Linear Arithmetic Satisfiability via Strategy Improvement

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- SMT solvers handle the ground fragment. Techniques for quantifiers:
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- Today: alternating quantifier satisfiability modulo linear rational (& integer) arithmetic via **strategy improvement**.

Game interpretation

$$\varphi \triangleq \underbrace{\exists w. \forall x. \exists y. \forall z.}_{\textit{quantifier prefix}} \underbrace{(y < 1 \vee 2w < y) \wedge (z < y \vee x < z)}_{\textit{matrix}}$$

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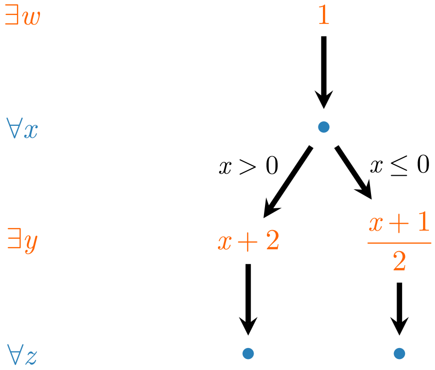
- φ is satisfiable $\iff$ SAT has a winning strategy

$$\varphi \triangleq \exists w. \forall x. \exists y. \forall z. (y < 1 \vee 2w < y) \wedge (z < y \vee x < z)$$



Mutual strategy improvement

S_0

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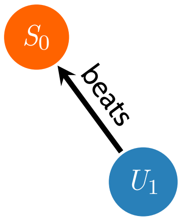

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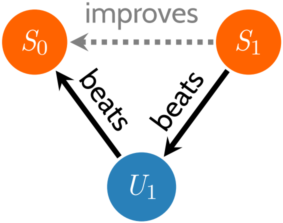

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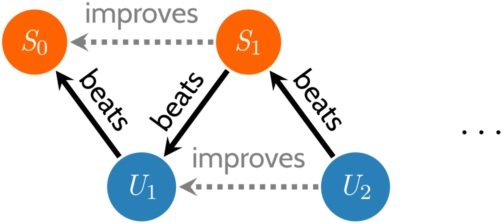

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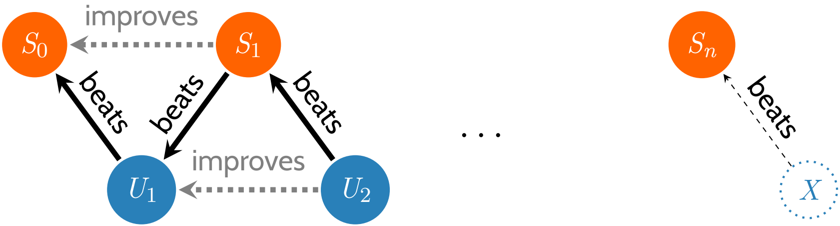

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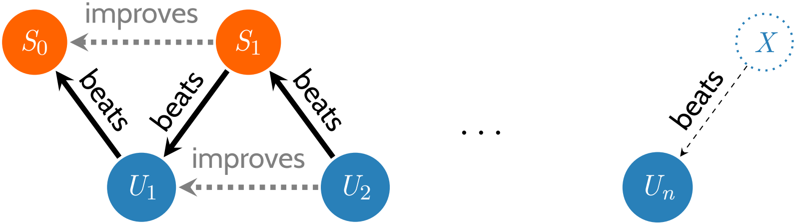

Two questions:

- What does it mean to *improve* a strategy?
- How can we find counter-strategies?

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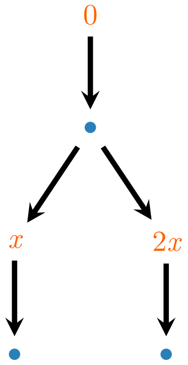

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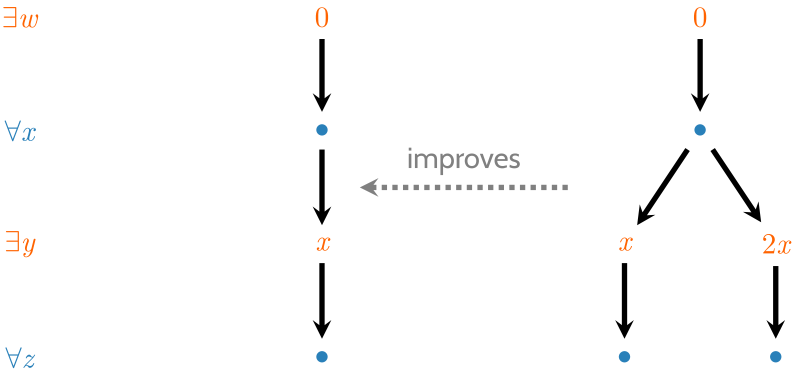

Counter strategy synthesis via ground satisfiability

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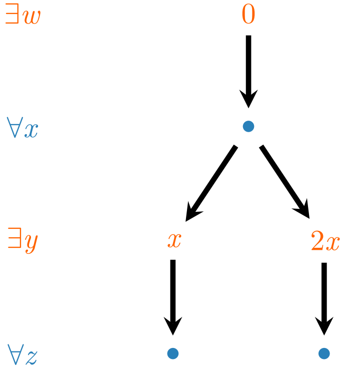

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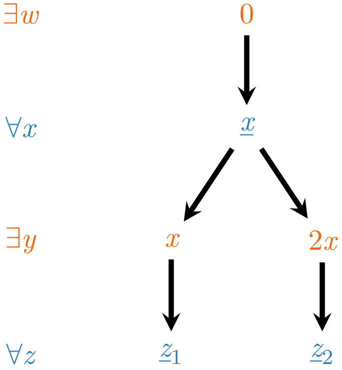

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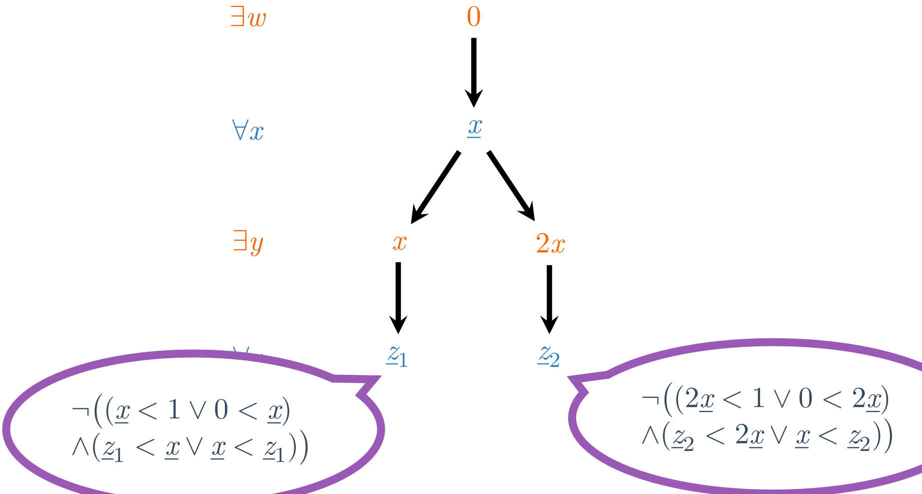

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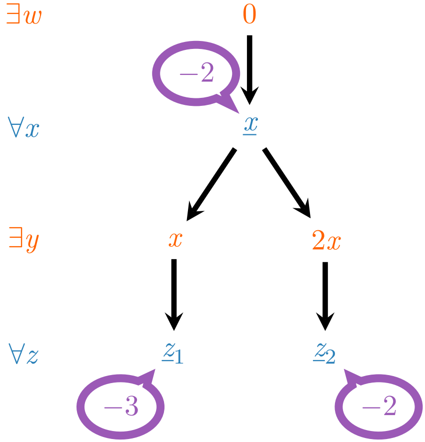

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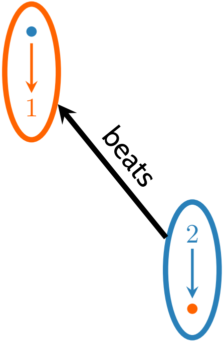

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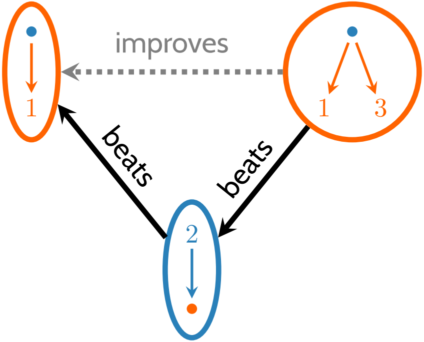

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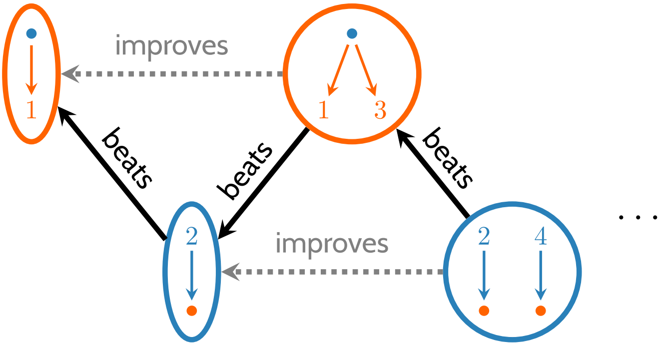

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Given:

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- model $m \models F$
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Idea: there is a set of terms T such that $\exists x. F$ is equivalent to $\bigvee_{t \in T} F[x \mapsto t]$.

Use model m to select the right disjunct.
(similar to model based projection - [Komuravelli, Gurfinkel, Chaki 2014]).

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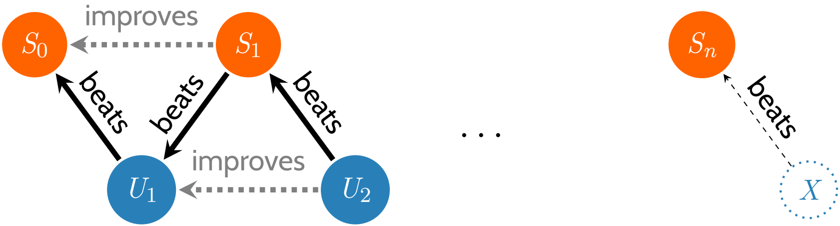
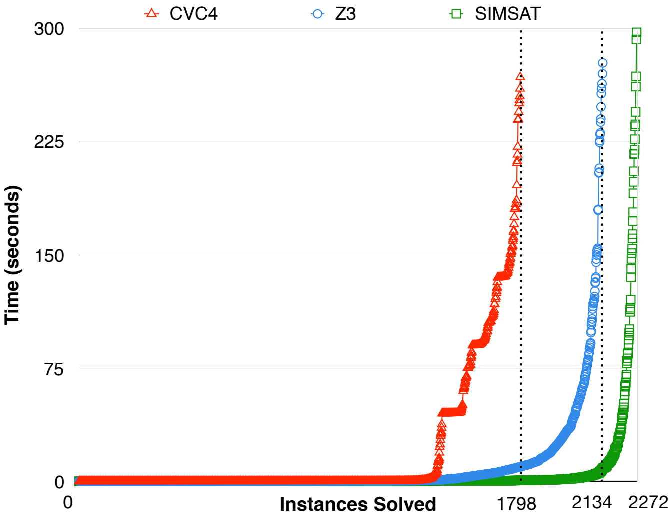


Experimental results



2421 instances drawn from SMTLIB2 & Mjollnir benchmark suite, 300s time limit.

Thanks!