An Improved LP-Based Approximation for Steiner Tree

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Joint work with

J. Byrka, T. Rothvoß, L. Sanità

The Steiner Tree Problem

Def (Steiner tree) Given an undirected graph $G = (V, E)$ with edge costs $c : E \longrightarrow \mathbb{R}_{>0}$, and a set of **terminal** nodes $R \subseteq V$, find the tree S spanning R of minimum cost $c(S) := \sum_{e \in S} c(e)$.

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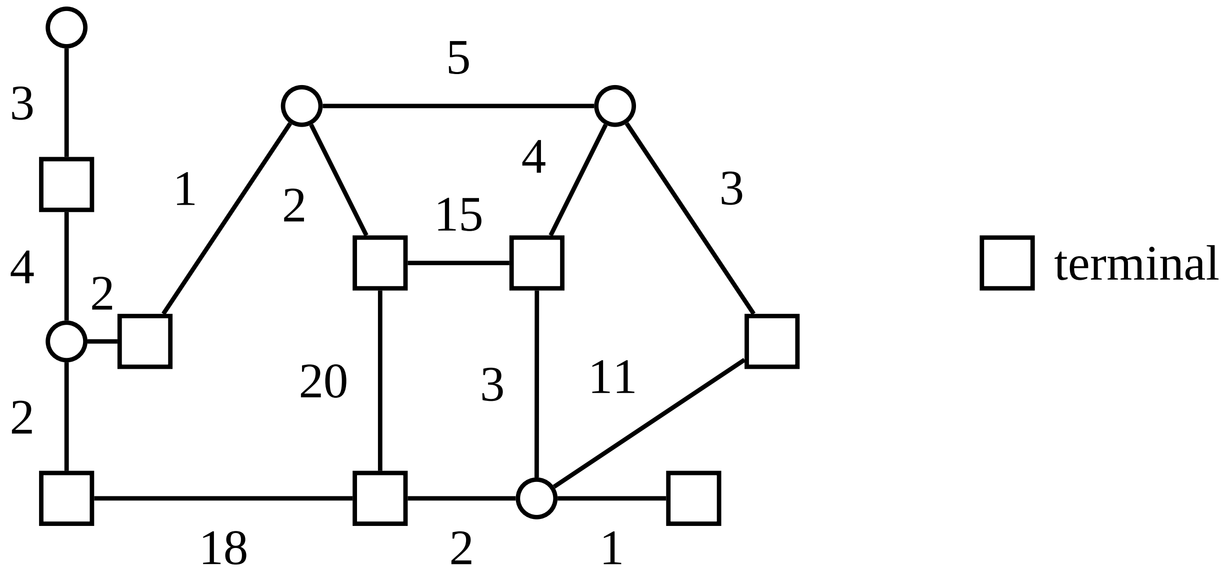

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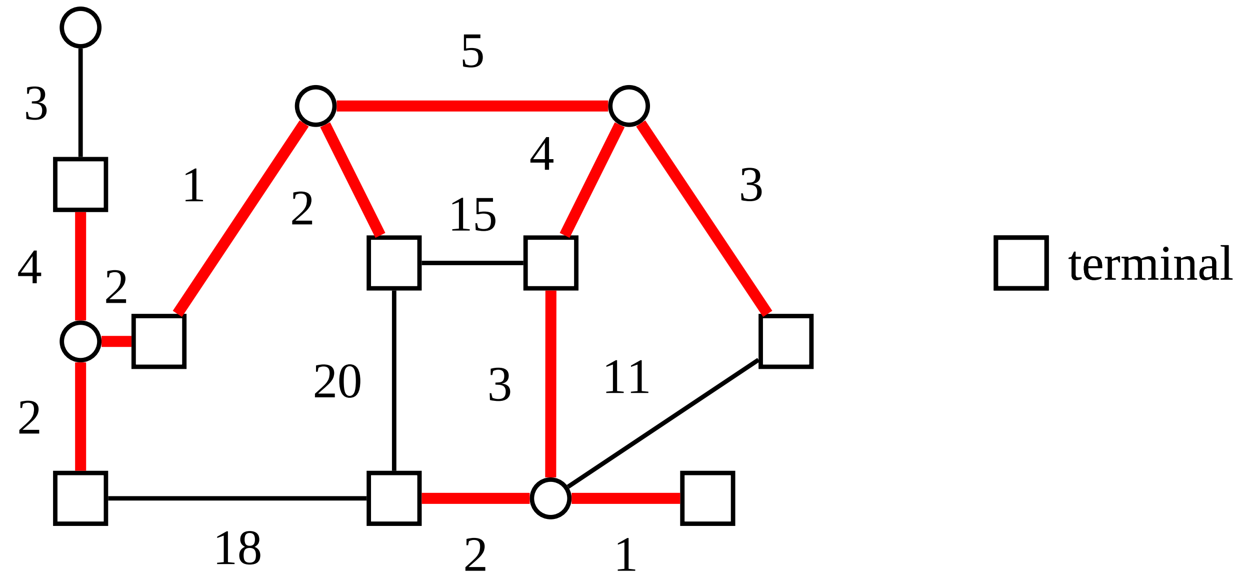

Known Results

Hardness:

- NP-hard even for edge costs in $\{1, 2\}$ [Bern&Plassmann'89]
- no $< \frac{96}{95}$-apx unless P=NP [Chlebik&Chlebikova'02]

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- 2-apx [minimum spanning tree heuristic]
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Integrality gap:

- ≤ 2 [Goemans&Williamson'95, Jain'98]

Our Results and Techniques

Thr There is an (LP-based) deterministic $\ln 4 + \varepsilon < 1.39$ approximation for the Steiner tree problem

- Here we show an expected $1.5 + \varepsilon$ apx

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- **Directed-Component Cut Relaxation**
  - bidirected cut relaxation
  - k-components
- **Iterative Randomized Rounding**
  - randomized rounding
  - iterative rounding

Directed-Component Cut Relaxation

Bidirected Cut Relaxation

- We select a **root** $r \in R$ and bi-direct the edges. Then

$$\min \sum_{e \in E} c(e) z_e \qquad \text{(BCR)}$$

$$\sum_{e \in \delta^+(U)} z_e \geq 1 \qquad \forall U \subseteq V - r, U \cap R \neq \emptyset$$

$$z_e \geq 0 \qquad \forall e \in E$$

- $\delta^+(U) = \{ab \in E : a \in U \text{ and } b \notin U\}$

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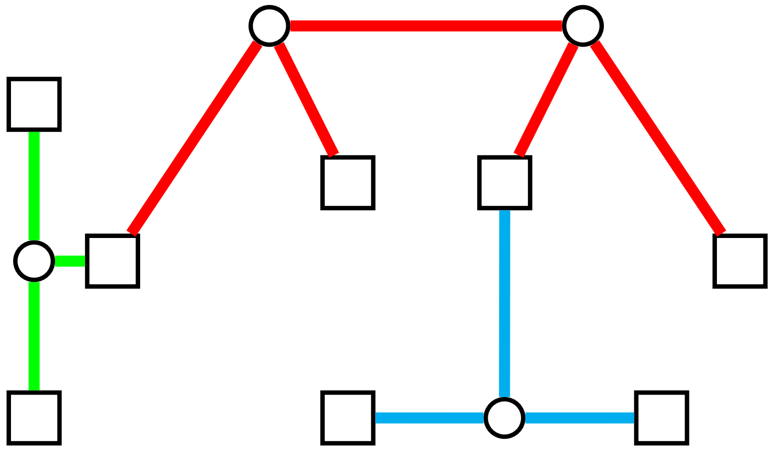
Thr [Edmonds'67] For $R = V$, BCR is integral

Rem the undirected version has integrality gap 2 even for $R = V$

Components

Def A **component** of a Steiner tree is a maximal subtree whose terminals coincide with its leaves

- A **k-component** is a component with at most k terminals
- A Steiner tree made of k-components is **k-restricted**.



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Thr [Borchers & Du'97] If opt_k and opt are the costs of an optimal k-restricted Steiner tree and an optimal Steiner tree, respectively, then

$$opt_k \leq \left(1 + \frac{1}{\lfloor \log_2 k \rfloor}\right) opt$$

Directing Components

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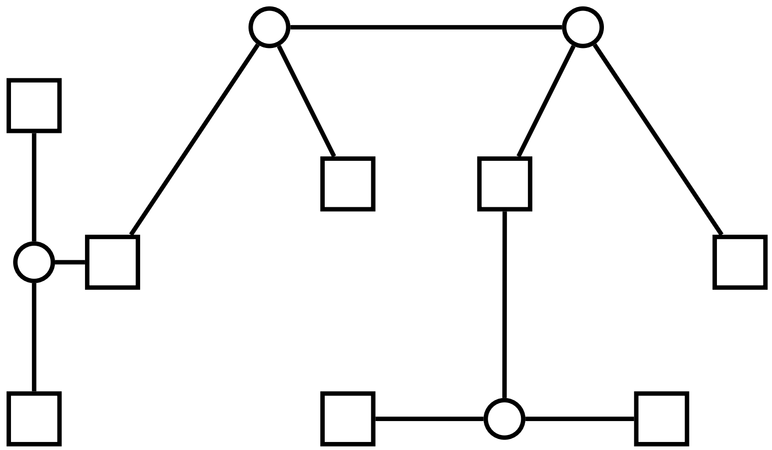

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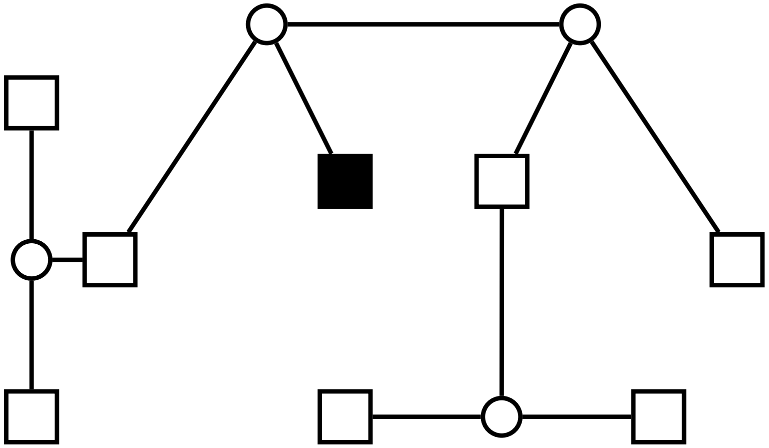

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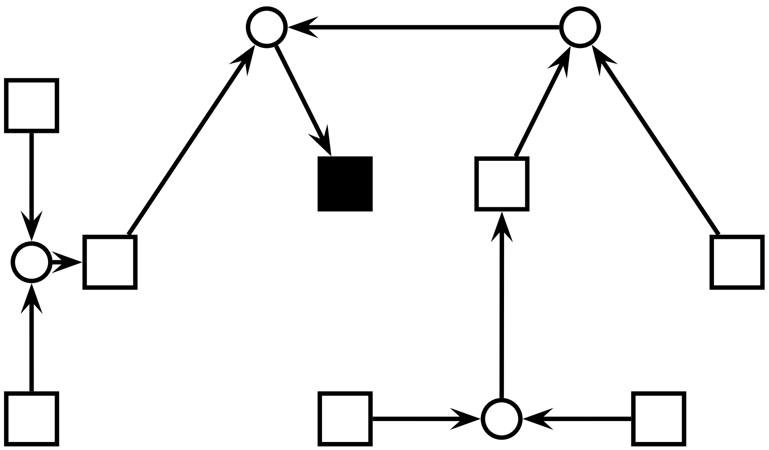

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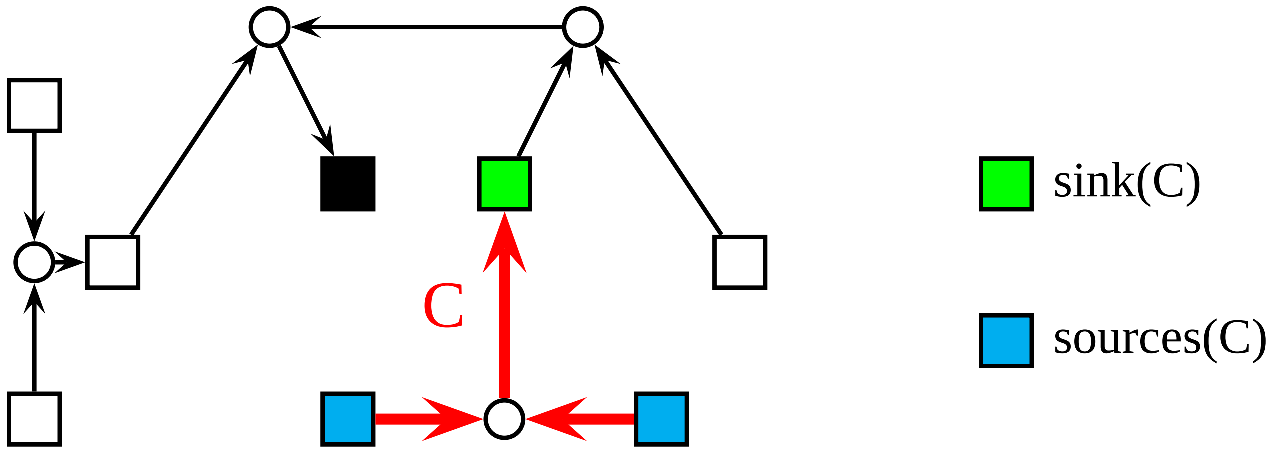

Directed-component Cut Relaxation

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Lem A $(1 + \varepsilon)$ approximation of the optimal fractional solution opt^f to DCR can be computed in polynomial time

Lem The cost of a minimum terminal spanning tree is $\leq 2\, opt^f$

Lem DCR is strictly stronger than BCR

Iterative
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- Sample one variable with probability proportional to its fractional value, and round it
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Rem In **randomized rounding** variables are rounded randomly and (typically) simultaneously

Rem In **iterative rounding** variables are rounded deterministically and (typically) one at a time

Algorithm IRR

- For $t = 1, 2, \ldots$
  - ◇ Compute a $(1+\varepsilon)$-apx solution x^t for DCR
  - ◇ Sample a component $C = C^t$ with probability $p_C^t := x_C^t / \sum_{D \in \mathcal{C}} x_D^t$
  - ◇ Contract C^t and update DCR consequently
  - ◇ If there is only one terminal, output the sampled components

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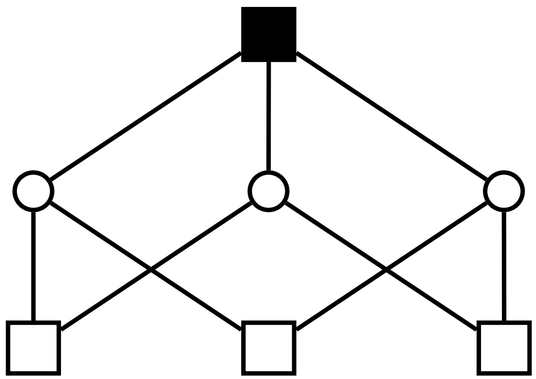

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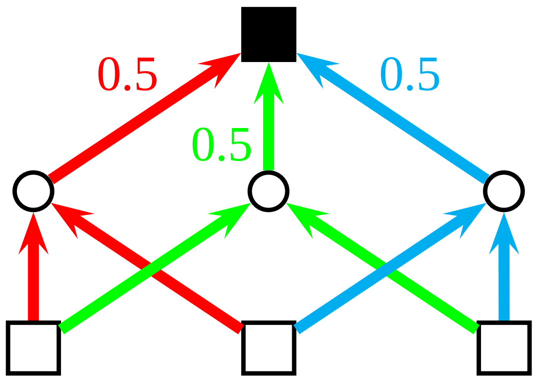

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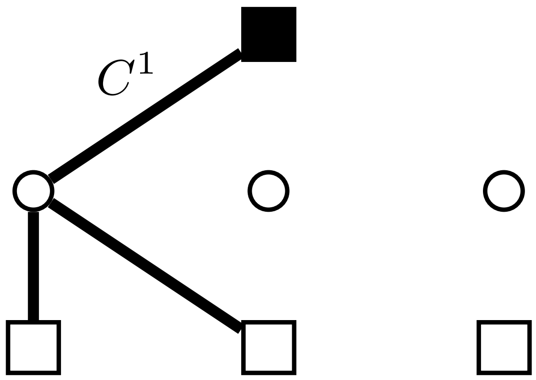

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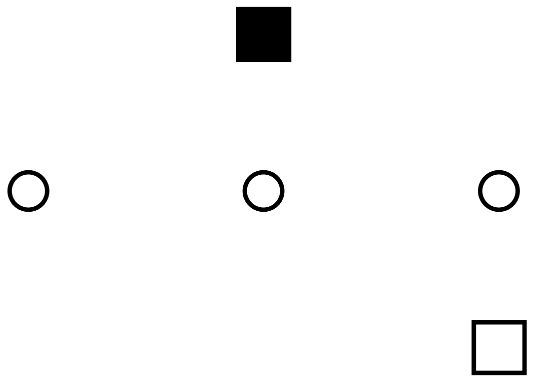

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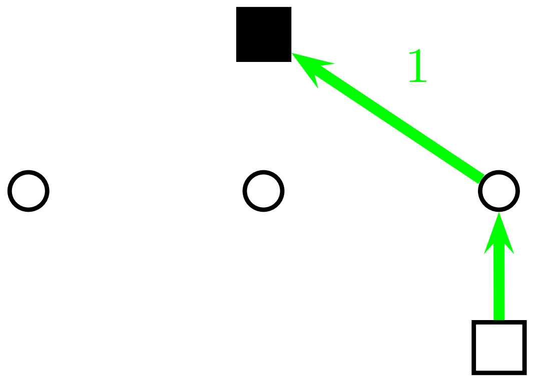

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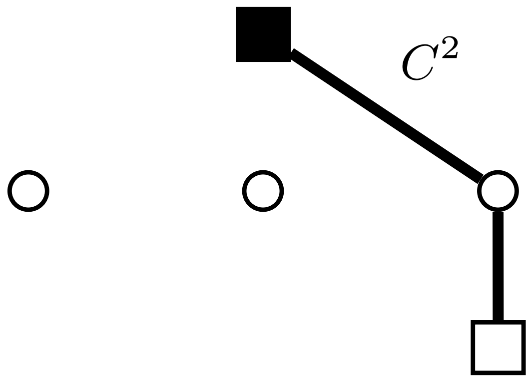

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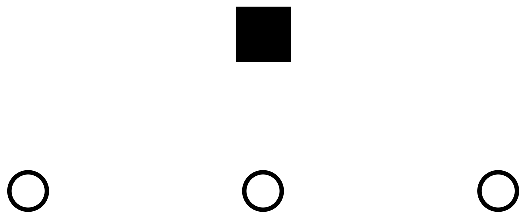

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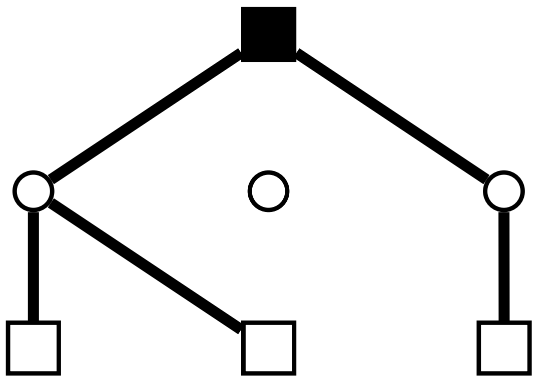

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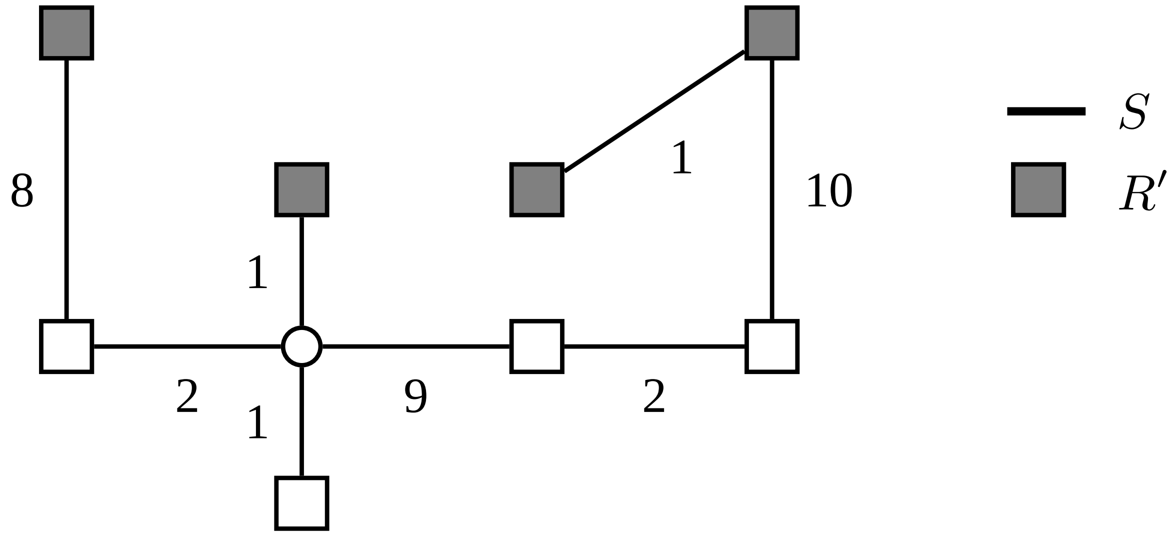
Rem By adding a dummy component in the root, we can assume w.l.o.g. that $M := \sum_{D \in \mathcal{C}} x^t_D$ is fixed for all t

Bridge Lemma

Bridges

Def Given a Steiner tree S and $R' \subseteq R$, the **bridges** $br_{S,c}(R')$ of S w.r.t. R' (and edge costs c) are the edges of S which do not belong to the minimum spanning tree of $V(S)$ after the contraction of R'



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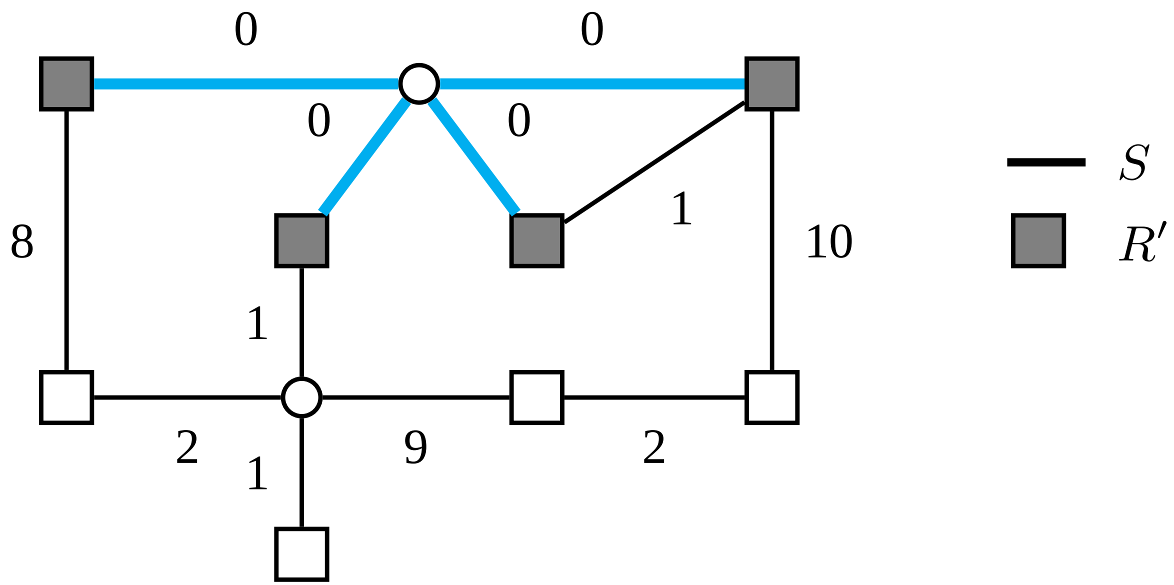

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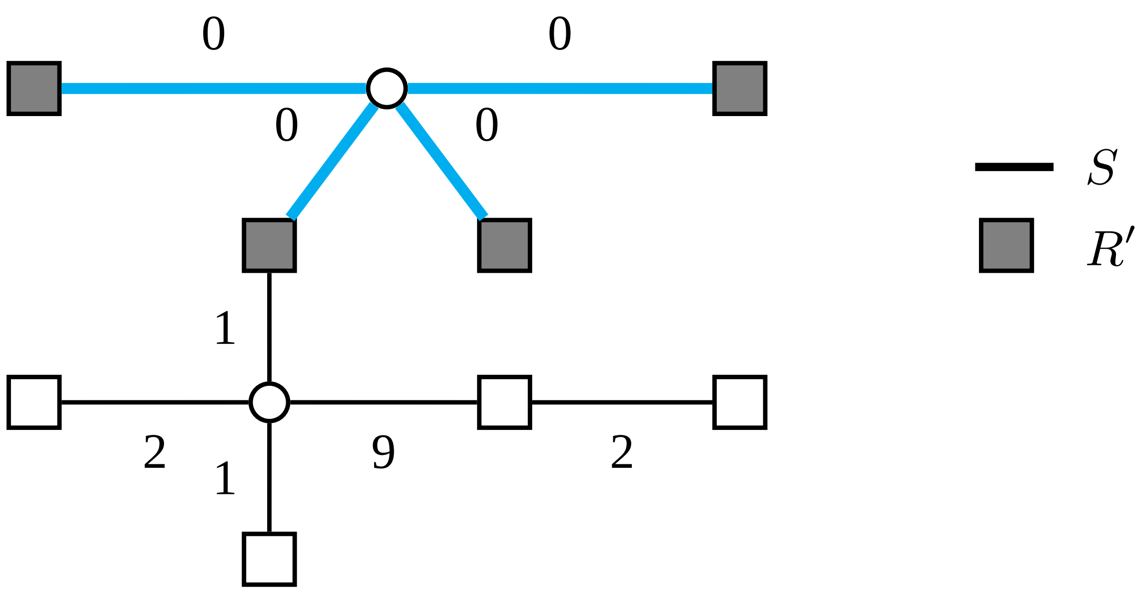

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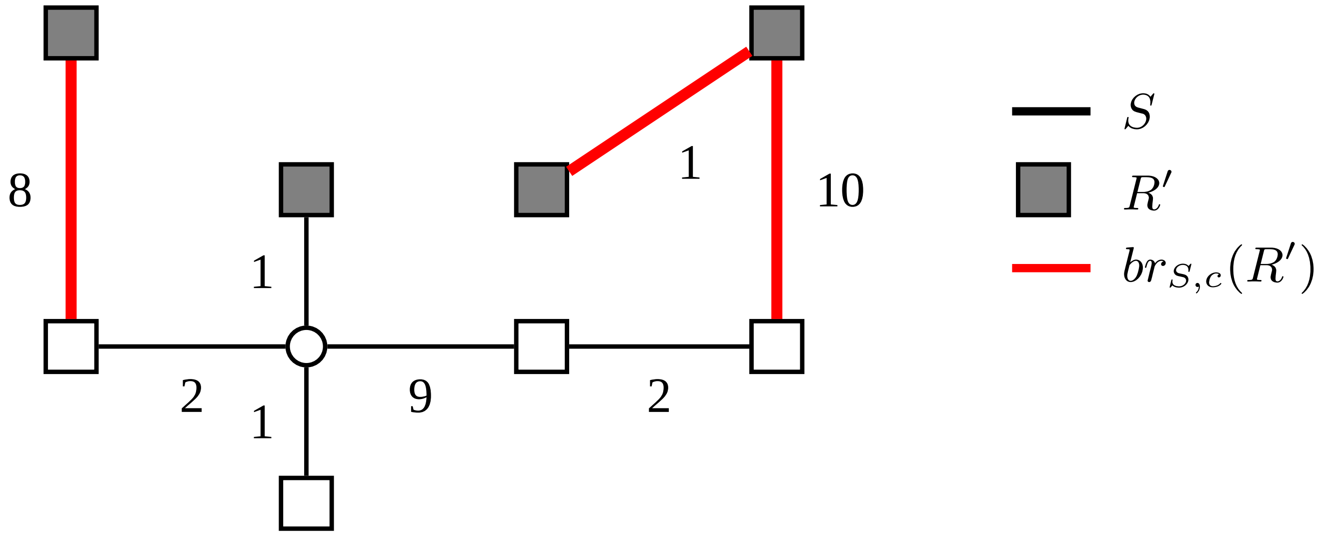

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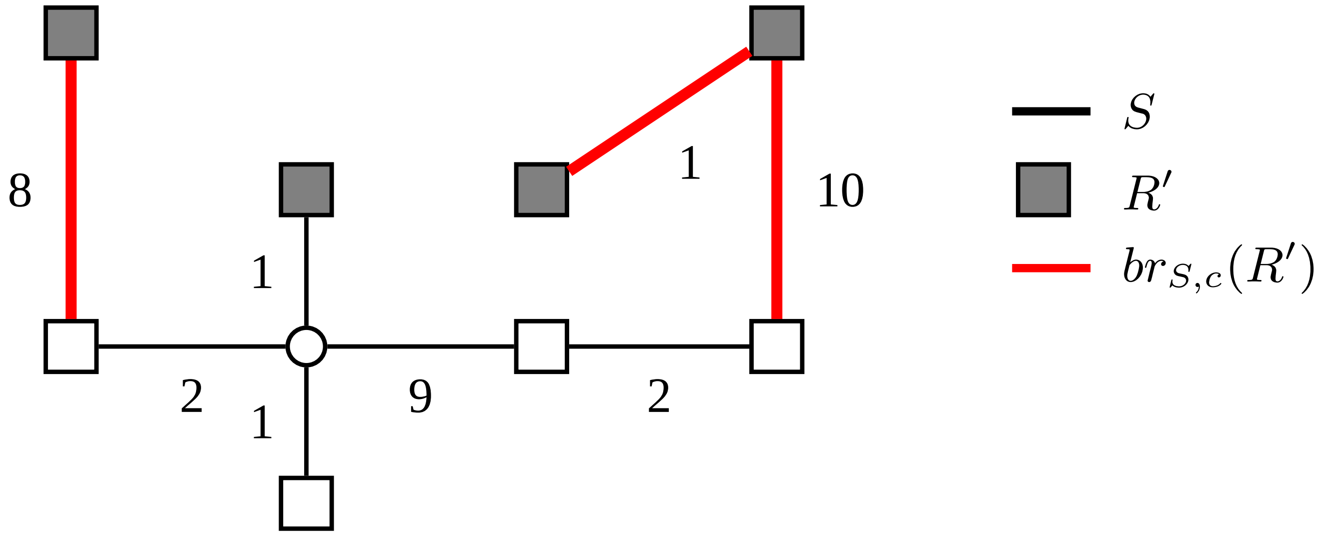

Rem The most expensive edge on a path between two gray nodes is a bridge

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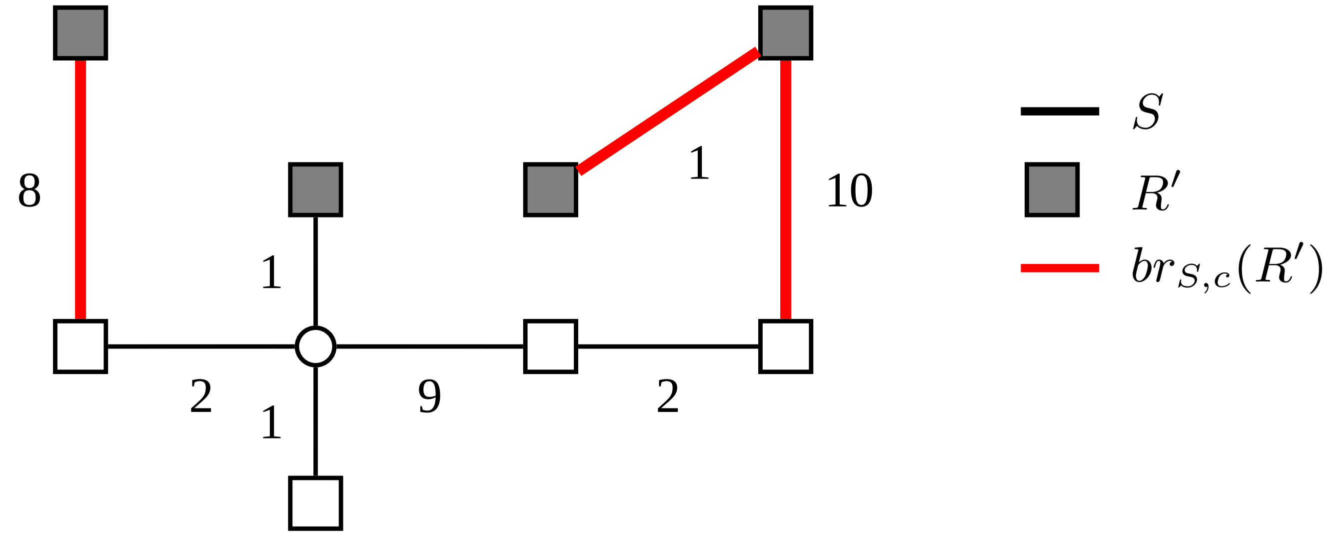
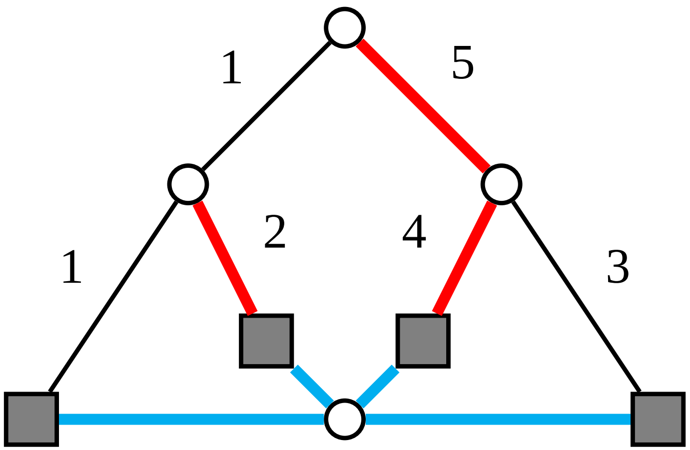


Rem Let $br_S(R')$=$br_{S,c}(R')$, $br_S(R')$:=$c(br_S(R'))$ and $br_S(C)$:=$br_S(R \cap C)$.

Bridges

Lem For any Steiner tree S on R, $br_S(R) \geq \frac{1}{2}c(S)$



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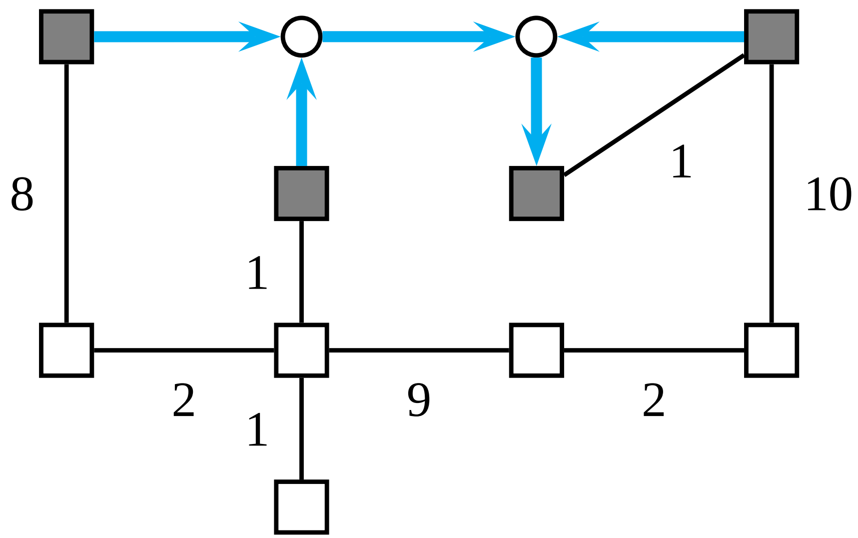

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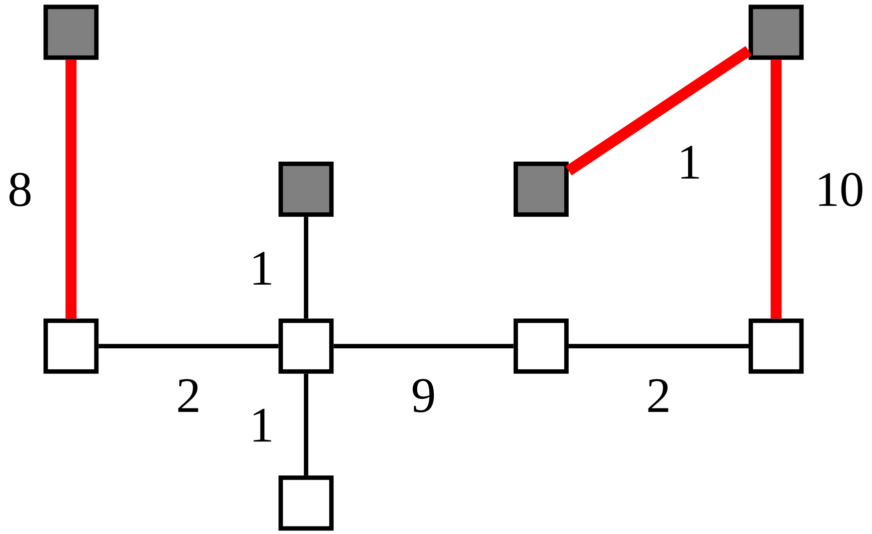

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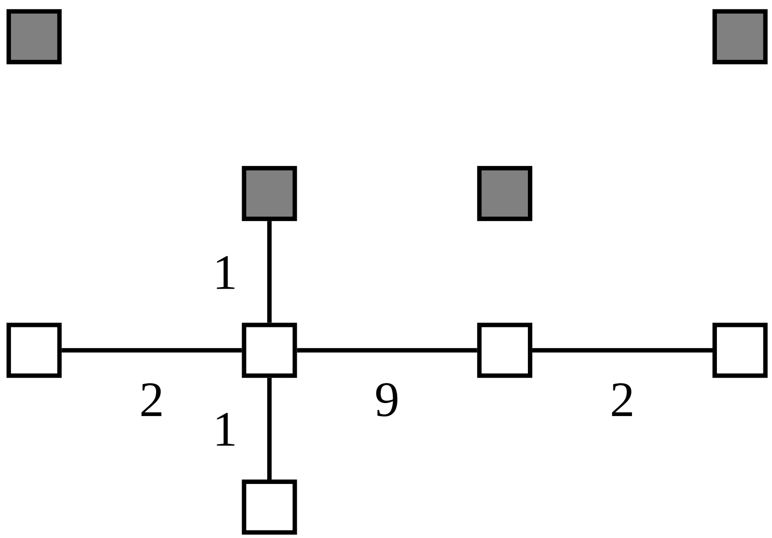

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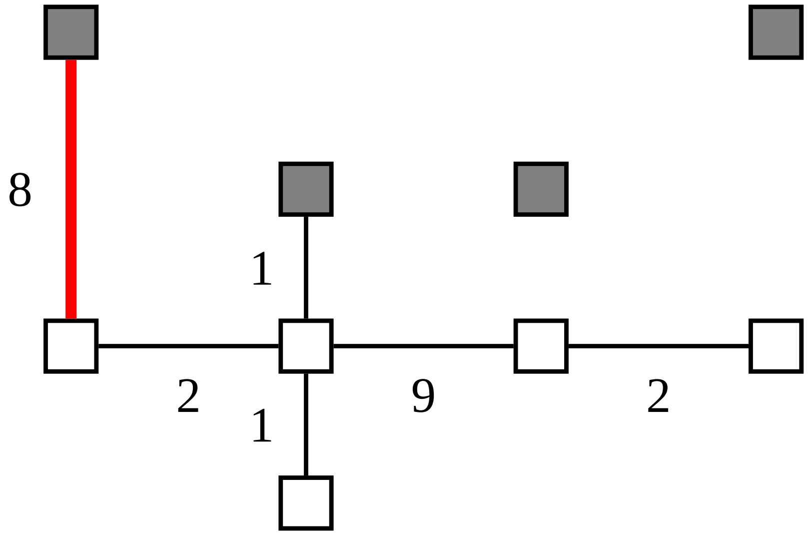

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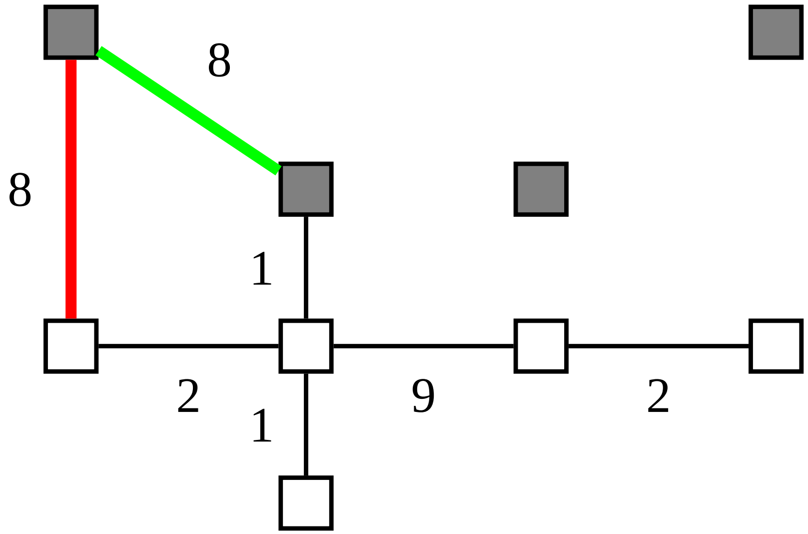

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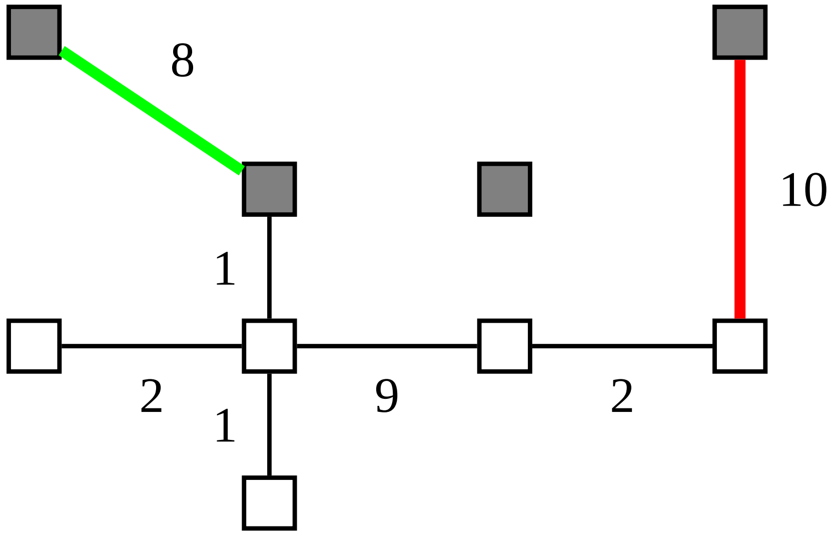

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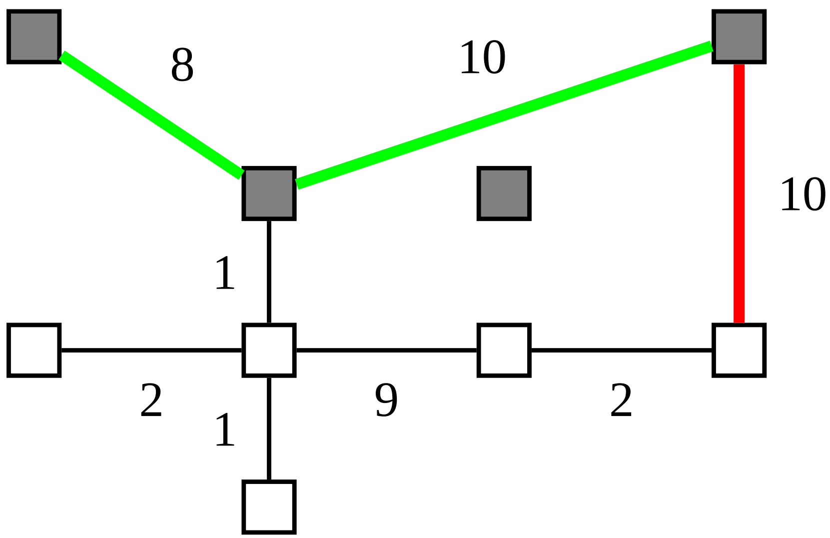

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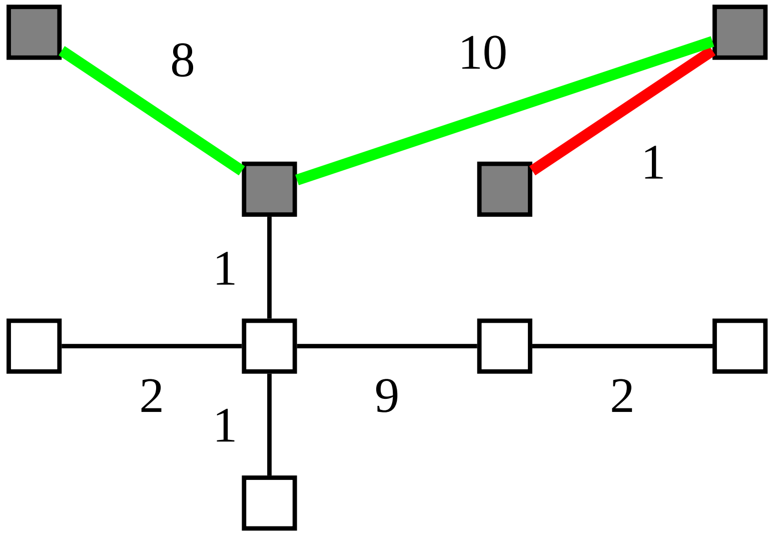

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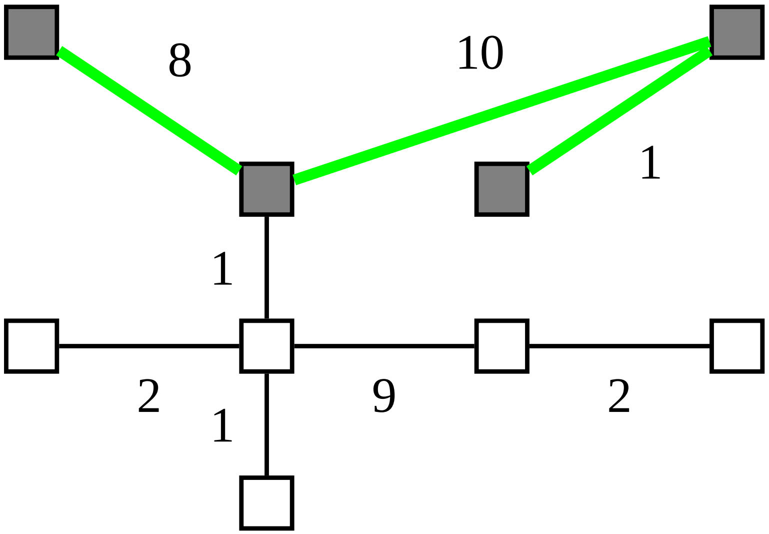

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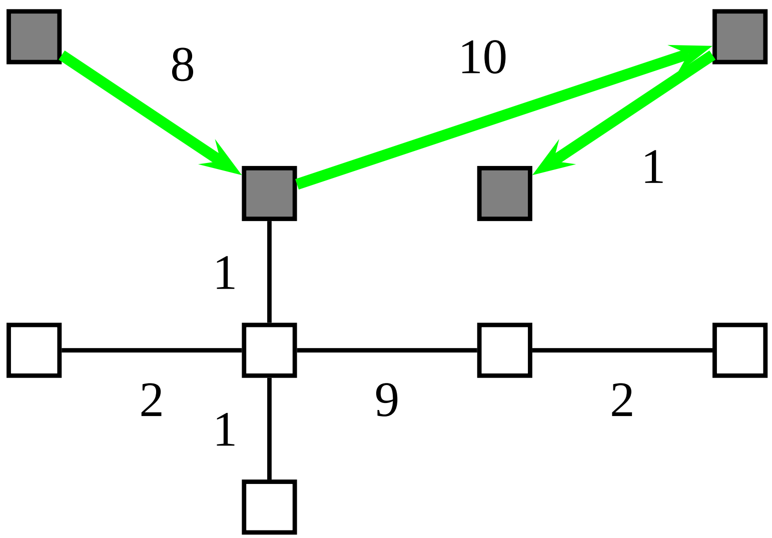

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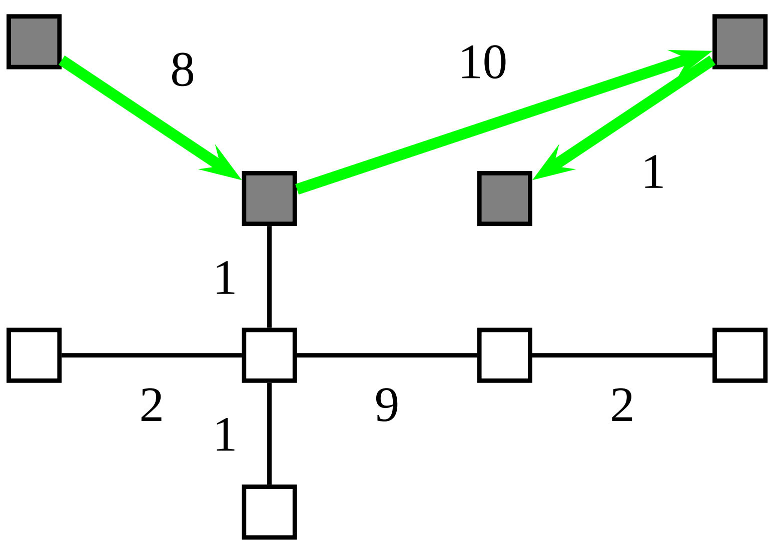

Rem Y_C supports the same flow to the root as C w.r.t. terminals

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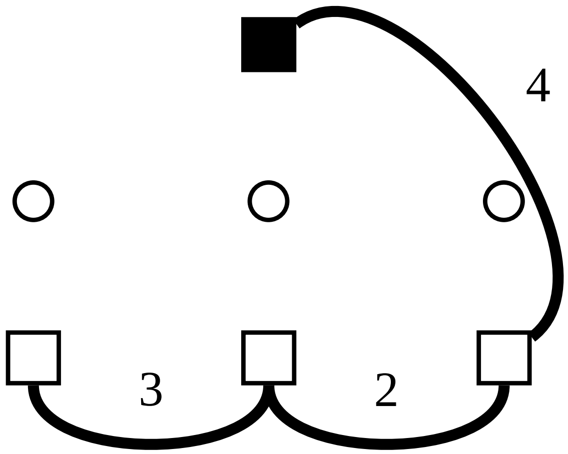

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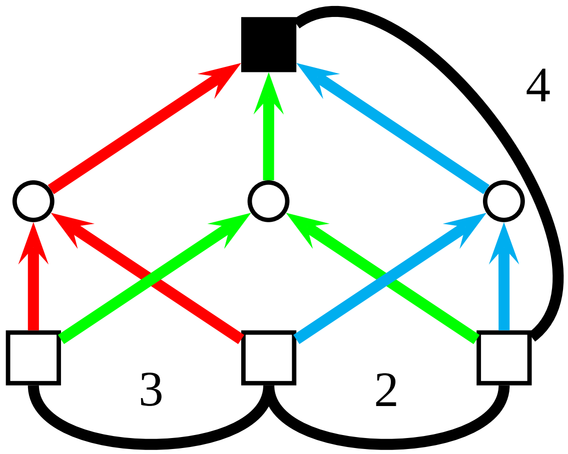

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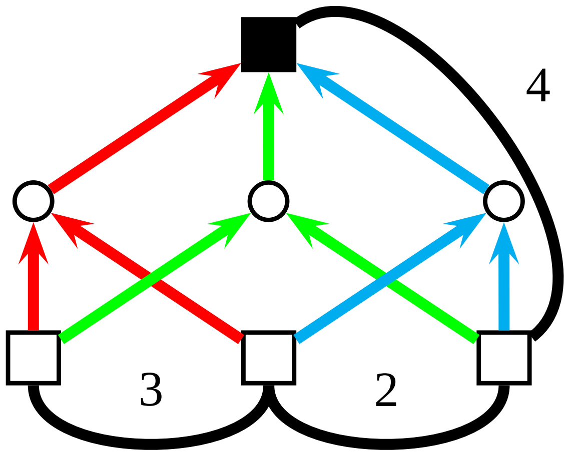

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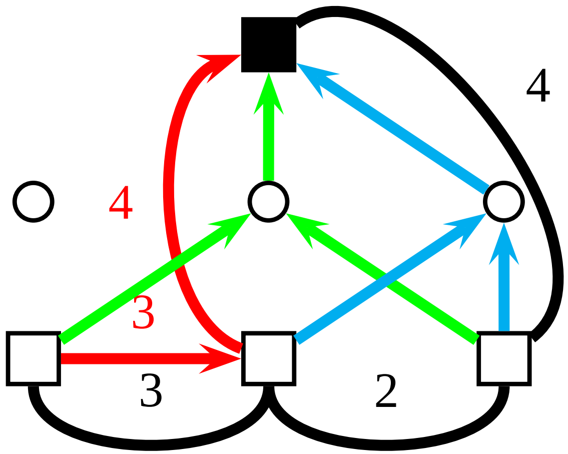

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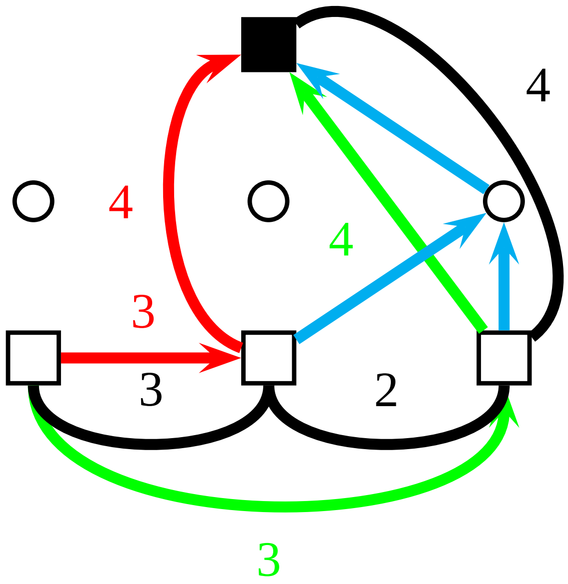

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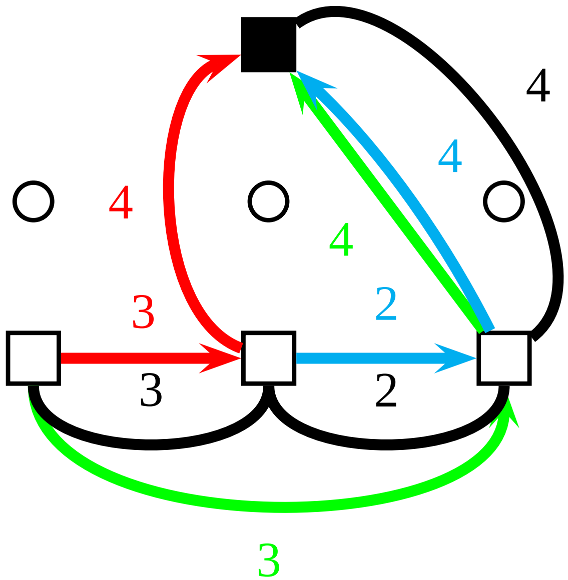

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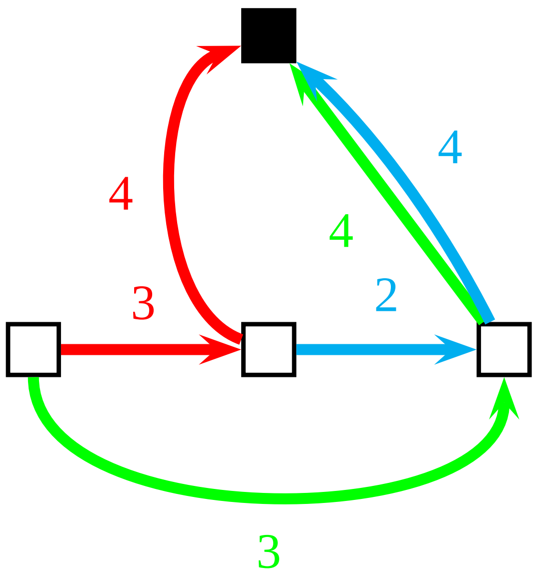

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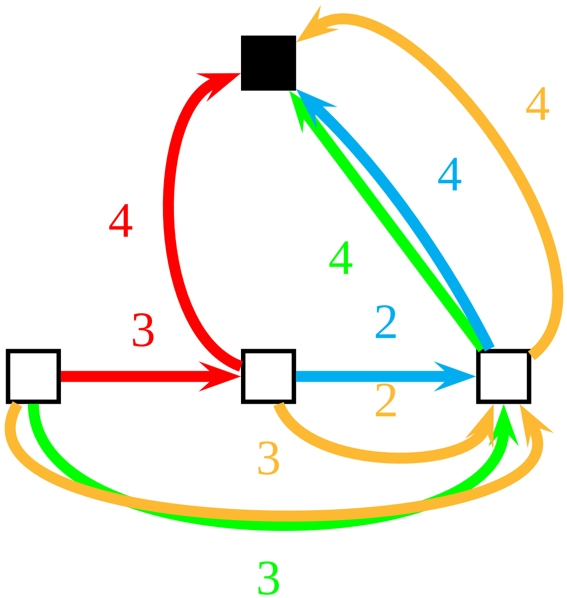

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$\Rightarrow$ By Edmod's thr there is a cheaper (w.r.t. w) integral directed terminal spanning tree F

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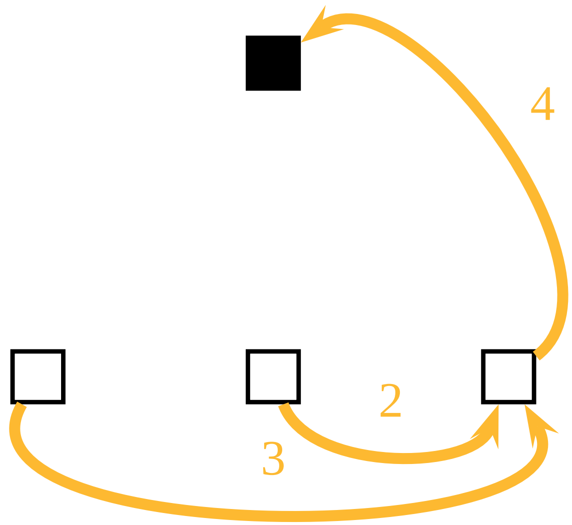

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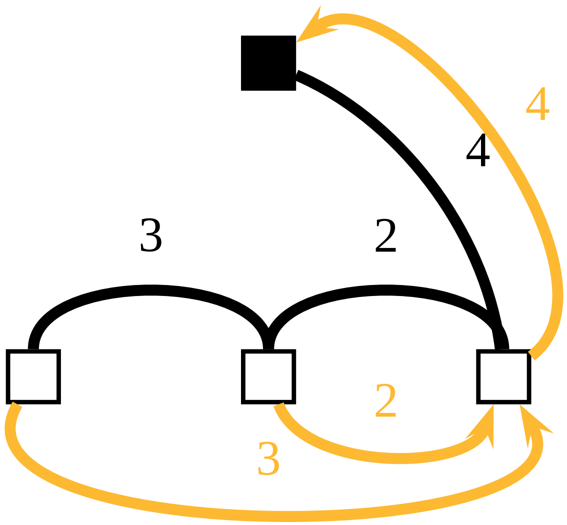

- The new terminal spanning tree F is more expensive than the original terminal spanning tree T by the cycle-rule

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- Summarizing

$$\sum_{C \in \mathcal{C}} x_C \cdot br_T(C) = \underbrace{\sum_{C \in \mathcal{C}} x_C \cdot w(Y_C)}_{\text{w-cost of fractional terminal spanning tree}} \geq \underbrace{w(F)}_{\text{w-cost of integral terminal spanning tree}} \geq c(T)$$

Approximation Factor

A First Bound

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Cor The integrality gap of DCR is at most $1 + \ln 2 < 1.7$

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$$E[apx] = \sum_{t \geq 1} E[c(C^t)] \leq \sum_{t \geq 1} E[\sum_C \frac{x_C^t}{M} c(C)] \leq \frac{1+\varepsilon}{M} \sum_{t \geq 1} E[opt^{f,t}]$$

$$\leq \frac{1+\varepsilon}{M} \sum_{t=1}^{M \ln 2} opt^f + \frac{1+\varepsilon}{M} \sum_{t > M \ln 2} E[c(T^t)]$$

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$$\leq opt^f (1+\varepsilon) \ln 2 + 2\, opt^f (1+\varepsilon) \sum_{t > M \ln 2} \frac{1}{M} \left(1 - \frac{1}{M}\right)^{t-1}$$

$$\leq (1+\varepsilon)(\ln 2 + 2e^{-\ln 2}) \cdot opt^f$$

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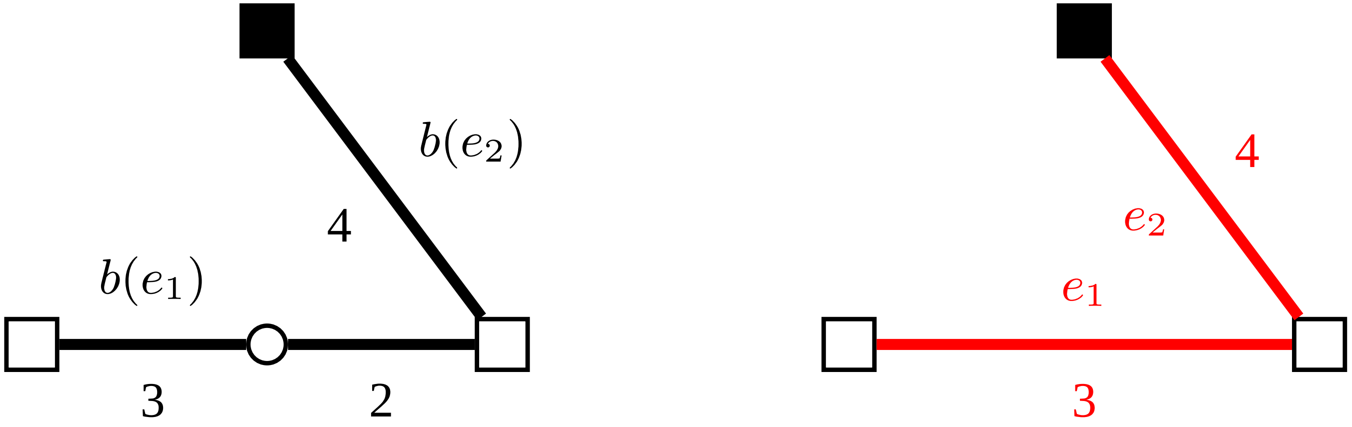

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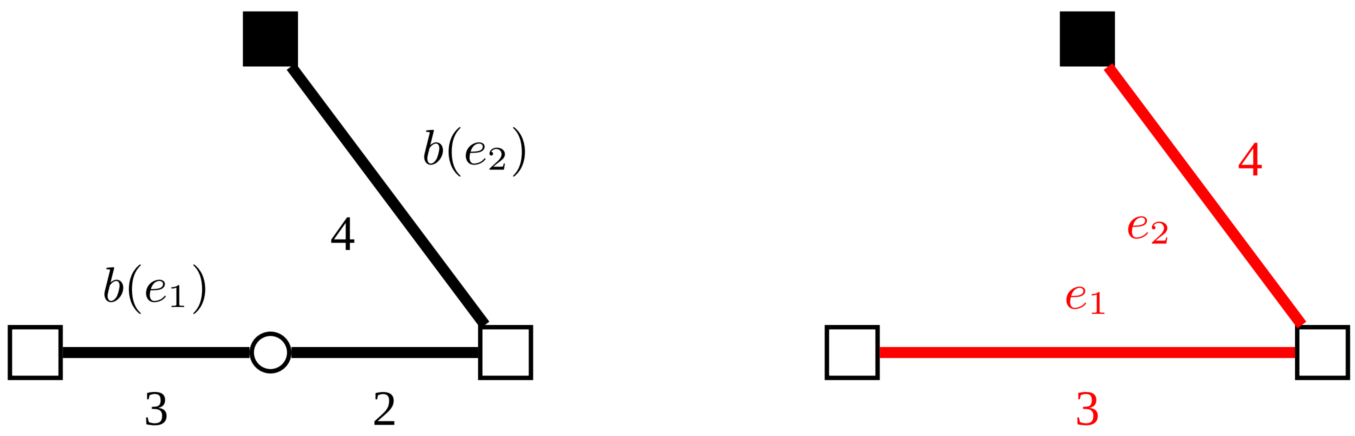

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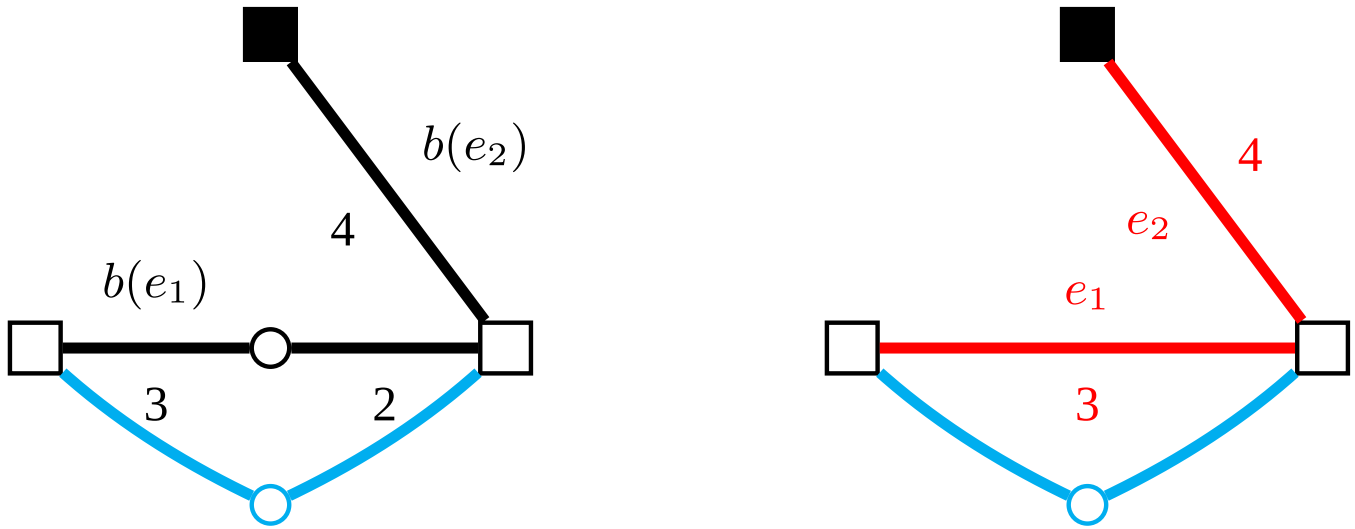

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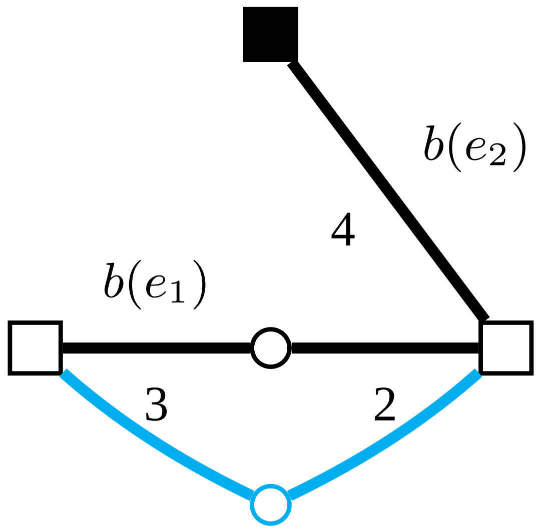

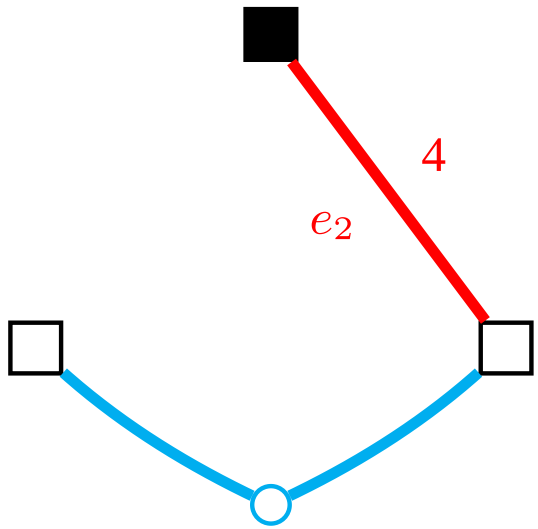

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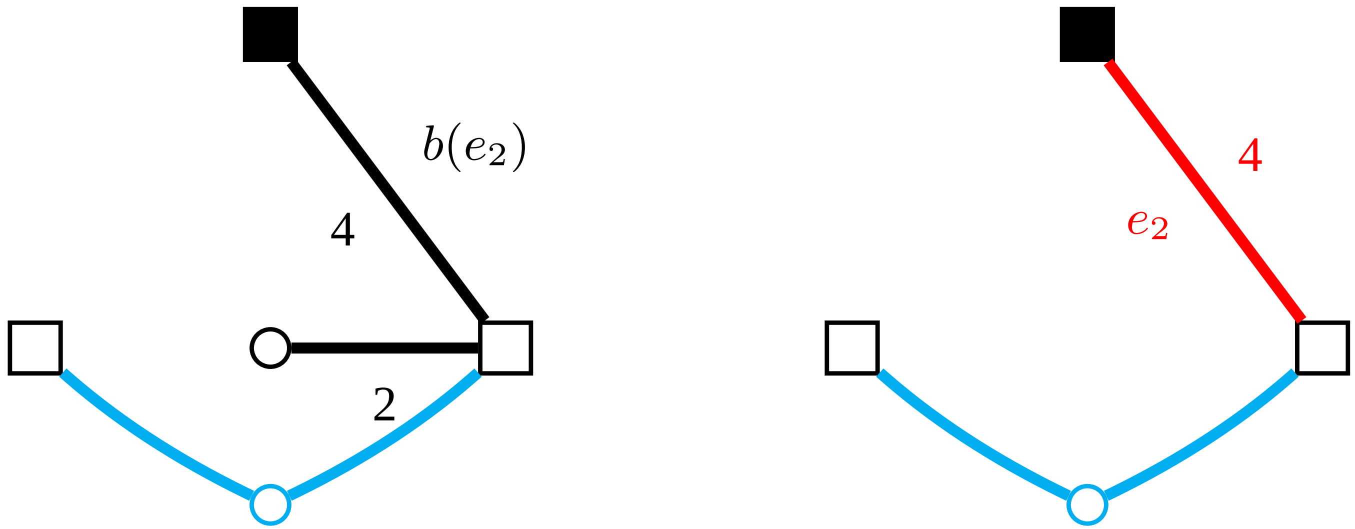

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&= c(S^t) - \frac{1}{M} \sum_C x_C^t br_{Y^t,\mathbf{w}}(C) \\
&\overset{\text{Bridge Lem}}{\leq} c(S^t) - \frac{1}{M} w(Y^t) \\
&= c(S^t) - \frac{1}{M} br_{S^t,\mathbf{c}}(R^t) \\
&\leq c(S^t) - \frac{1}{M} \frac{c(S^t)}{2}
\end{aligned}
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$$\leq (1+\varepsilon)(2 - 2e^{-\ln(4)/2} + 2e^{-\ln(4)}) \cdot opt$$

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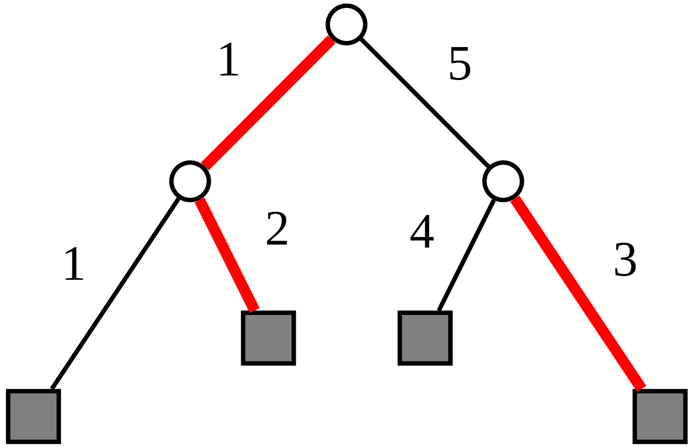

- We define a random terminal spanning tree W (*witness* tree)

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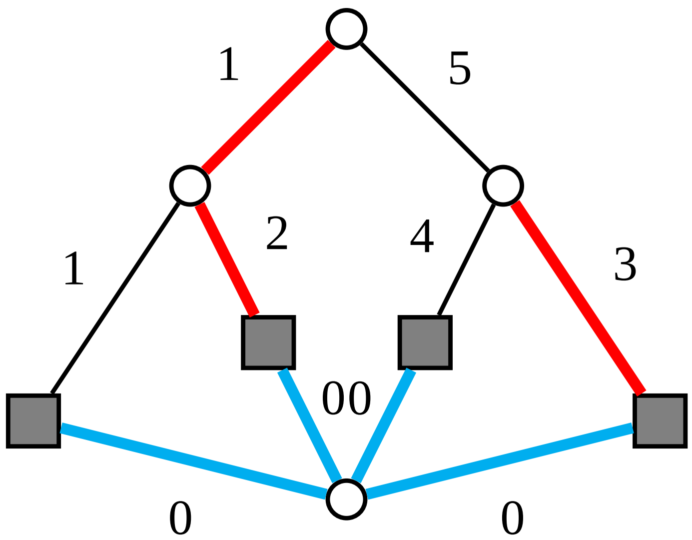

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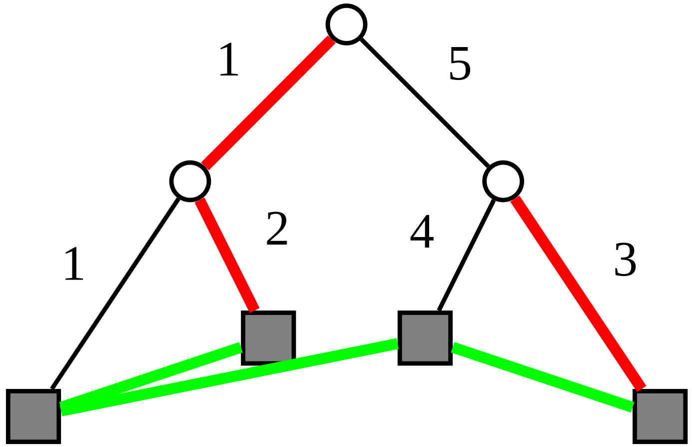

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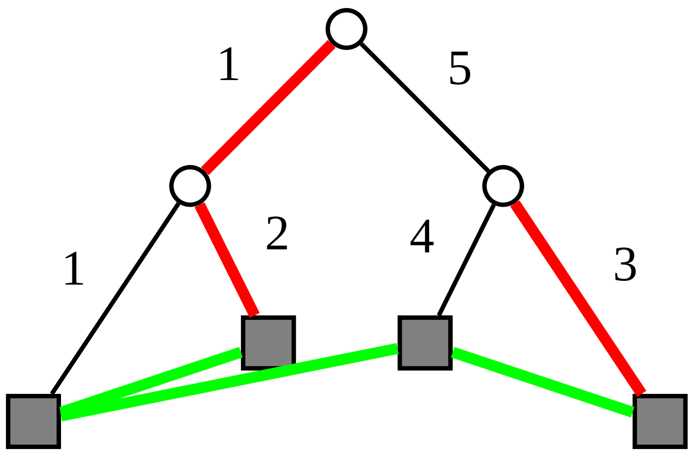

- We associate to each e in the Steiner tree S the edges $W(e)$ of W such that the corresponding path in S contains e
- Observe that $|W(e)|$ is 1, 2... with probability $\frac{1}{2}$, $\frac{1}{4}$, ...

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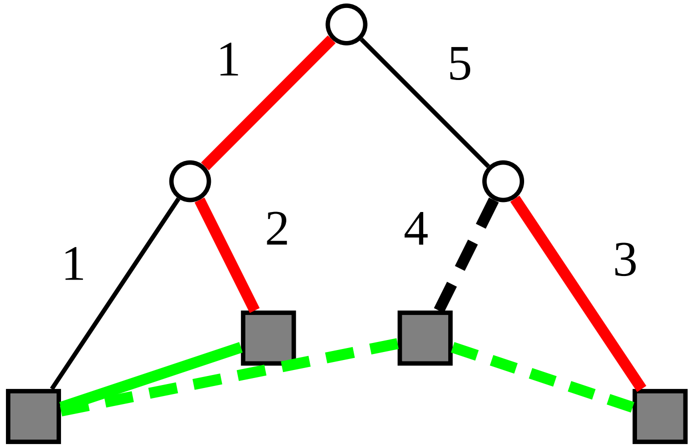

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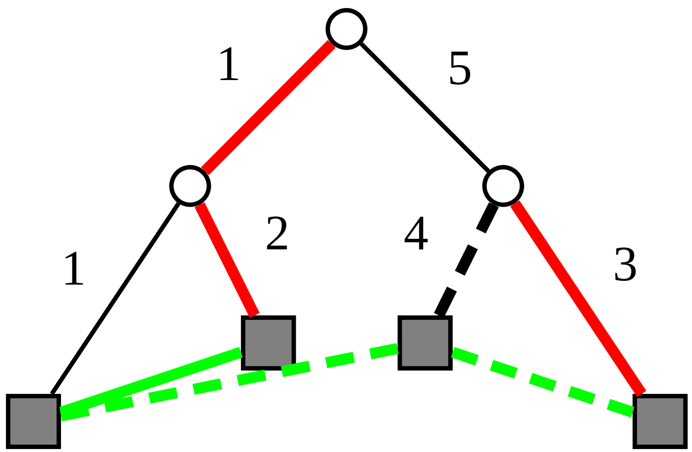

- For any sampled component C^t, we delete from W a random set of bridges such that each edge of W is deleted with probability $\geq 1/M$ ($\Leftarrow$ Farkas' lemma+Bridge lemma)
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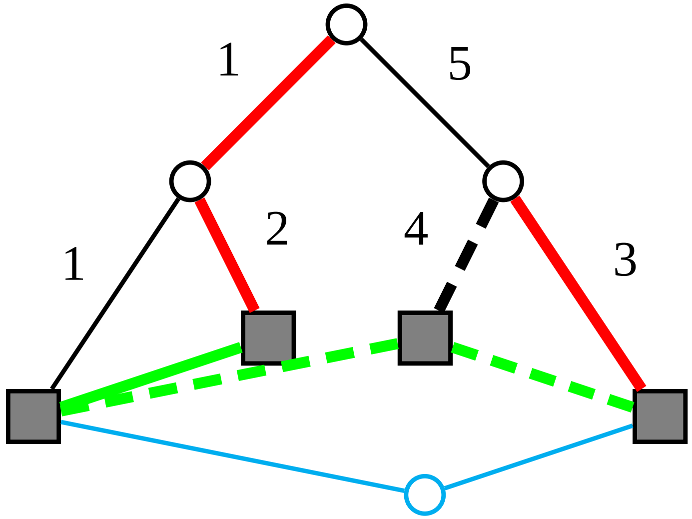

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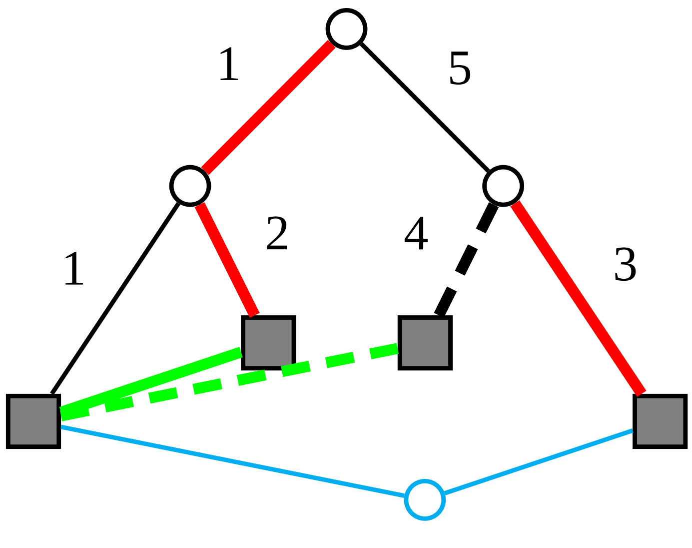
An Even Better Bound

Thr Algorithm `IRR` computes a solution of expected cost $\leq (\ln 4 + \varepsilon)\, opt$



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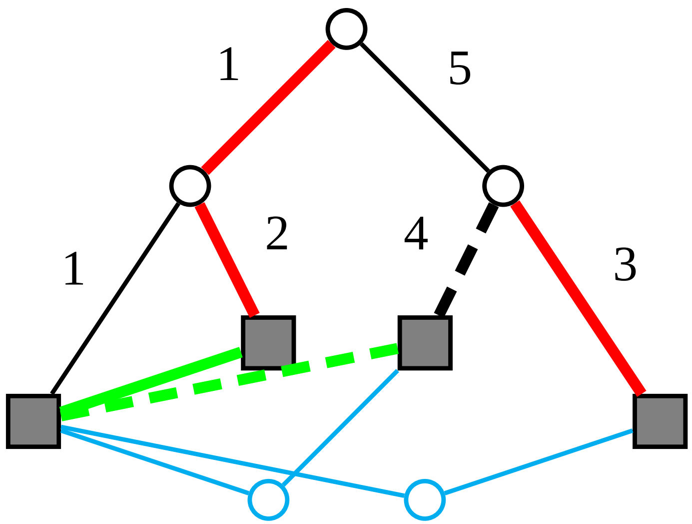

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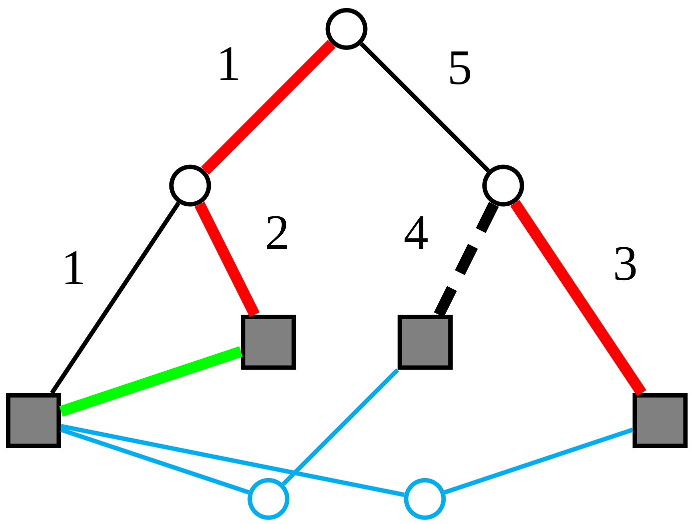

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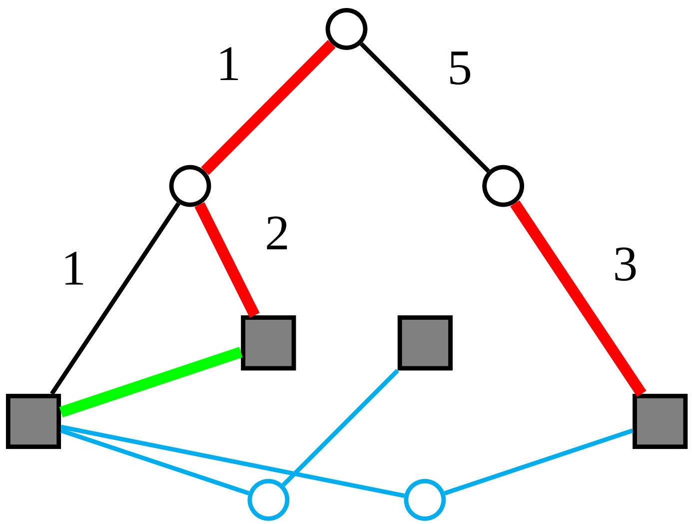

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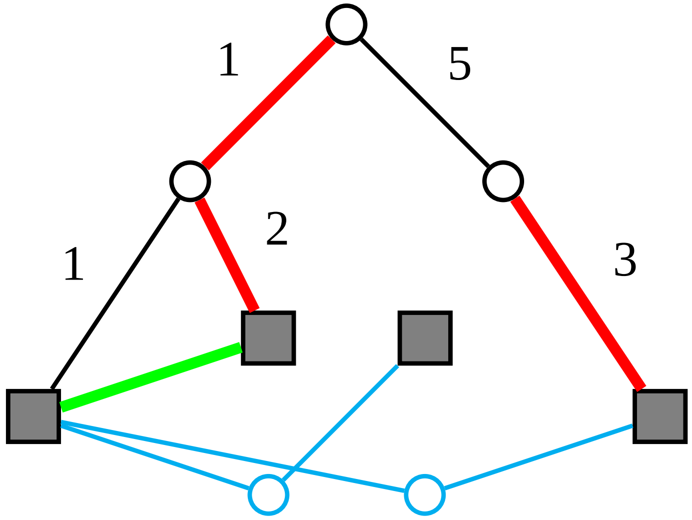

- Each $e \in S$ survives in expectation $M \cdot \ln 4$ rounds

Derandomization

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- We define a phase-based randomized algorithm, with $1/\varepsilon^2$ phases s
- At each phase, we sample a proper number of components (without updating the LP)
- It is sufficient to guarantee that, at each phase:
  - ◇ Each component is sampled with probability $O(\varepsilon) x_C^s$
  - ◇ Each edge of the witness tree W is marked with probability $\Omega(\varepsilon)$
- This can be done by using only $O(\log n)$ random bits per phase

Open Problems

- The best 1.39 (and even 1.5) bound is w.r.t. the optimal integral solution. Does is hold w.r.t. the fractional one?
- Other applications of iterative randomized rounding?
  - Prize-collecting Steiner tree
  - k-MST
  - Single-Sink Rent-or-Buy
  - $\ldots$

THANKS!!!