Smoothers

- Disappointing convergence rates observed for stationary iterative methods are *asymptotic*
- Much better progress may be made initially before eventually settling into slow asymptotic phase
- Many stationary iterative methods tend to reduce high-frequency (i.e., oscillatory) components of error rapidly, but reduce low-frequency (i.e., smooth) components of error much more slowly, which produces poor asymptotic rate of convergence
- For this reason, such methods are sometimes called *smoothers*

< interactive example >

Multigrid Methods

- Smooth or oscillatory components of error are relative to mesh on which solution is defined
- Component that appears smooth on fine grid may appear oscillatory when sampled on coarser grid
- If we apply smoother on coarser grid, then we may make rapid progress in reducing this (now oscillatory) component of error
- After few iterations of smoother, results can then be interpolated back to fine grid to produce solution that has both higher-frequency and lower-frequency components of error reduced

Multigrid Methods, continued

- *Multigrid methods* : This idea can be extended to multiple levels of grids, so that error components of various frequencies can be reduced rapidly, each at appropriate level
- Transition from finer grid to coarser grid involves *restriction* or *injection*
- Transition from coarser grid to finer grid involves *interpolation* or *prolongation*

Residual Equation

- If $\hat{\boldsymbol{x}}$ is approximate solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, with residual $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\hat{\boldsymbol{x}}$, then error $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ satisfies equation $\boldsymbol{A}\boldsymbol{e} = \boldsymbol{r}$
- Thus, in improving approximate solution we can work with just this *residual equation* involving error and residual, rather than solution and original right-hand side
- One advantage of residual equation is that zero is reasonable starting guess for its solution

Two-Grid Algorithm

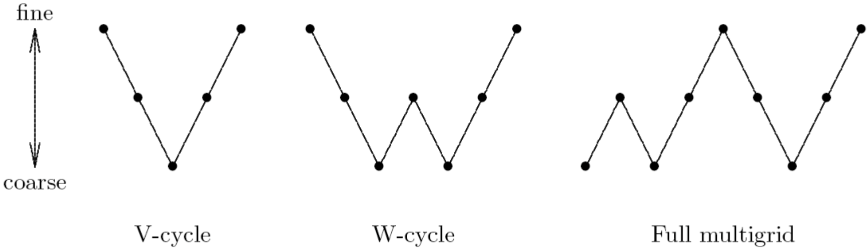
1. On fine grid, use few iterations of smoother to compute approximate solution $\hat{\boldsymbol{x}}$ for system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
2. Compute residual $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\hat{\boldsymbol{x}}$
3. Restrict residual to coarse grid
4. On coarse grid, use few iterations of smoother on residual equation to obtain coarse-grid approximation to error
5. Interpolate coarse-grid correction to fine grid to obtain improved approximate solution on fine grid
6. Apply few iterations of smoother to corrected solution on fine grid

Multigrid Methods, continued

- *Multigrid method* results from recursion in Step 4: coarse grid correction is itself improved by using still coarser grid, and so on down to some bottom level
- Computations become progressively cheaper on coarser and coarser grids because systems become successively smaller
- In particular, direct method may be feasible on coarsest grid if system is small enough

Cycling Strategies

Common strategies for cycling through grid levels



Cycling Strategies, continued

- *V-cycle* starts with finest grid and goes down through successive levels to coarsest grid and then back up again to finest grid
- *W-cycle* zig-zags among lower level grids before moving back up to finest grid, to get more benefit from coarser grids where computations are cheaper
- *Full multigrid* starts at coarsest level, where good initial solution is easier to come by (perhaps by direct method), then bootstraps this solution up through grid levels, ultimately reaching finest grid

Multigrid Methods, continued

- By exploiting strengths of underlying iterative smoothers and avoiding their weaknesses, multigrid methods are capable of extraordinarily good performance, linear in number of grid points in best case
- At each level, smoother reduces oscillatory component of error rapidly, at rate independent of mesh size h, since few iterations of smoother, often only one, are performed at each level
- Since all components of error appear oscillatory at some level, convergence rate of entire multigrid scheme should be rapid and independent of mesh size, in contrast to other iterative methods

Multigrid Methods, continued

- Moreover, cost of entire cycle of multigrid is only modest multiple of cost of single sweep on finest grid
- As result, multigrid methods are among most powerful methods available for solving sparse linear systems arising from PDEs
- They are capable of converging to within truncation error of discretization at cost comparable with fast direct methods, although latter are much less broadly applicable