Vertex-Connectivity Survivable Network Design

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Survivable Network Design (SNDP)

Input: A graph $G(V,E)$ with costs on edges, and pairwise connectivity requirements $r(u,v)$.

Goal: Minimum cost subset $E' \subseteq E$ s.t. $G(V,E')$ has $r(u,v)$ disjoint paths for each pair u,v.

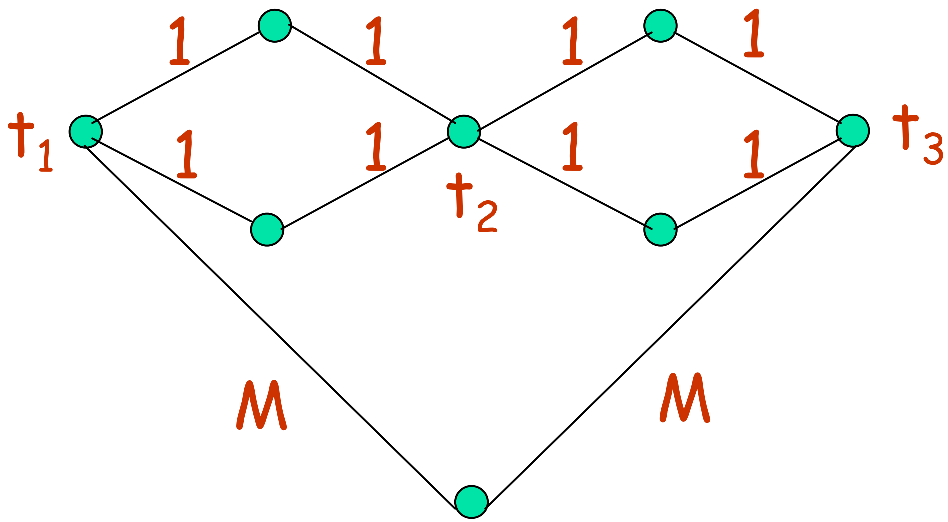
EC-SNDP: $r(u,v)$ edge-disjoint paths.

VC-SNDP: $r(u,v)$ vertex-disjoint paths.

k = Max connectivity requirement for any pair.

SNDP Example

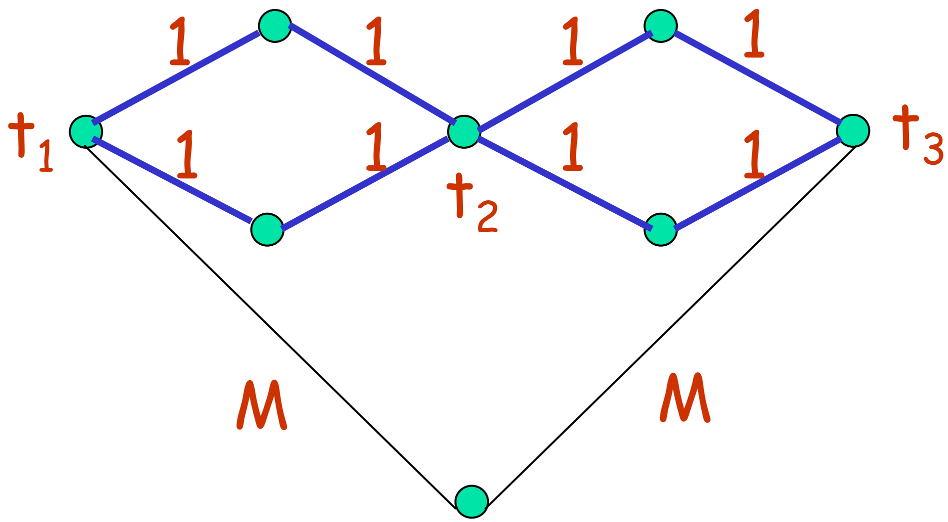
$r(t_1,t_2) = r(t_2,t_3) = r(t_1,t_3) = 2$



$M >> 1$

EC-SNDP Solution

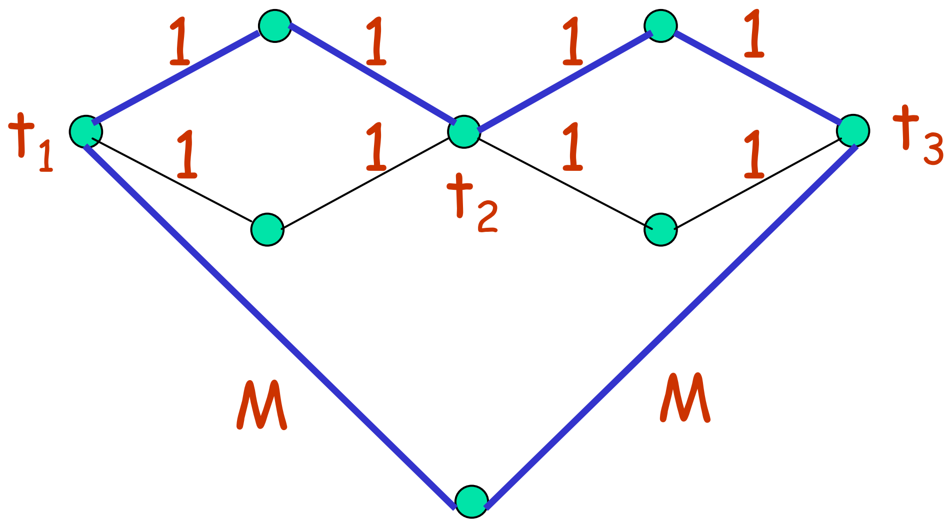
$r(t_1,t_2) = r(t_2,t_3) = r(t_1,t_3) = 2$



Cost = 8

VC-SNDP Solution

$r(t_1,t_2) = r(t_2,t_3) = r(t_1,t_3) = 2$



Cost = 2M + 4 >> 8

Directed Graphs

[Dodis-K '99]

If the underlying graph is directed, then even for $k=1$, SNDP is $2^{\log^{1-\varepsilon} n}$-hard to approximate for any $\epsilon > 0$.

We focus on the undirected case from here on.

EC-SNDP Results

- 2-approximation for $k = 1$ via a primal-dual approach [Agrawal, Klein, Ravi '95].
- Extended to higher connectivity values. [Goemans, Mihail, Vazirani, Williamson '95], [Goemans,Goldberg, Plotkin, Shmoys, Williamson'94].
- 2-approximation by iterative LP-rounding. [Jain '98].

VC-SNDP Results

- $2^{\log^{1-\varepsilon} n}$ hardness for any $\varepsilon > 0$, for $k=\text{poly}(n)$ [Kortsarz, Krauthgamer, Lee '03].
- $k^{\Omega(1)}$-hardness [Chakraborty, Chuzhoy, K '08].

- 2-approximation if $k = 2$ [Fleischer, Jain, Williamson '06].
- $O((\log k).\log (n/n-k))$-approx. if $r(u,v) = k \ \forall\, u,v$. [Cheriyan, Vempala, Vetta '03], [Kortsarz, Nutov '04], [Fakcharoenphol, Laekhanukit '08], [Nutov '08].

Single-Source Vertex Connectivity

All vertex connectivity requirements are between a source s and a set T of terminals.

- $O(k^2 \log n)$-approximation
  [Chuzhoy,K '08]; [Nutov' 08].

- $O(k \log n)$-approximation if $r(s,t) = k \ \forall\, t \in T$
  [Chuzhoy,K'08]; (an elegant simpler proof
  [Chekuri, Korula '08].)

Our Results

Nothing better than $O(n \log n)$ approximation for VC-SNDP even for $k=3$.

Theorem 1: VC-SNDP has a randomized $O(k^3 \log n)$ approximation algorithm for any k.

Theorem 2: Single-source VC-SNDP has a randomized $O(k^2 \log n)$-approximation algorithm.

Rest of This Talk …

- Element connectivity problem.
- Resilient set systems.
- Algorithm for general VC-SNDP.
- Improved algorithm for single-source VC-SNDP.
- Further developments.

Element Connectivity SNDP

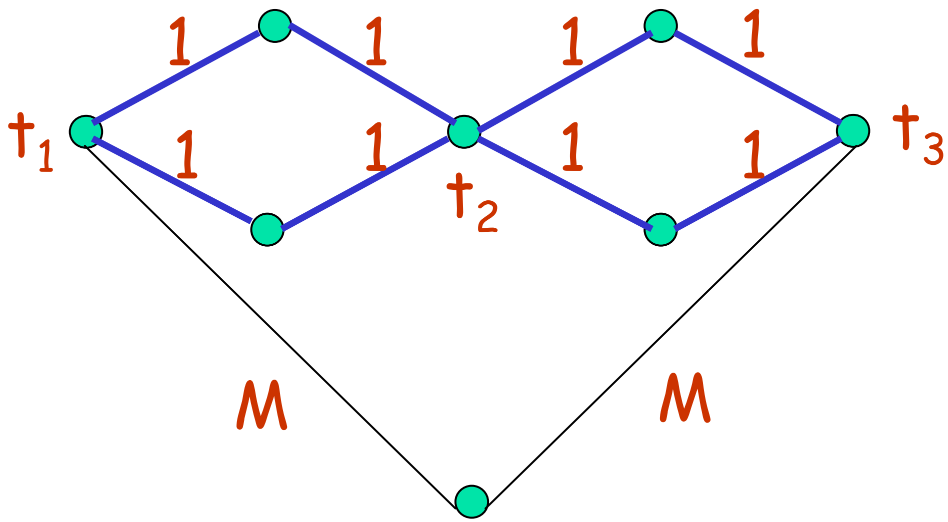
Terminal: A vertex u s.t. $r(u,v) > 0$ for some v.
Element: An edge or a non-terminal vertex.

Goal: A minimum cost subset $E' \subseteq E$ s.t. $G(V,E')$ has $r(u,v)$ element-disjoint paths for each u, v.

If u and v are k-element connected then they are k-edge connected as well, but may not be k-vertex connected.

Element Connectivity SNDP

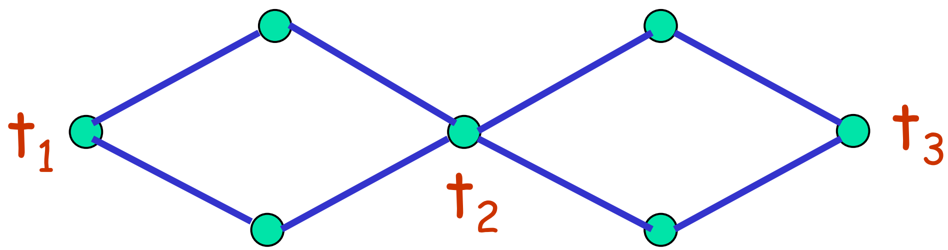
$r(t_1,t_2) = r(t_2,t_3) = r(t_1,t_3) = 2$



Cost = 8

Element Connectivity is Transitive

t_1 and t_2 are k-element connected, and t_2 and t_3 are k-element connected, then so are t_1 and t_3.

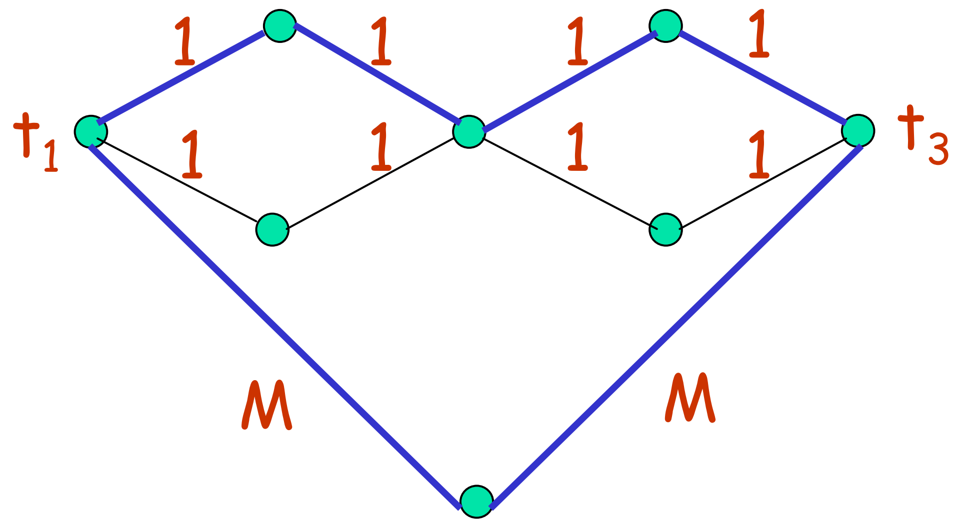


Only true for terminals.

Edge-connectivity is also transitive but not vertex-connectivity.

Element Connectivity is not Monotone

$r(t_1,t_3) = 2$



Cost = 2M + 4 >> 8

Element Connectivity SNDP

- [Jain,Mandoiu,Vazirani,Williamson '99]
  - Introduced element connectivity SNDP.
  - A primal-dual O(log k)-approximation.

- 2-approximation via iterative LP-rounding
  [Fleischer, Jain, Williamson '01]
  [Cheriyan, Vempala, Vetta '03]

- Matches approximation ratio for EC-SNDP.

k-Resilient Family of Sets

A family $\{T_1, T_2, \ldots, T_p\}$ of vertex subsets is k-resilient if for every pair (s,t) of vertices and every subset $X \subseteq V \setminus \{s,t\}$ of size $(k-1)$, there is a T_i such that

$$s,t \in T_i \text{ and } X \cap T_i = \emptyset.$$

If s and t are k-element connected w.r.t. terminal set T_i, then X can not disconnect s from t.

Algorithm for General VC-SNDP

Input: A graph G and a set P of pairs of vertices with connectivity requirements between 1 from k.

- Let $\{T_1, T_2, \ldots, T_p\}$ be a k-resilient family.
- Let $P_i = \{ (s,t) \in P \mid s, t \in T_i \}$ for $1 \leq i \leq p$.
- Solve the element-connectivity instance defined by P_i on the input graph G.
- Let G_i be the 2-approximate solution obtained for P_i.
- Output $G_1 \cup G_2 \ldots \cup G_p$.

Feasibility of the Solution

Fix any pair (s,t) with vertex-connectivity requirement $r(s,t) \in [1..k]$.

- Consider any $X \subseteq V \setminus \{s,t\}$ s.t. $|X| \leq r(s,t)-1 \leq (k-1)$.
- Since $\{T_1, T_2, \ldots, T_p\}$ is a k-resilient family, there exists T_i s.t. $s,t \in T_i$ and $X \cap T_i = \emptyset$.
- So X is a set of non-terminals in the instance for T_i.

The pair (s,t) is $r(s,t)$-element connected in G_i, so X can not disconnect s from t.

Cost Analysis of the Solution

- Let OPT denote the cost of an optimal solution for the given VC-SNDP instance.
- Then cost of each element connectivity instance P_i is at most OPT.
- Cost of each solution G_i is at most $2 \cdot OPT$.

Thus we get an $O(p)$-approximation.

Constructing a k-Resilient Family

- Set $p \approx 2k^3 \log n$.
- Each terminal $t \in T$ selects uniformly at random $q \approx p/2k \approx k^2 \log n$ indices from $\{1, 2, \dots, p\}$.
- Let $\phi(t)$ be the set of indices chosen by t.
- Define $T_i = \{t \mid i \in \phi(t)\}$.

Lemma: With high probability, $\{T_1, T_2, \dots, T_p\}$ is a k-resilient family.

Proof of k-Resiliency

- Fix a pair (s,t).
- Let X be any set of $\leq$ (k-1) vertices in V\{s,t}.
- Bad event E(s,t, X): $\phi(s) \cap \phi(t) \subseteq \cup_{t' \in X \cap T} \phi(t')$.

- Probability of E(s,t,X) $\leq n^{-4k}$.
  - Key observation: $|\cup_{t' \in X \cap T} \phi(t')| \leq p/2$.
  - So w.h.p. $\phi(s) \cap \phi(t)$ contains an index outside the union $\cup_{t' \in X \cap T} \phi(t')$.
- By union bounds, w.h.p. bad event E(s,t,X) does not occur for any s,t, X.

(w,r)-Cover-Free Families

- A family F of sets is (w,r)-cover-free if for any distinct $A_1, \ldots, A_w \in F$, and any other $B_1, \ldots, B_r \in F$ if we have

$$A_1 \cap A_2 \cap \ldots \cap A_w \not\subseteq B_1 \cup B_2 \cup \ldots \cup B_r.$$

- Set $w = 2$, $r = (k-1)$.
- Then $\{ T_1, T_2, \ldots, T_p \}$ is k-resilient $\Leftrightarrow$ $F = \{ \phi(t) \mid t \in T \}$ is a (2,k-1)-cover-free family on elements $\{1, 2, \ldots, p\}$.

Implication for k-Resilient Family

Need a (2,k-1)-cover-free family with n sets.
How small can we make the universe size p?

Theorem [Stinson, Wei, and Zhu '00]
A (2,k-1)-cover-free family with n sets exists on a universe of p elements only if $p = \Omega((k^3 \log n) / \log k)$.

The simple randomized construction is tight to within an $O(\log k)$ factor.

Single-Source VC-SNDP

Same algorithm but more efficient k-resilient family.

- Set $p = 4(k^2 \log n)$, each terminal $t \in T$ selects uniformly at random $q = p/(2k)$ indices from $\{1, 2, \ldots, p\}$.
- Let $\phi(t)$ be the set of indices chosen by t.
- Define $T_i = \{ t \mid i \in \phi(t) \}$.

Lemma: W.h.p. $\{T_1, T_2, \ldots, T_p\}$ is a k-resilient family.

We get an $O(k^2 \log n)$-approximation algorithm.

When Costs are on Vertices …

- Provably harder: even for $k = 1$, $\Omega(\log n)$-hard while a 2-approximation exists for edge costs.

- Our approach is oblivious to edge costs vs. vertex costs issue: α-approximation for element connectivity SNDP in vertex cost model gives an $O(\alpha k^3 \log n)$-approximation for VC-SNDP with vertex costs.

Further Developments …

[Nutov '09]

- For edge costs:
  - $O(k \log k)$-approximation for single-source VC-SNDP.
- For vertex costs:
  - $O(k \log n)$-approx. for element connectivity SNDP.
  - $O(k^4 \log^2 n)$-approximation for general SNDP.

Can be used with our approach to get same bound for vertex costs.

Concluding Remarks

- Simple reduction from vertex-connectivity to element-connectivity problem.
- Highlights an interesting connection between distinct notions of connectivity.

- Single-source case:
  - $\Omega(\log^{2-\varepsilon} n)$-hardness[Kortsarz,Krauthgamer,Lee'03] [Lando,Nutov '08].
  - $O(k \log k)$-approximation.
  - Poly-logarithmic approximation when k is large?

Thank You!