PDE Stability Analysis

COS 323

Lax Equivalence Theorem

- For a well-posed linear PDE, necessary and sufficient conditions for solver to converge:
  - Consistency: local truncation error goes to zero
  - Stability: solution remains bounded

- Consistency derived from soundness of approximation to derivatives as $\Delta t \to 0$
- Stability: exact analysis often difficult

Von Neumann Stability Analysis

- Valid under assumptions (linear PDE, periodic boundary conditions), but often good starting point
- Fourier expansion (!) of solution

$$u(x,t) = \sum a_k(n\Delta t) e^{ikj\Delta x}$$

- Assume

$$a_k(n\Delta t) = (\xi_k)^n$$

  - Valid for linear PDEs, otherwise locally valid
  - Will be stable if magnitude of ξ is less than 1: errors decay, not grow, over time

Von Neumann: Diffusion Equation, FTCS

$$u_t = k u_{xx}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = k \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

$$\frac{\xi - 1}{\Delta t} = k \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{(\Delta x)^2}$$

$$\xi = 1 + \frac{k\Delta t}{(\Delta x)^2}(2\cos k\Delta x - 2)$$

$$\xi = 1 - \frac{4k\Delta t}{(\Delta x)^2}\sin^2(k\Delta x / 2)$$

Von Neumann: Diffusion Equation, FTCS

- Since $\sin^2$ between 0 and 1, to have amplitude less than 1 we need

$$1-\frac{4k\Delta t}{(\Delta x)^2} > -1$$

$$\Delta t < \frac{(\Delta x)^2}{2k}$$

- Scheme is conditionally stable

Von Neumann: Advection Equation, FTCS

$$u_t = -v u_x$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$\frac{\xi - 1}{\Delta t} = -v \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}$$

$$\xi = 1 - \frac{iv\Delta t}{\Delta x} \sin k\Delta x$$

- Magnitude always ≥ 1: unconditionally unstable