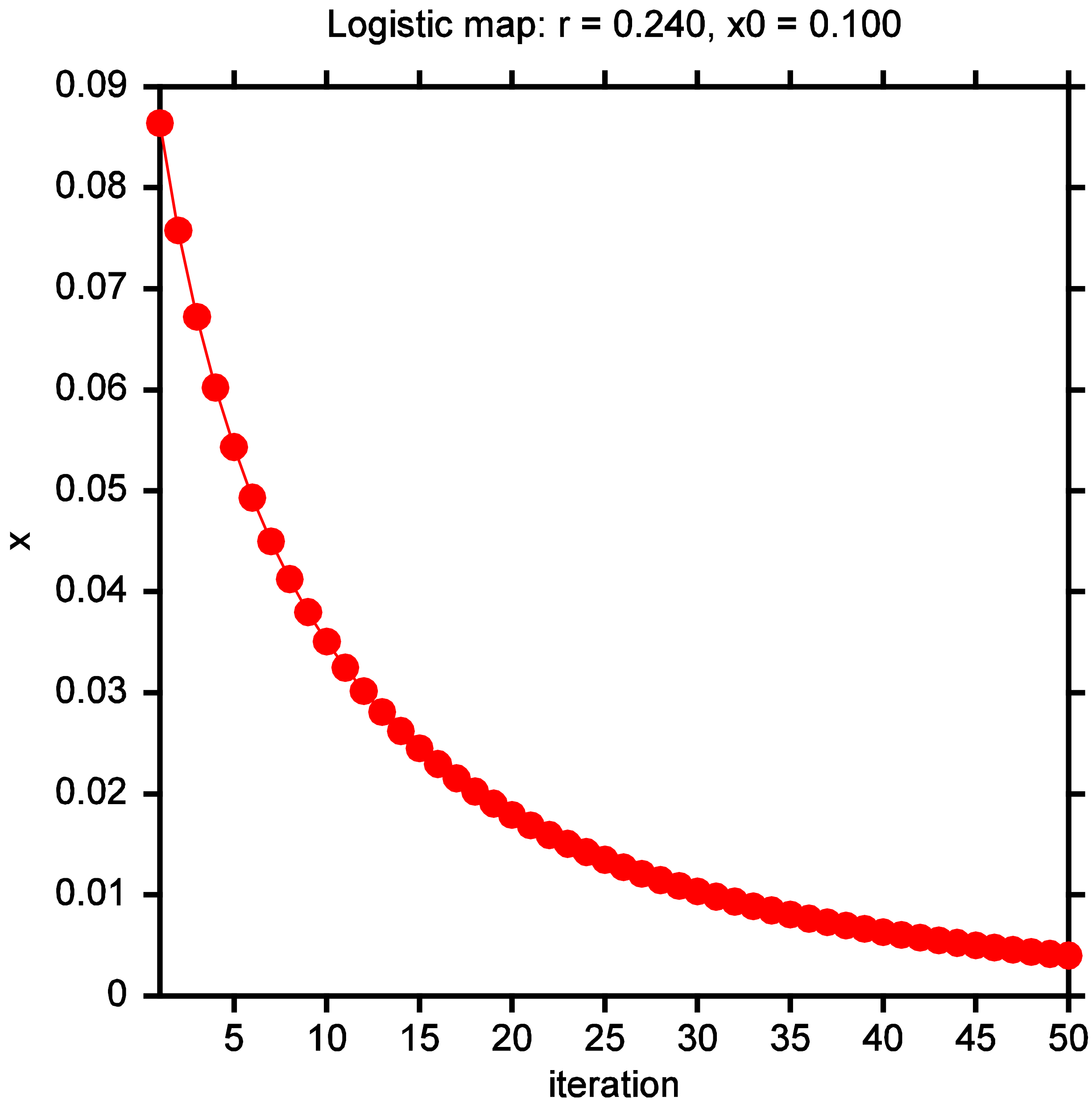
- Verhulst equation:

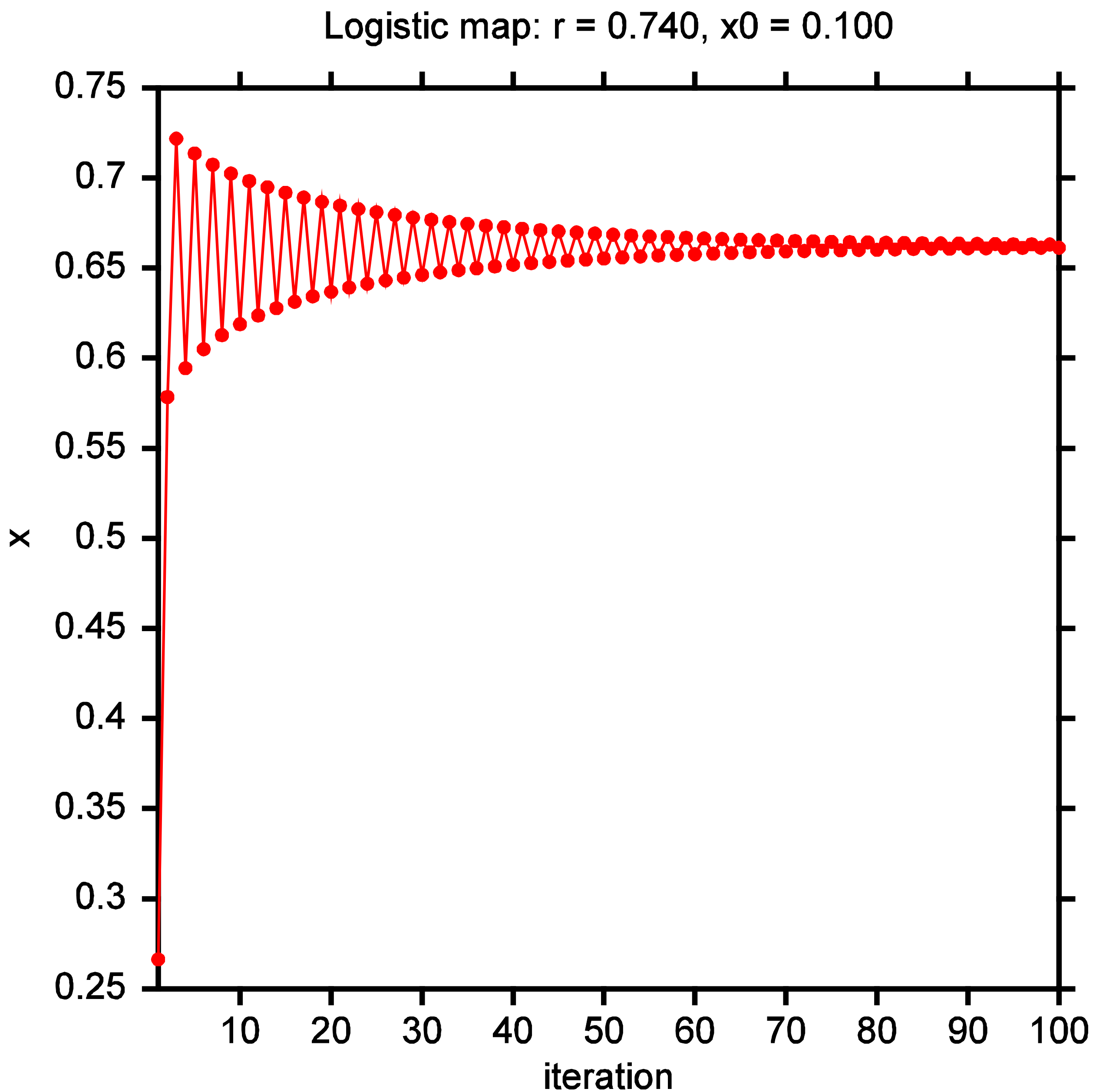
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

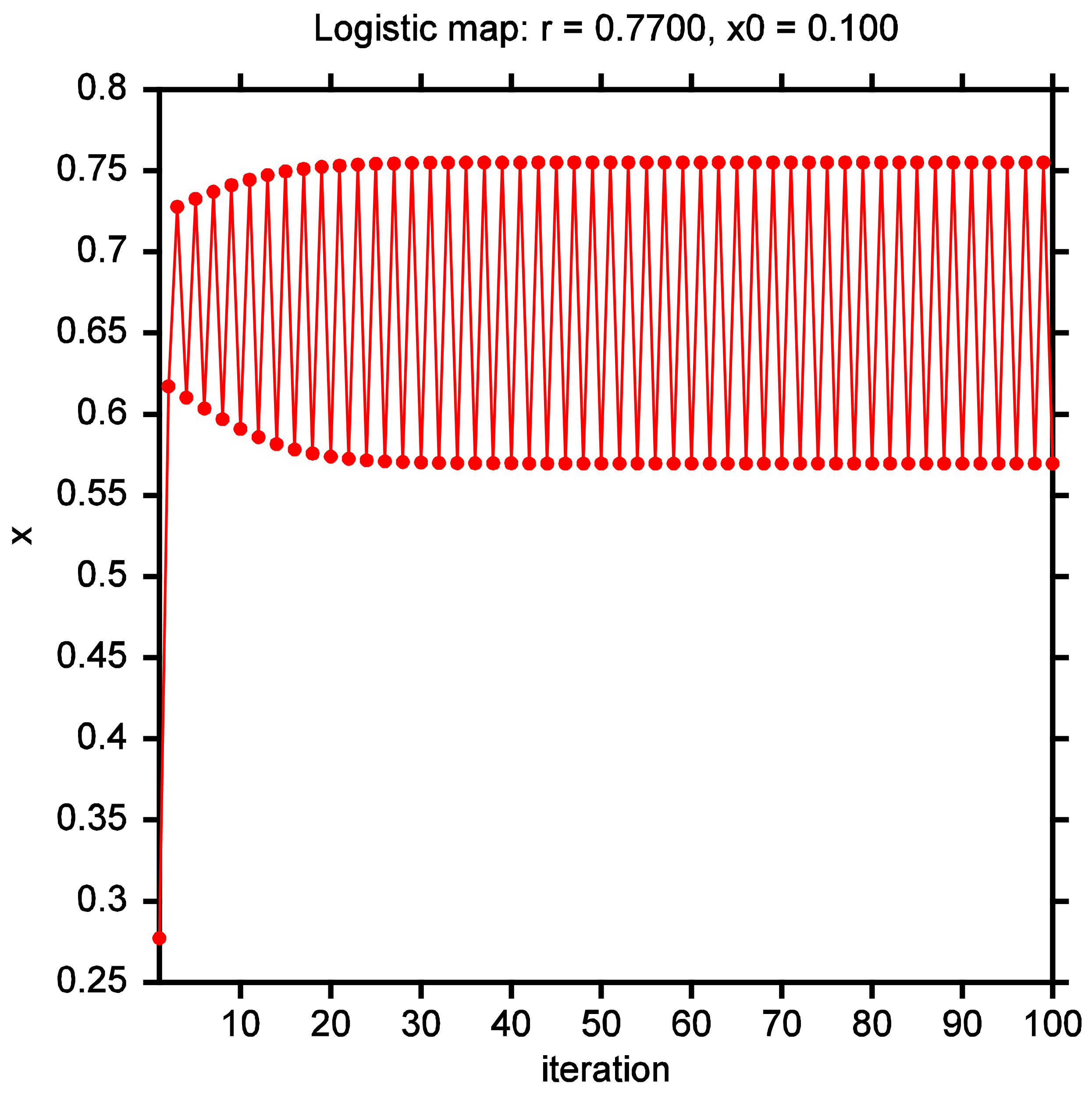
- Logistic map:

$$x_{n+1} = 4rx_n(1 - x_n)$$

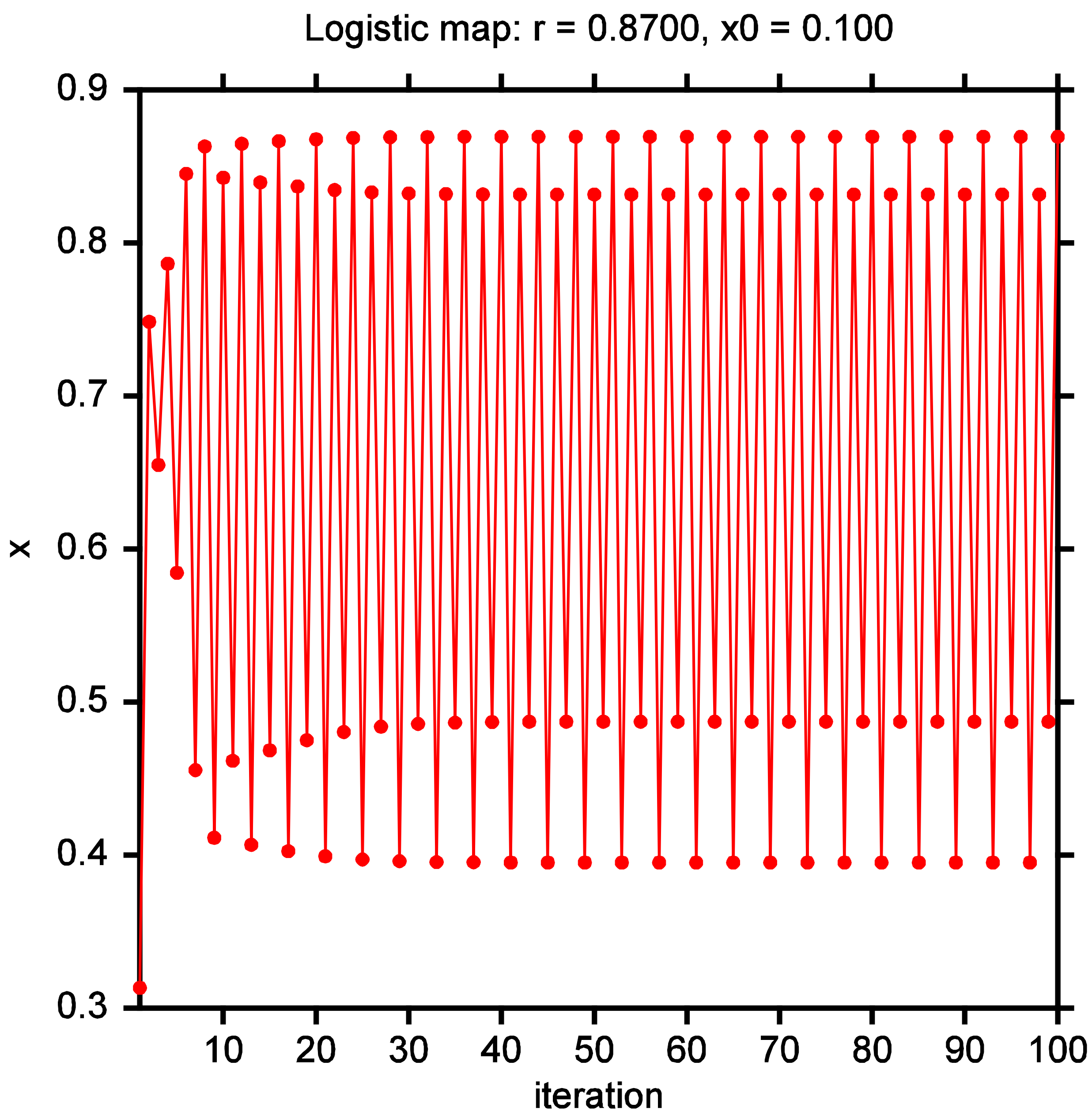
Maps [0,1] → [0,1]

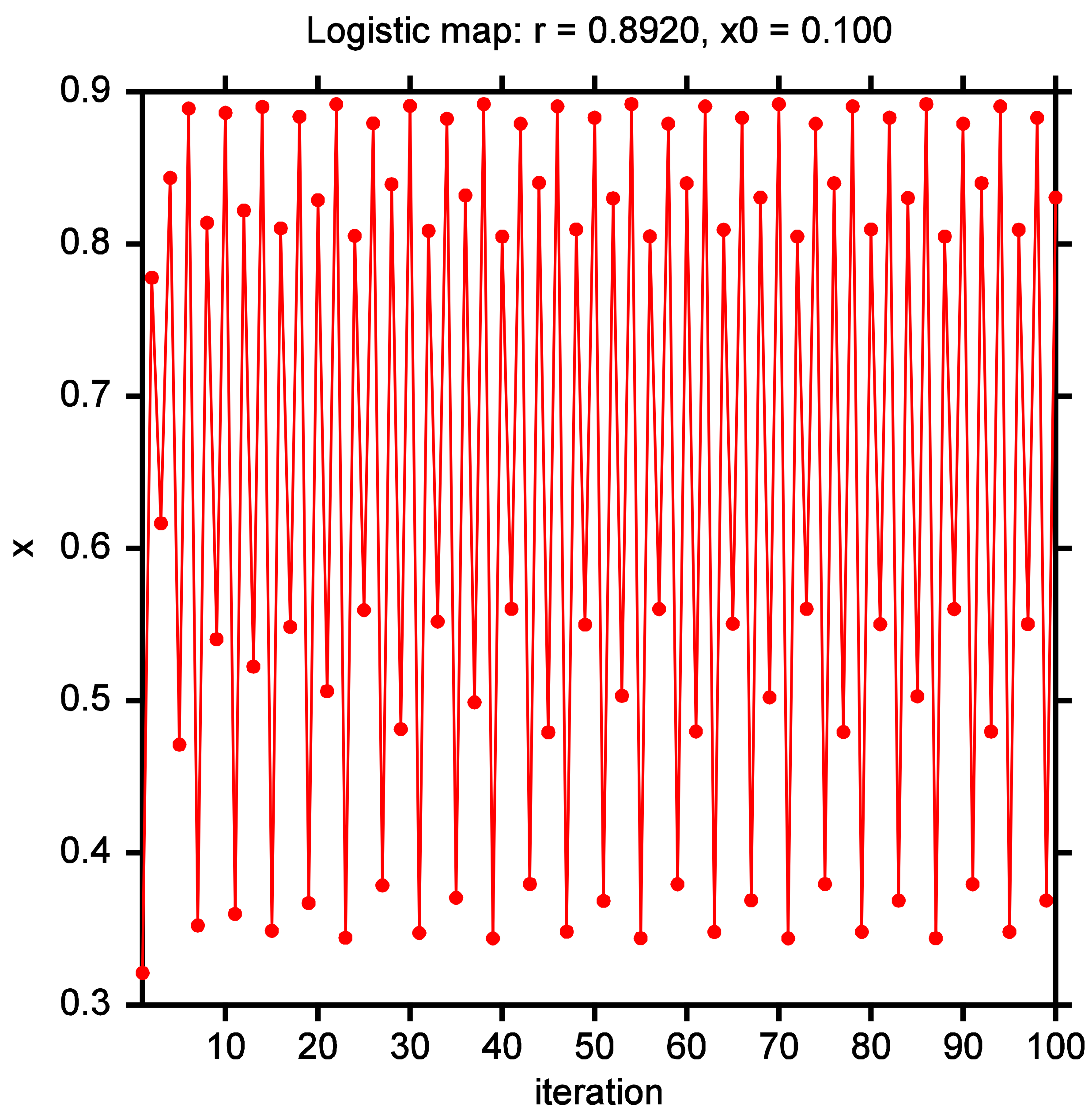

Logistic map: r = 0.240, x0 = 0.100
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
x
5
10
15
20
25
30
35
40
45
50
iteration


Logistic map: r = 0.740, x0 = 0.100
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
x
10
20
30
40
50
60
70
80
90
100
iteration

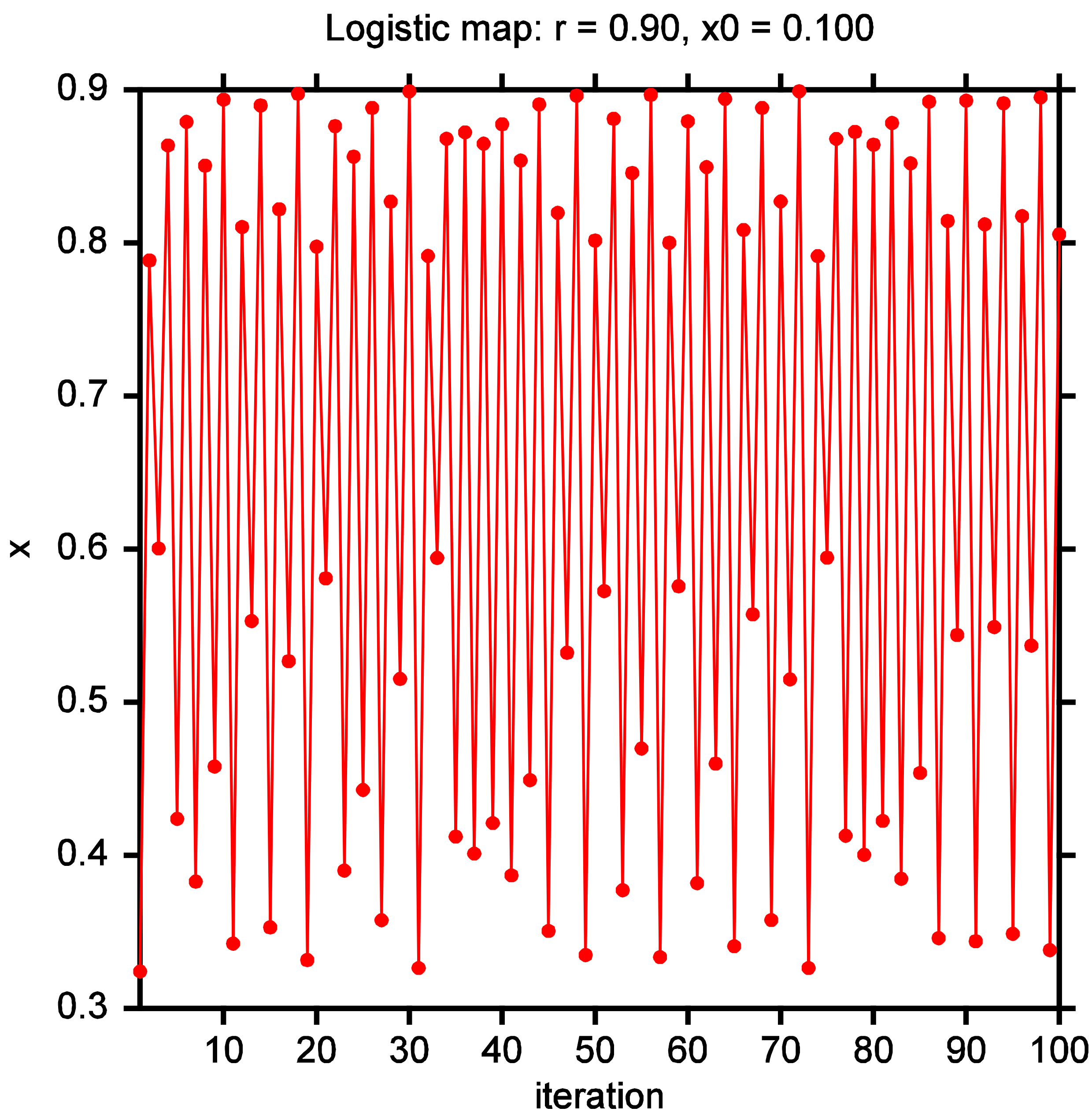


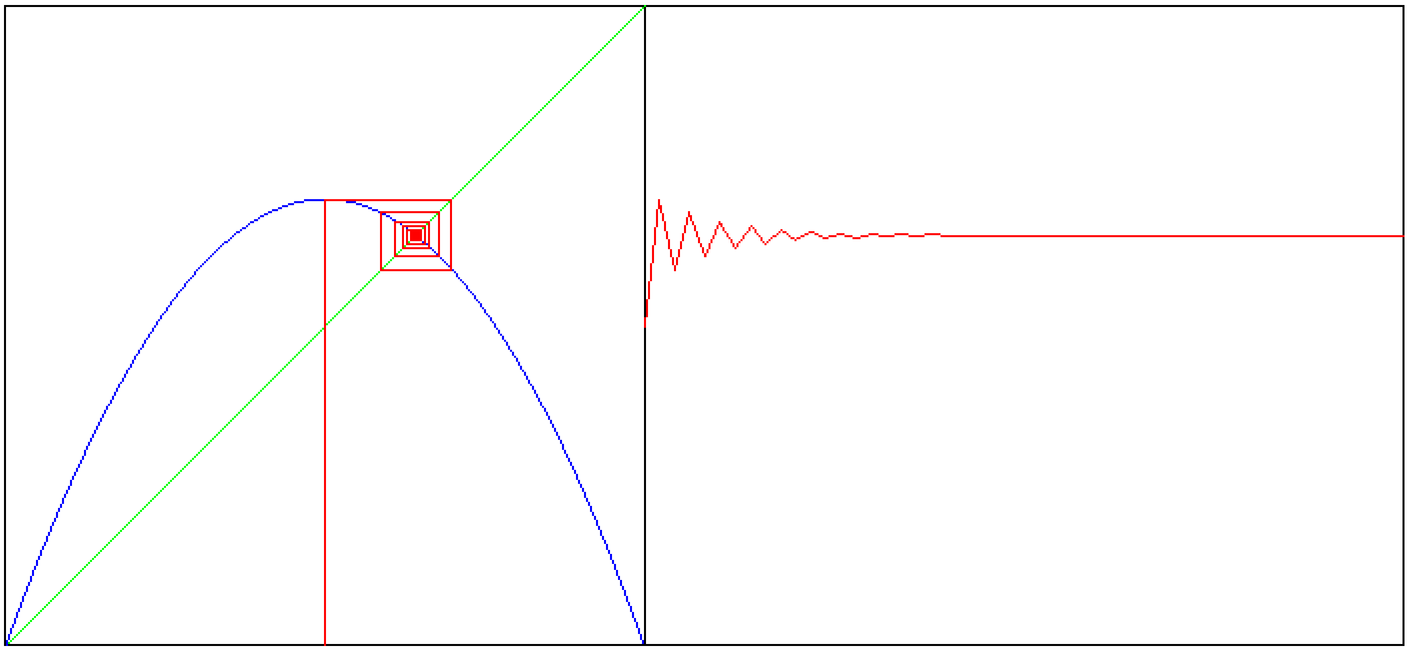
Logistic map: r = 0.7700, x0 = 0.100
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
x
10
20
30
40
50
60
70
80
90
100
iteration


Logistic map: r = 0.8700, x0 = 0.100
0.9
0.8
0.7
0.6
0.5
0.4
0.3
x
10
20
30
40
50
60
70
80
90
100
iteration



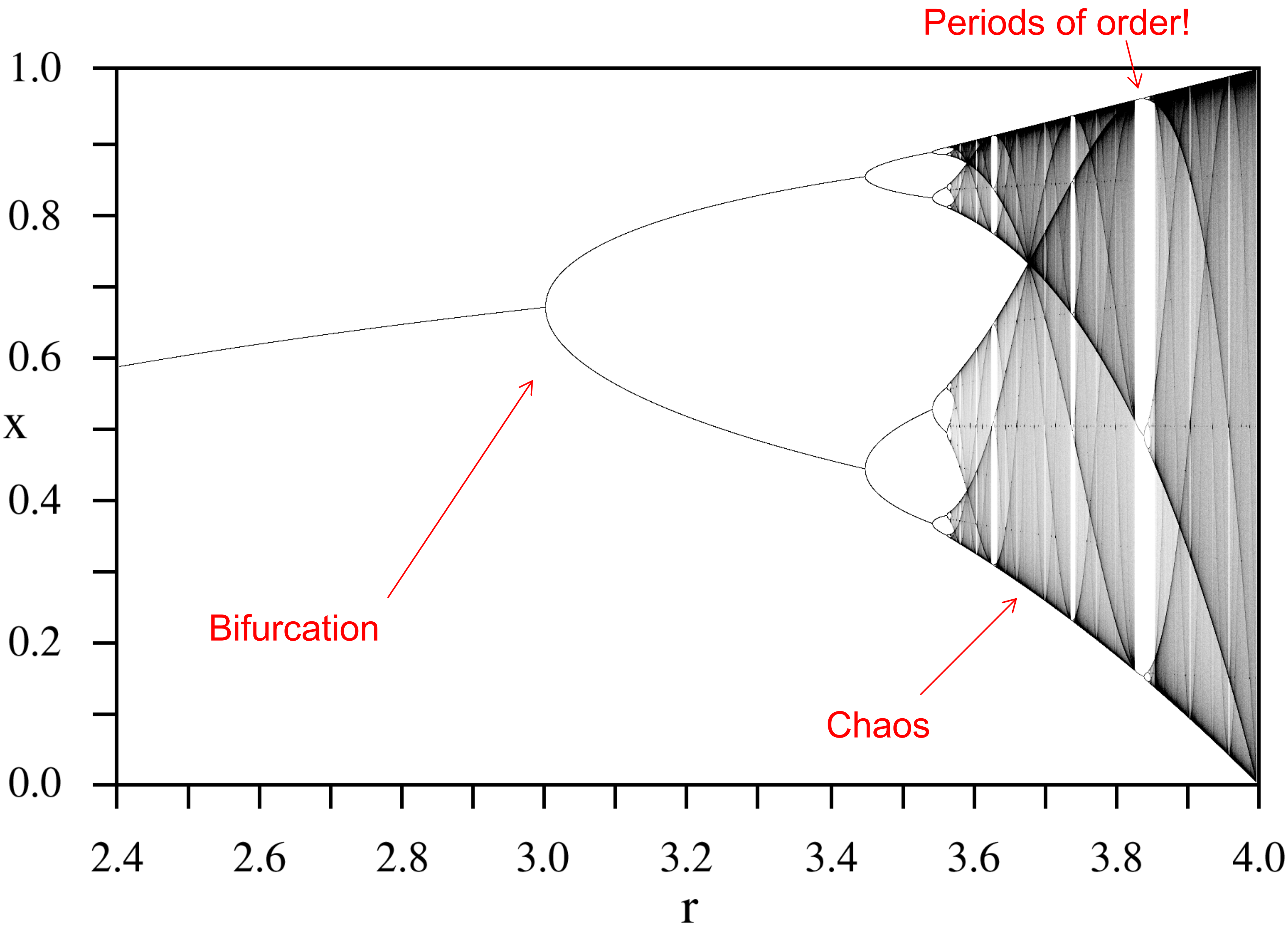
Logistic map: r = 0.8920, x0 = 0.100
x
0.9
0.8
0.7
0.6
0.5
0.4
0.3
10
20
30
40
50
60
70
80
90
100
iteration


Logistic map: r = 0.90, x0 = 0.100
0.9
0.8
0.7
0.6
0.5
0.4
0.3
x
10
20
30
40
50
60
70
80
90
100
iteration



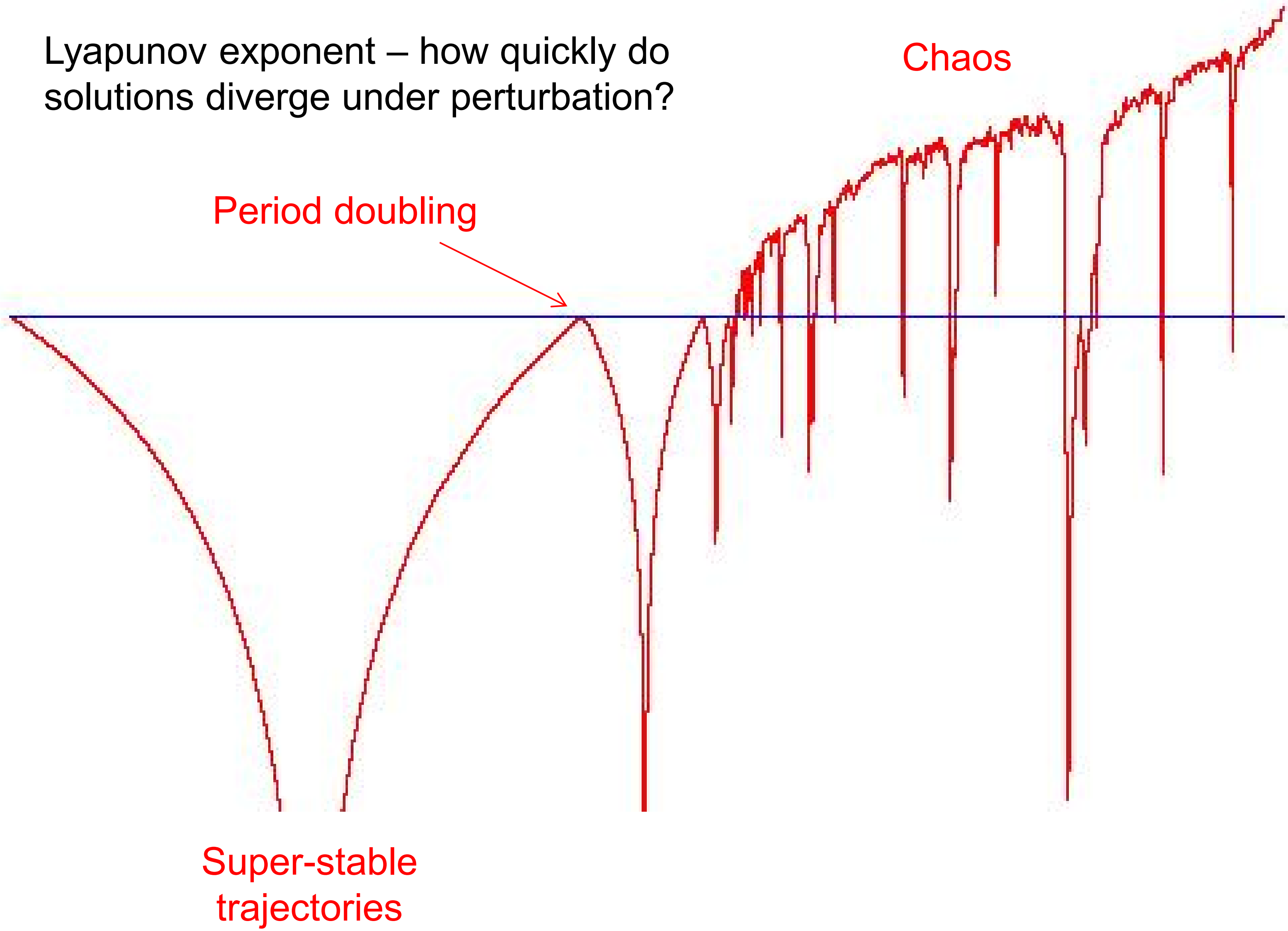
Iterated Logistic Map Demo

http://ibiblio.org/e-notes/MSet/Logistic.htm

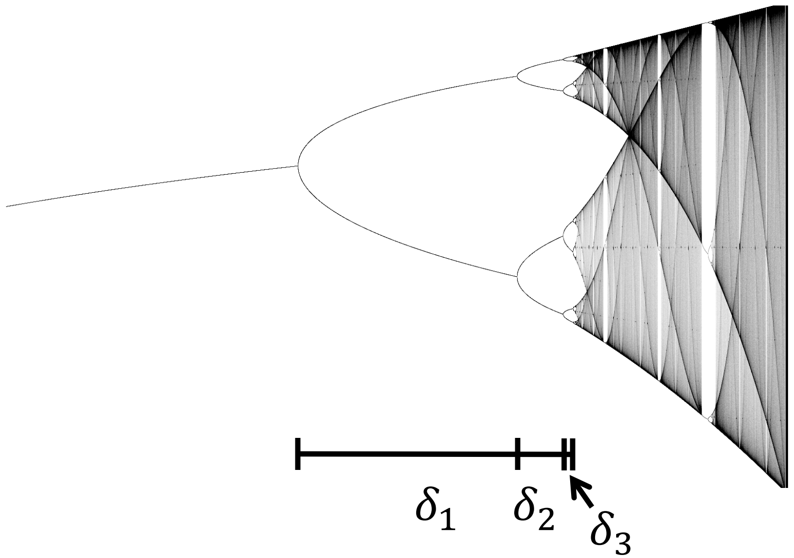


Bifurcation diagram

Lyapunov exponent – how quickly do solutions diverge under perturbation?



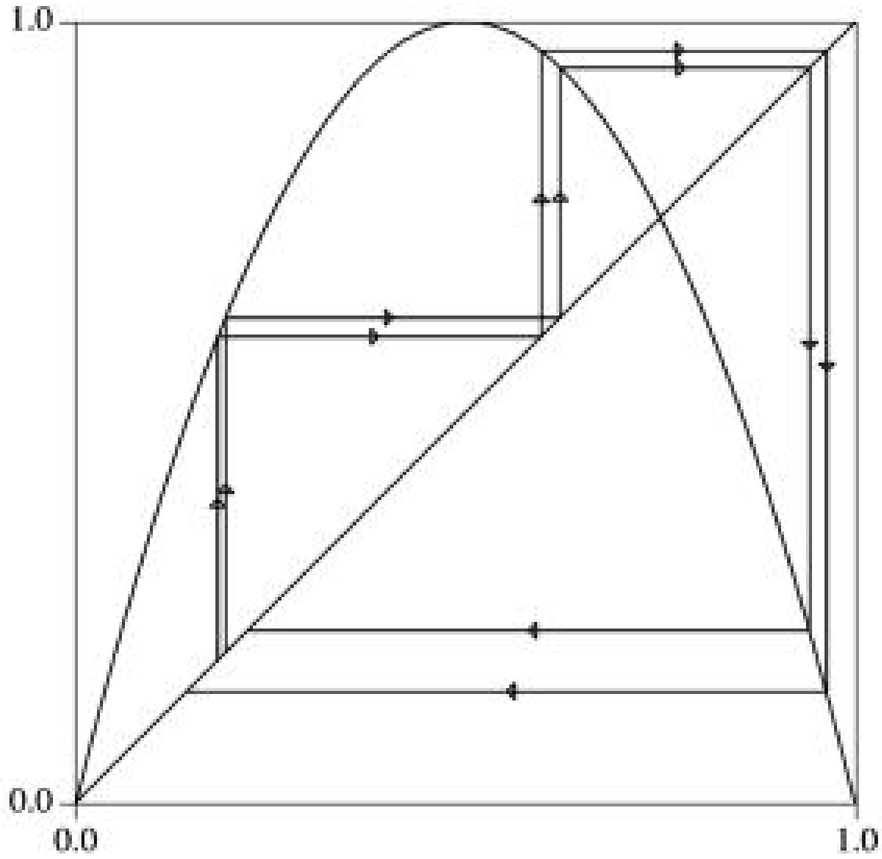
Doubling route to chaos



Intervals between doublings get smaller and smaller.
The limit $\delta = \lim \frac{\delta_k}{\delta_{k+1}}$ is known as Feigenbaum's constant.

- δ = 4.669 201 609 102 990 671 853 203 821 578 …
- Independent of shape of map, as long as there's a simple quadratic maximum
- Universal "route to chaos": examples in electrical circuits (ODEs), water flow (PDEs), …

Iterated logistic map



Economic applications: see Medio 92, Puu 03

Corn-Hog cycle:

[Corn-Hog cycle (William King, Drexel)]