Allocating Goods to Maximize Fairness

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Max Min Allocation

Input:

- Set ***A*** of m agents
- Set ***I*** of n items
- Utilities $u_{A,i}$ of agent A for item i.

Output: assignment of items to agents.

- Utility of agent A: $\sum u_{A,i}$ for items i assigned to agent A.

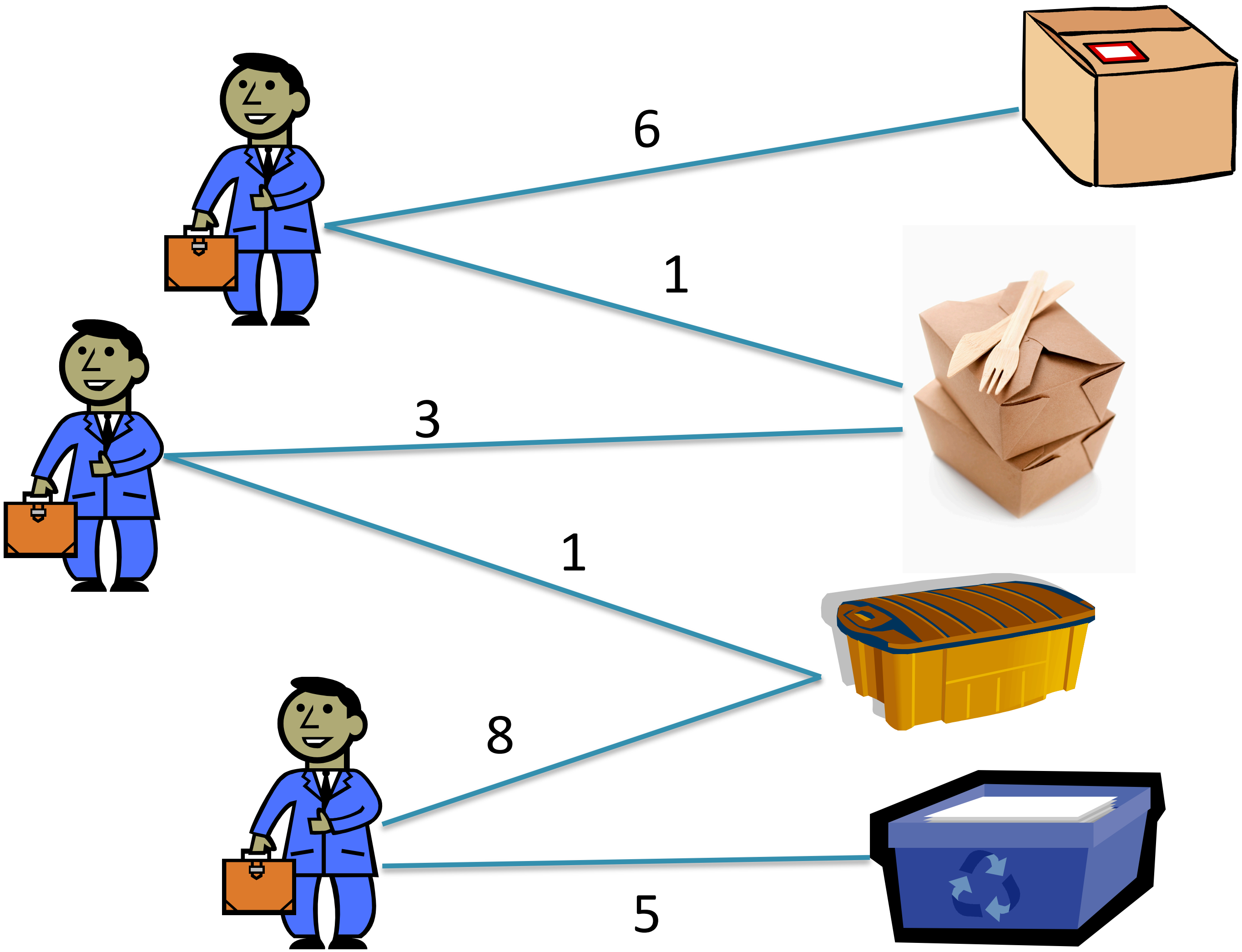
Goal: Maximize minimum utility of any agent.

Notation

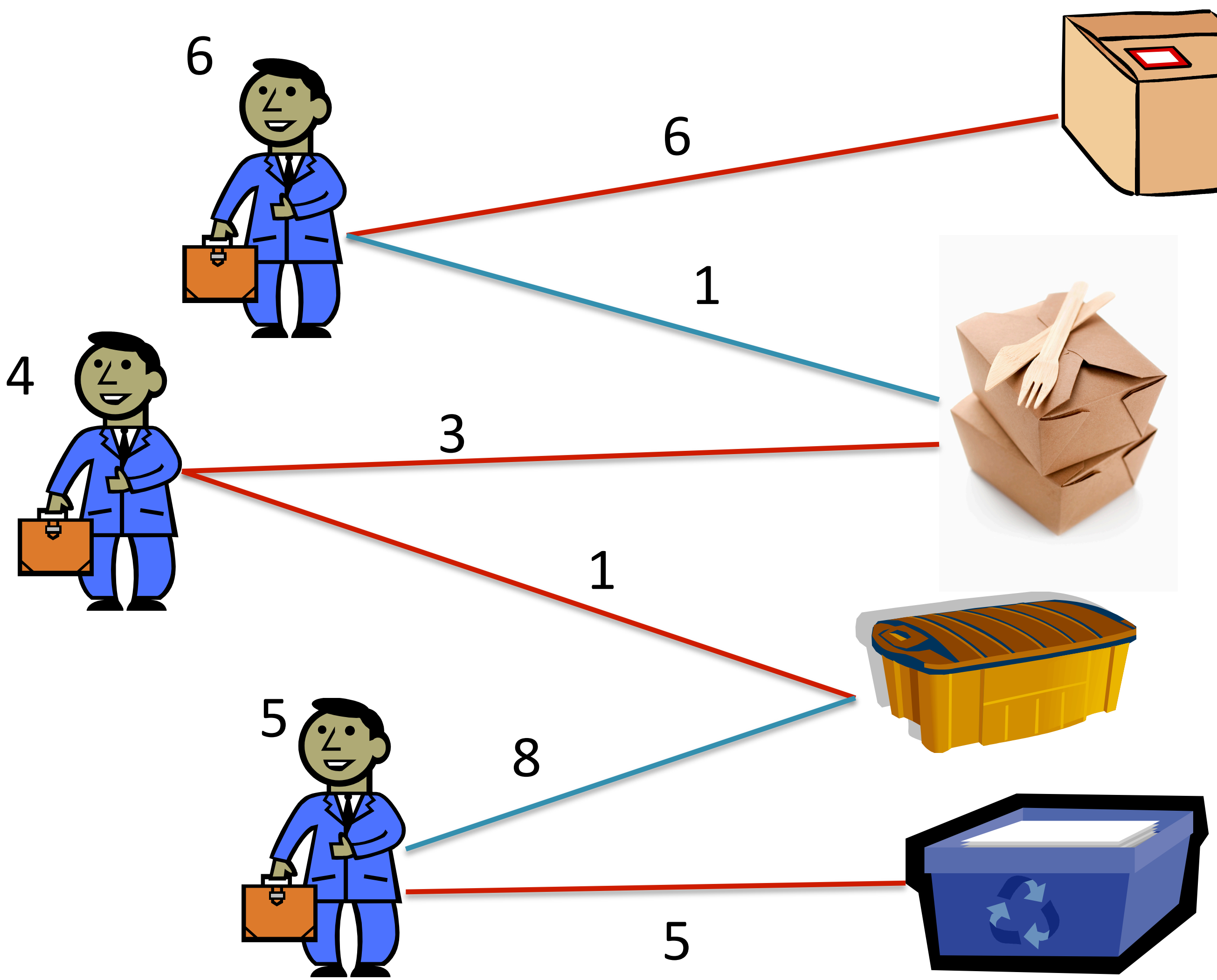
n - number of items

m - number of agents

Example



Example

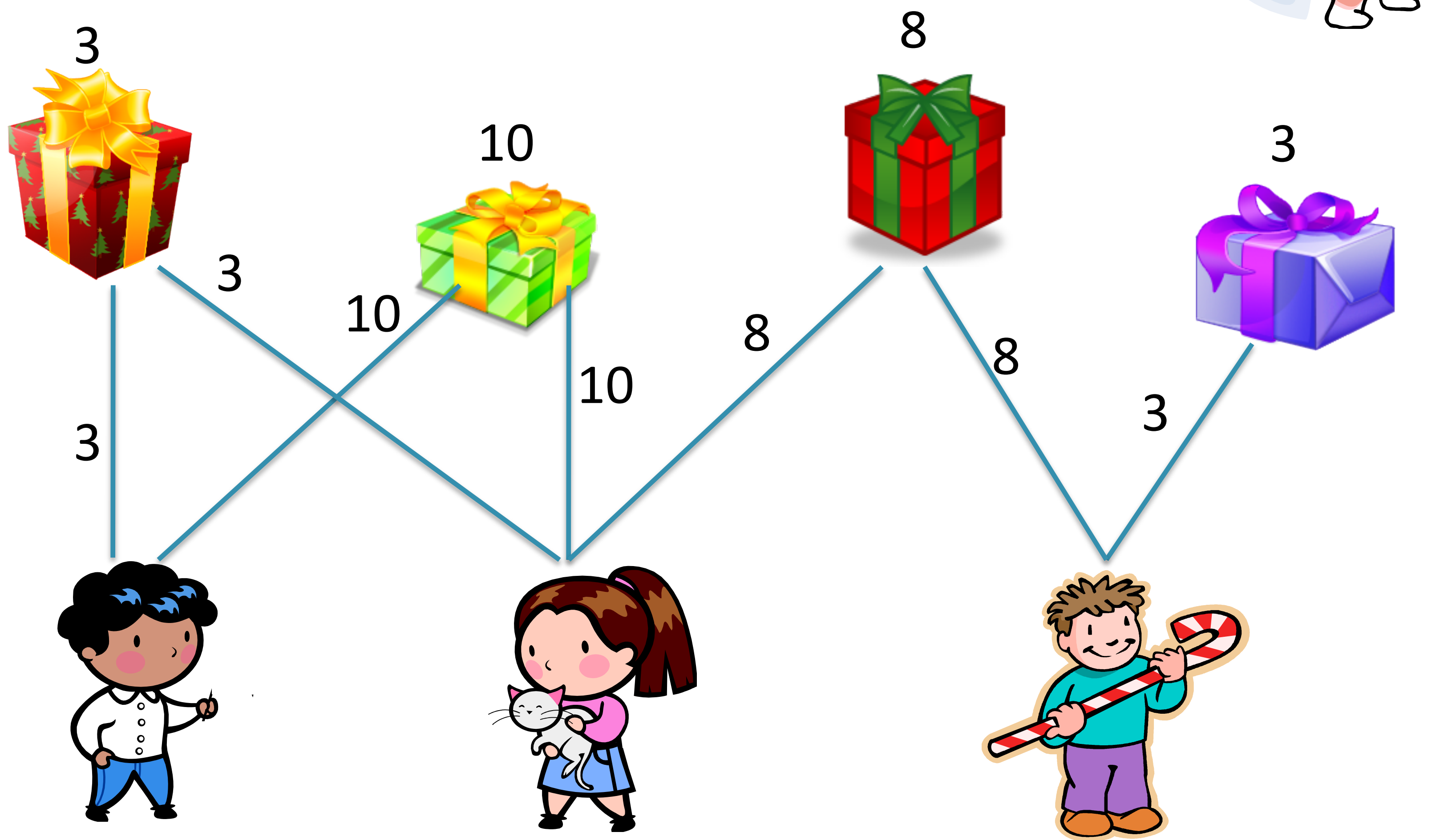


Solution value: 4

Max-Min Allocation

- Captures a natural notion of fairness in allocation of indivisible goods.
- Is related to the cake cutting theory.
- Approximation is still poorly understood.
- An interesting special case: Santa Claus problem.

The Santa Claus Problem



Santa Claus: Known Results

- Natural LP has $\Omega(m)$ integrality gap.
- [Bansal, Sviridenko '06]:
  - Introduced a new configuration LP
  - O(log log m/logloglog m)-approximation algorithm
- Non-constructive constant upper bounds on integrality gap of the LP [Feige '08], [Asadpour, Feige, Saberi '08].
- Constant approximation [Haeupler, Saha, Srinivasan '10]

Bad news: Configuration LP has $\Omega(\sqrt{m})$ integrality gap for Max-Min Allocation [Bansal, Sviridenko '06].

Known Results for Max Min Allocation

- (n-m+1)-approximation [Bezakova, Dani '05].
- $\tilde{O}(\sqrt{m})$-approximation via the configuration LP [Asadpour, Saberi '07].
- Configuration LP has $\Omega(\sqrt{m})$ integrality gap [Bansal, Sviridenko '06].
- Best current hardness of approximation factor: 2 [Bezakova, Dani '05]

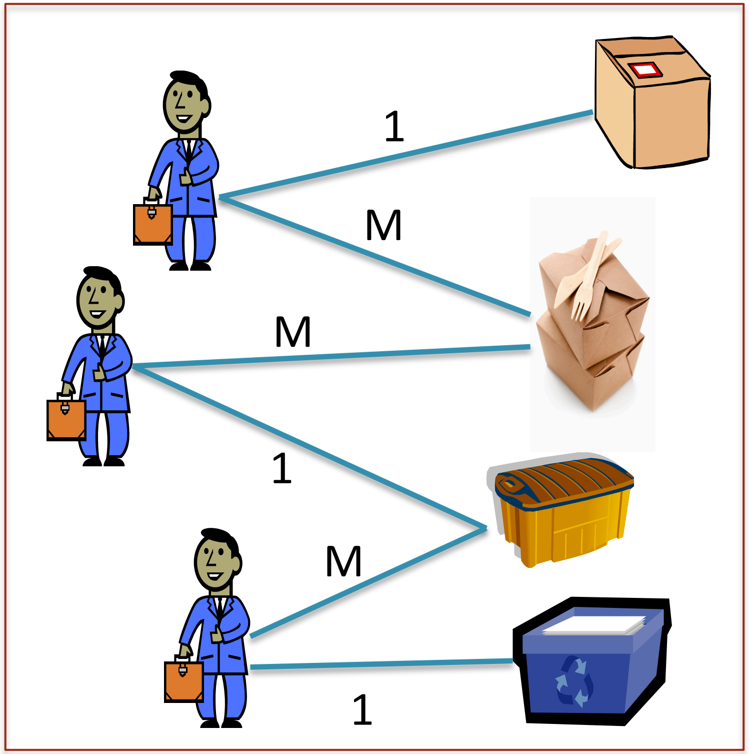
Our Results

- $\tilde{O}\left(n^{\epsilon}\right)$-approximation algorithm in time $n^{O(1/\epsilon)}$
  - Poly-logarithmic approximation in quasi-polynomial time.
  - n^{ϵ}-approximation in poly-time for any constant ϵ .
- We use an LP with $\Omega(\sqrt{m})$ integrality gap as a building block.

Independent Work

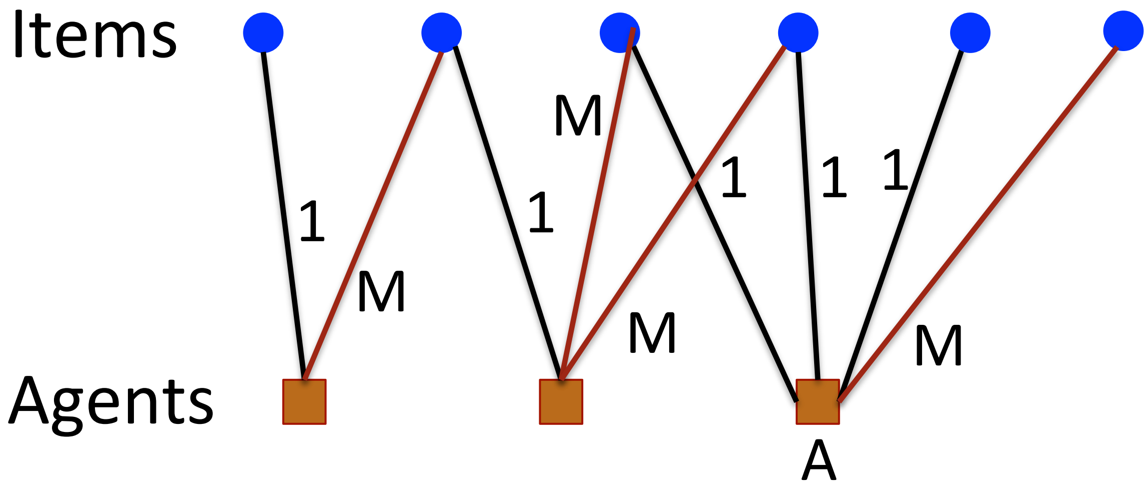
[Bateni, Charikar, Guruswami ‘09] obtained similar results for special cases of the problem:

- All utilities are in {0, 1, M}, where M=OPT.
  – All items have degree at most 2
  – Graph contains no cycles
- An $\tilde{O}\left(n^{\epsilon}\right)$-approximation in time $n^{O(1/\epsilon)}$



The $\tilde{O}\left(n^{\epsilon}\right)$-Approximation Algorithm

For simplicity, assume all utilities are in {0,1,M} where M=OPT.



OPT=M

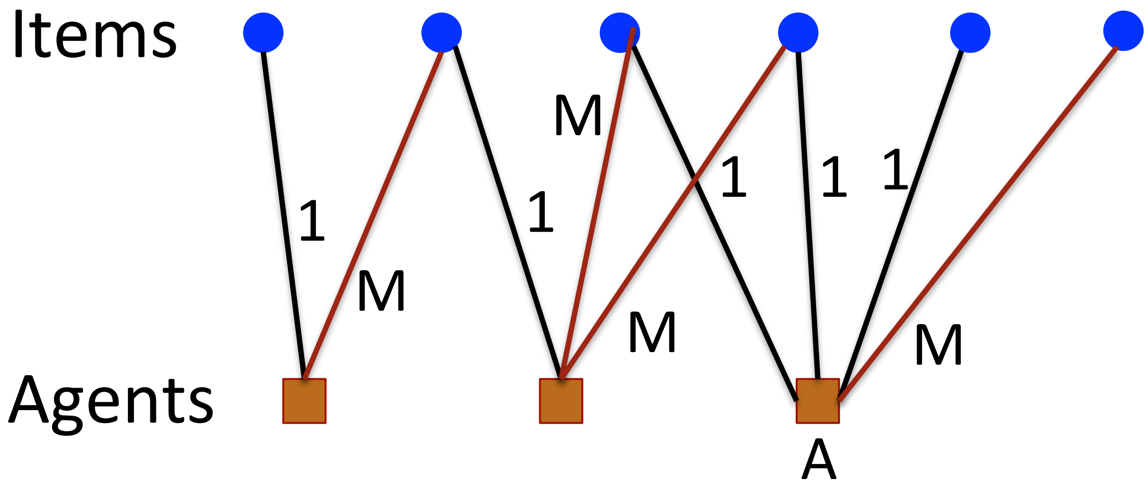
utility 1

utility M

Optimal solution

Each agent A is assigned:

- One utility-M item or
- M utility-1 items



OPT=M

utility 1

utility M

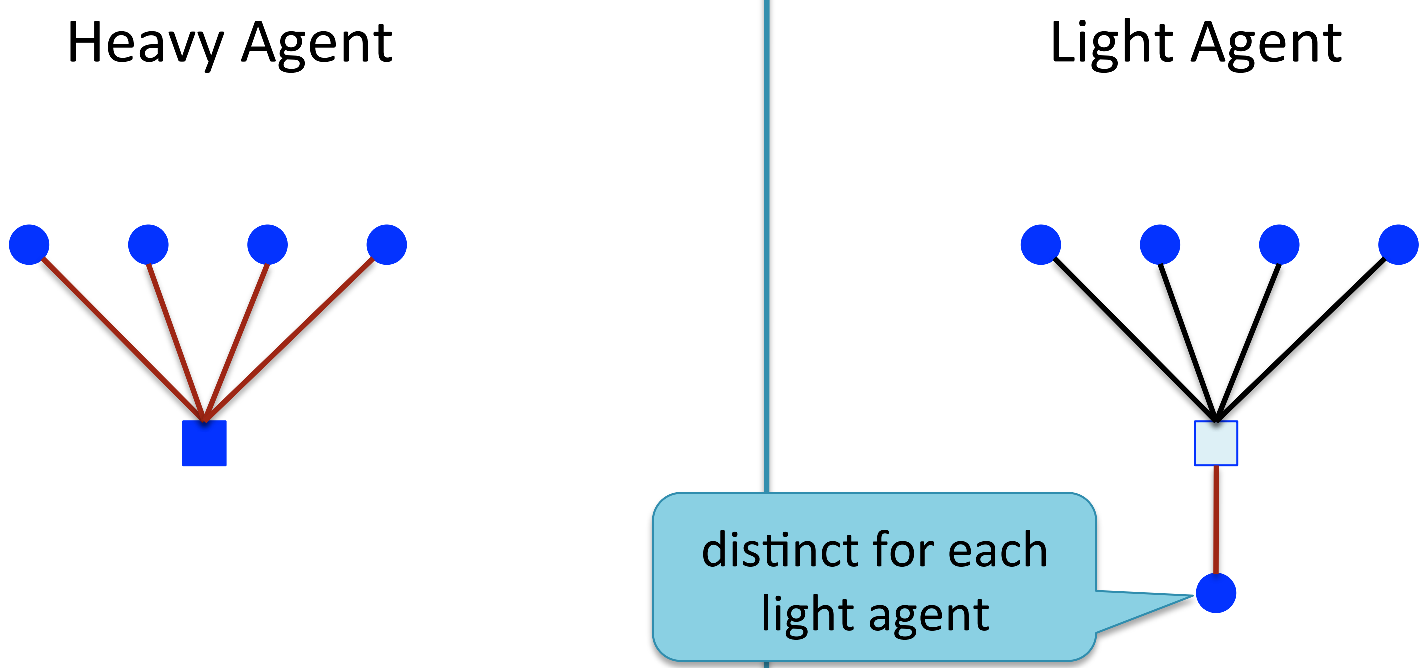
α-approximate solution

Each agent A is assigned:

- One utility-M item or
- ~~M~~ M/α utility-1 items

Canonical Instances

All agents are either heavy or light.

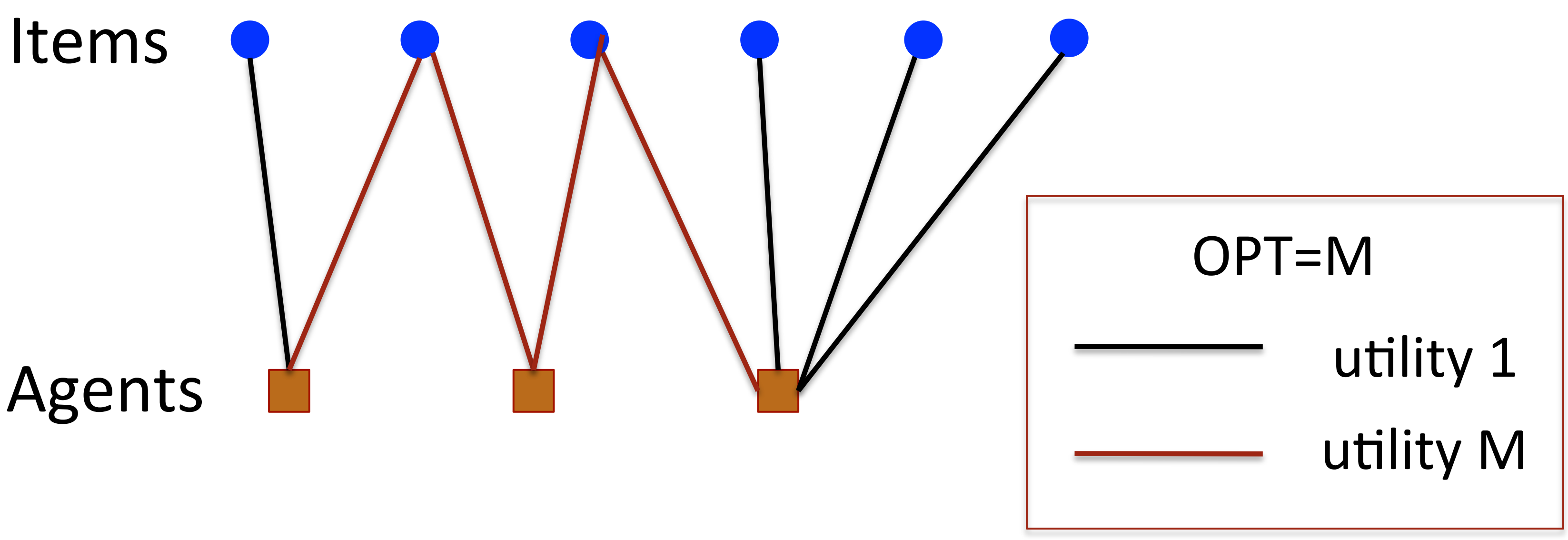


Can assume w.l.o.g. we are given a canonical instance.

Step 1: Turn the Problem into a Flow Problem!

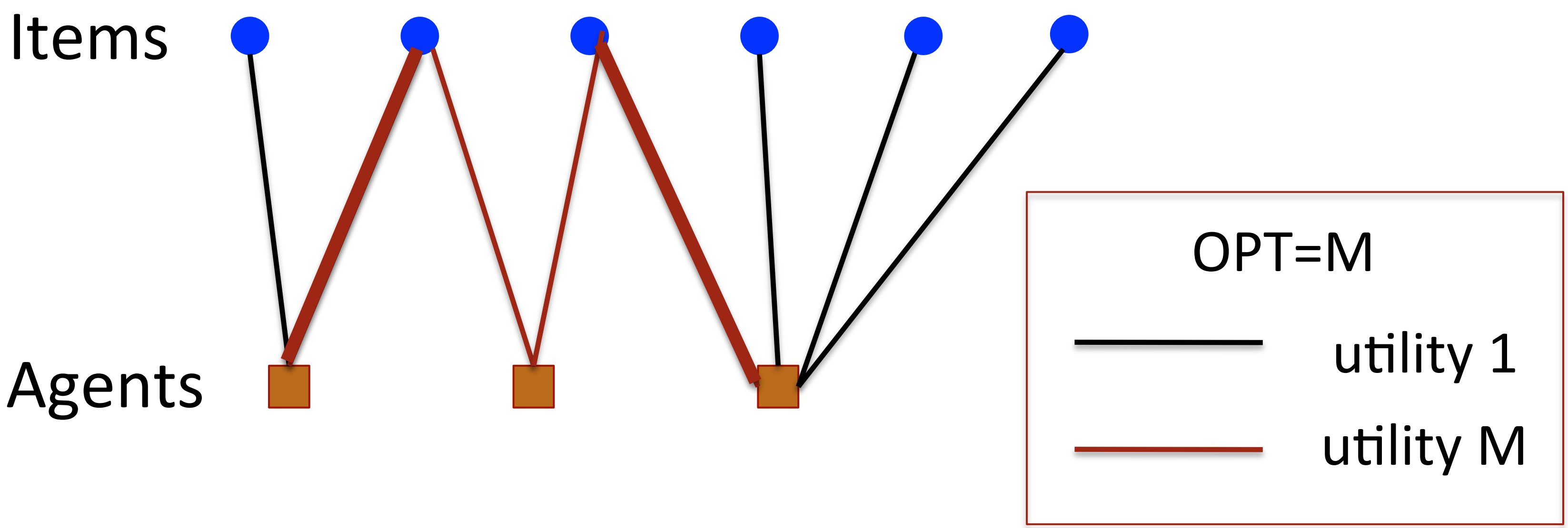
Main Idea

- Temporarily assign private items to agents



Main Idea

- Temporarily assign private items to agents
  - Item can be private for at most one agent
  - If i is private for A then $u_{A,i}$=M
  - Every light agent gets a private item



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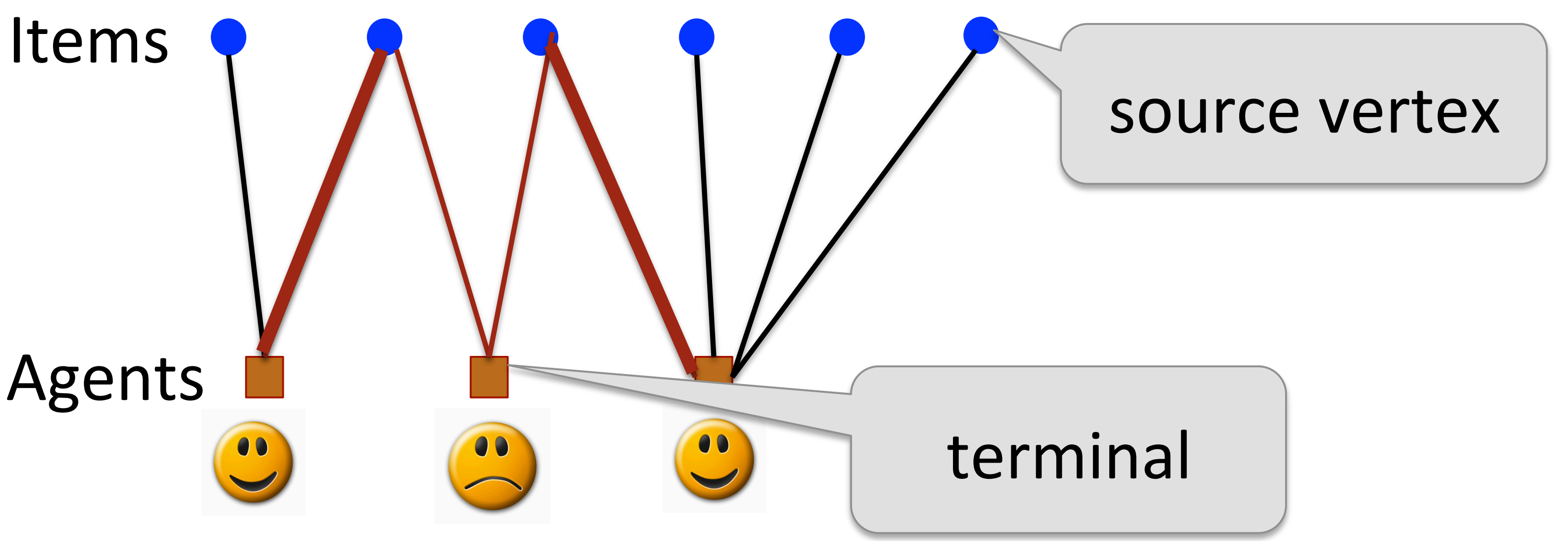


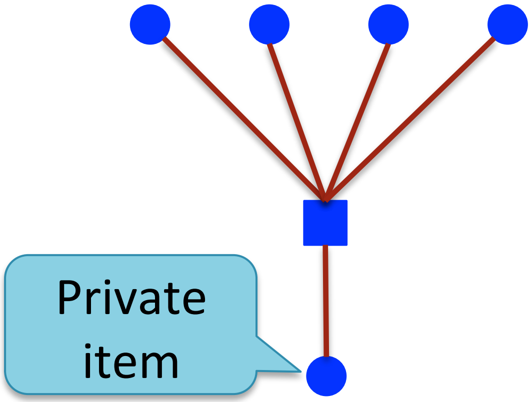
Re-Assignment of Items

- Use flow from source vertices towards terminals.
- An agent releases its private item iff it is satisfied by other items.
- Goal: find flow satisfying the terminals.

The Flow Network

The Flow Network

Heavy agent w. private item



The Flow Network

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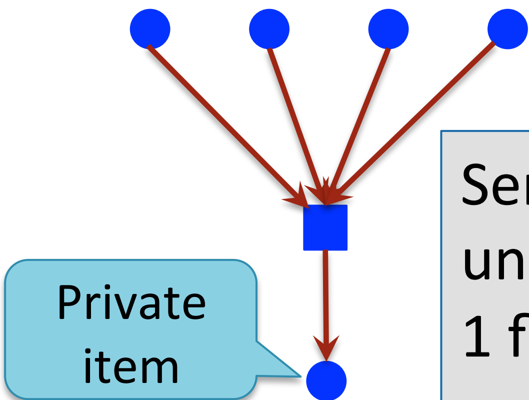
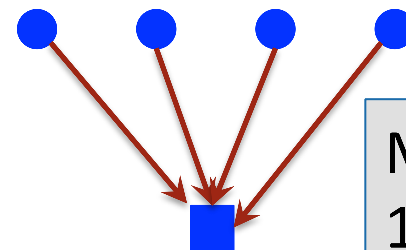


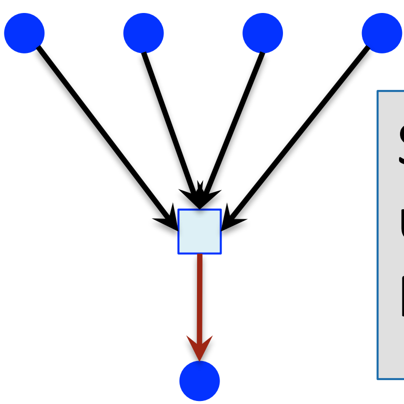
Sends 1 flow unit iff receives 1 flow unit

Terminal



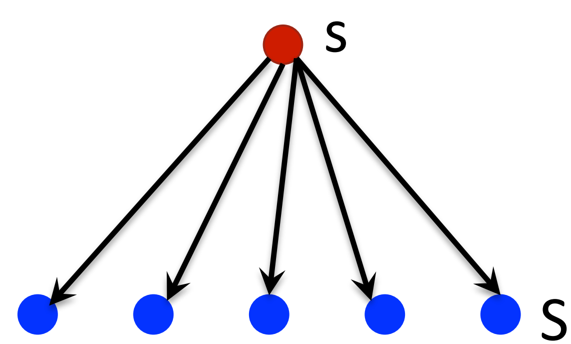
Must receive 1 flow unit

Light Agent

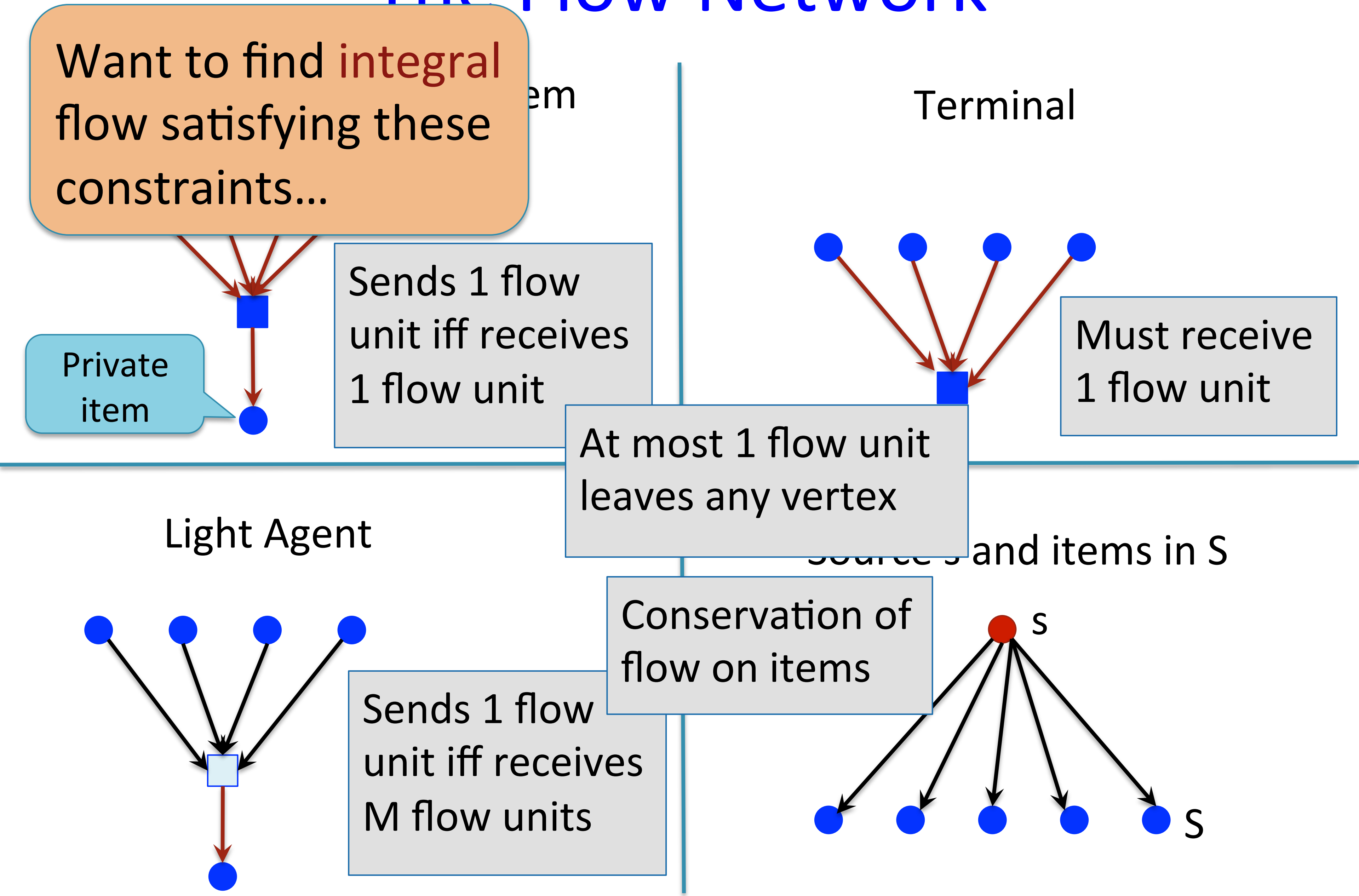


Sends 1 flow unit iff receives M flow units

Source s and items in S



The Flow Network



Interpretation of Flow

Edge e carries 1 flow unit	⟷	Lies in the symmetric difference of OPT and our assignment of private items

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No flow sent through agent A	⟷	A is assigned its private item
Flow from item i to agent A	⟷	Item i is assigned to A

Interpretation of Flow

Edge e carries 1 flow unit

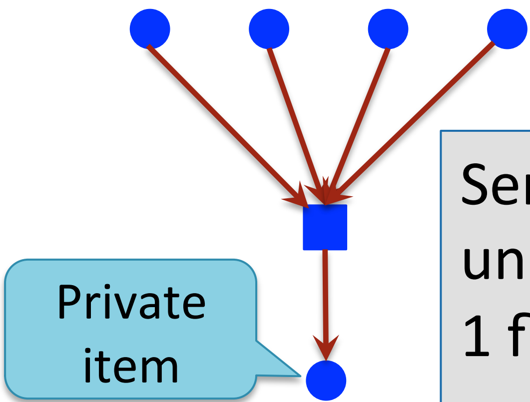
⟷

Lies in the symmetric difference of OPT and our assignment of private items

- If OPT=M then such flow always exists!

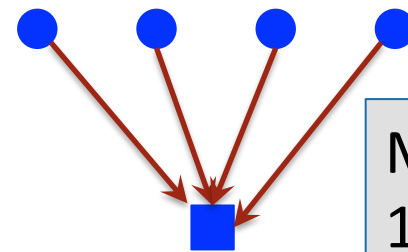
The Flow Network

Heavy agent w. private item



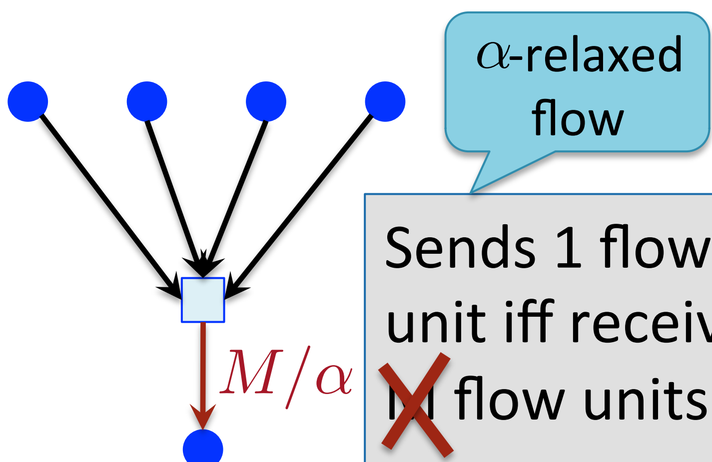
Sends 1 flow unit iff receives 1 flow unit

Terminal



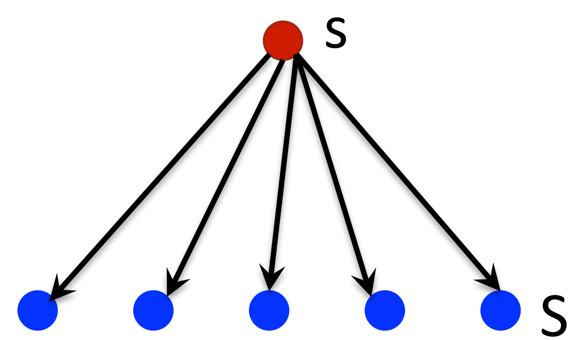
Must receive 1 flow unit

Light Agent



Sends 1 flow unit iff receives ~~M~~ flow units

Source s and items in S



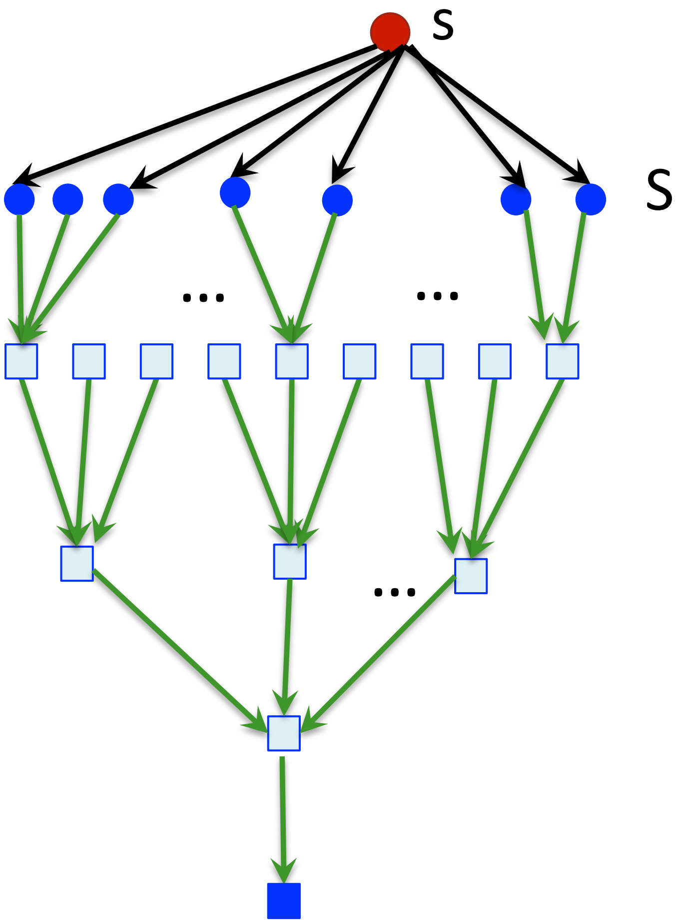
Interpretation of Flow

Edge e carries 1 flow unit ⟷ Lies in the symmetric difference of OPT and our assignment of private items

- If OPT=M then such flow always exists!
- An α-relaxed flow gives an α-approximation!

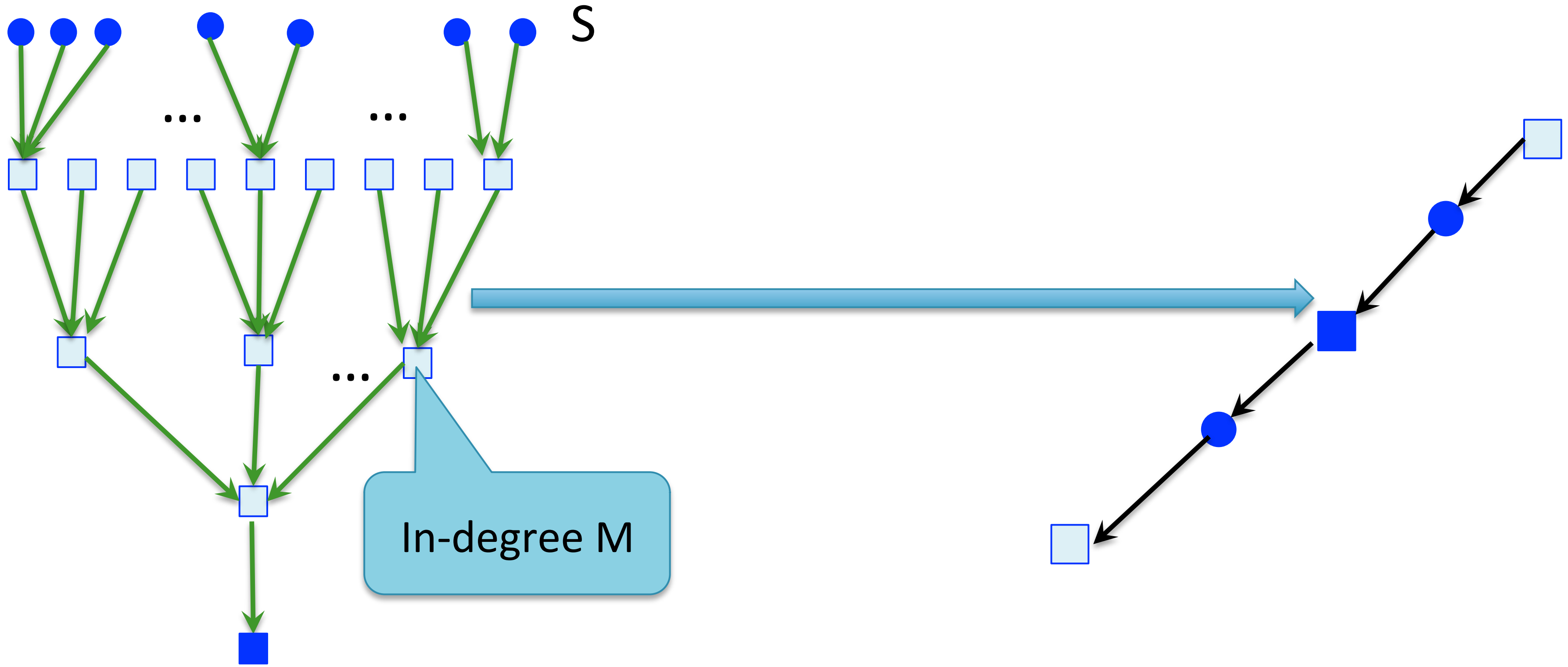
What Does a Feasible Flow Look Like?

A collection of structures like this:

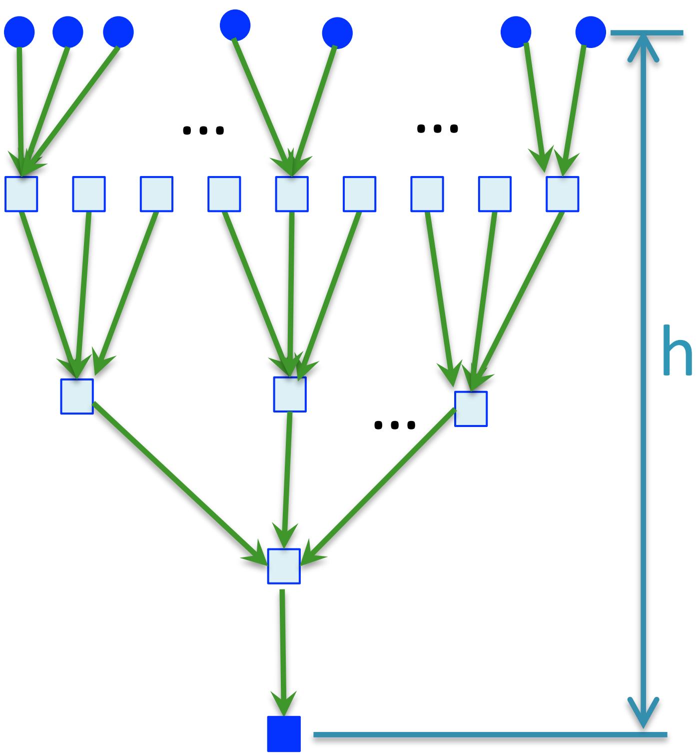


What Does a Feasible Flow Look Like?

A collection of trees like this:



Equivalent Problem Statement



Find a collection of such disjoint trees!

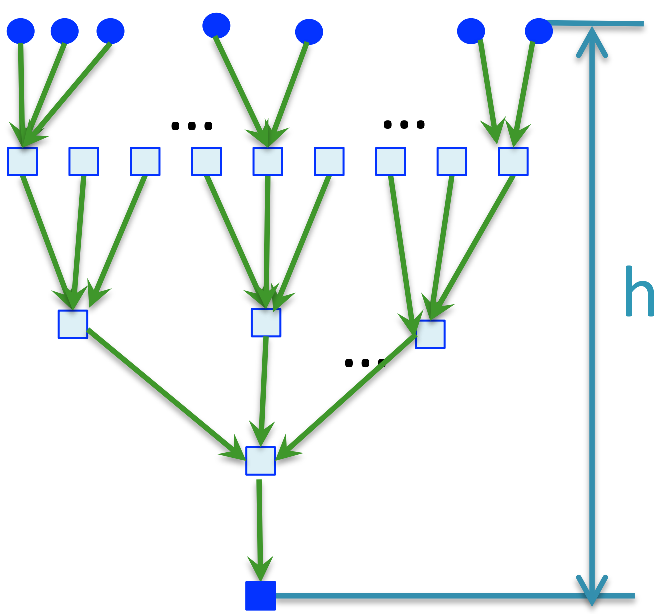
- A tree for each terminal
- Solution value = min degree of a light agent.
- If we only want $\tilde{O}\left(n^{\epsilon}\right)$-approximation, can assume that $h \leq 1/\epsilon$

Rest of the Algorithm

- LP and its rounding
- Use the LP-rounding as a sub-routine to get final solution.

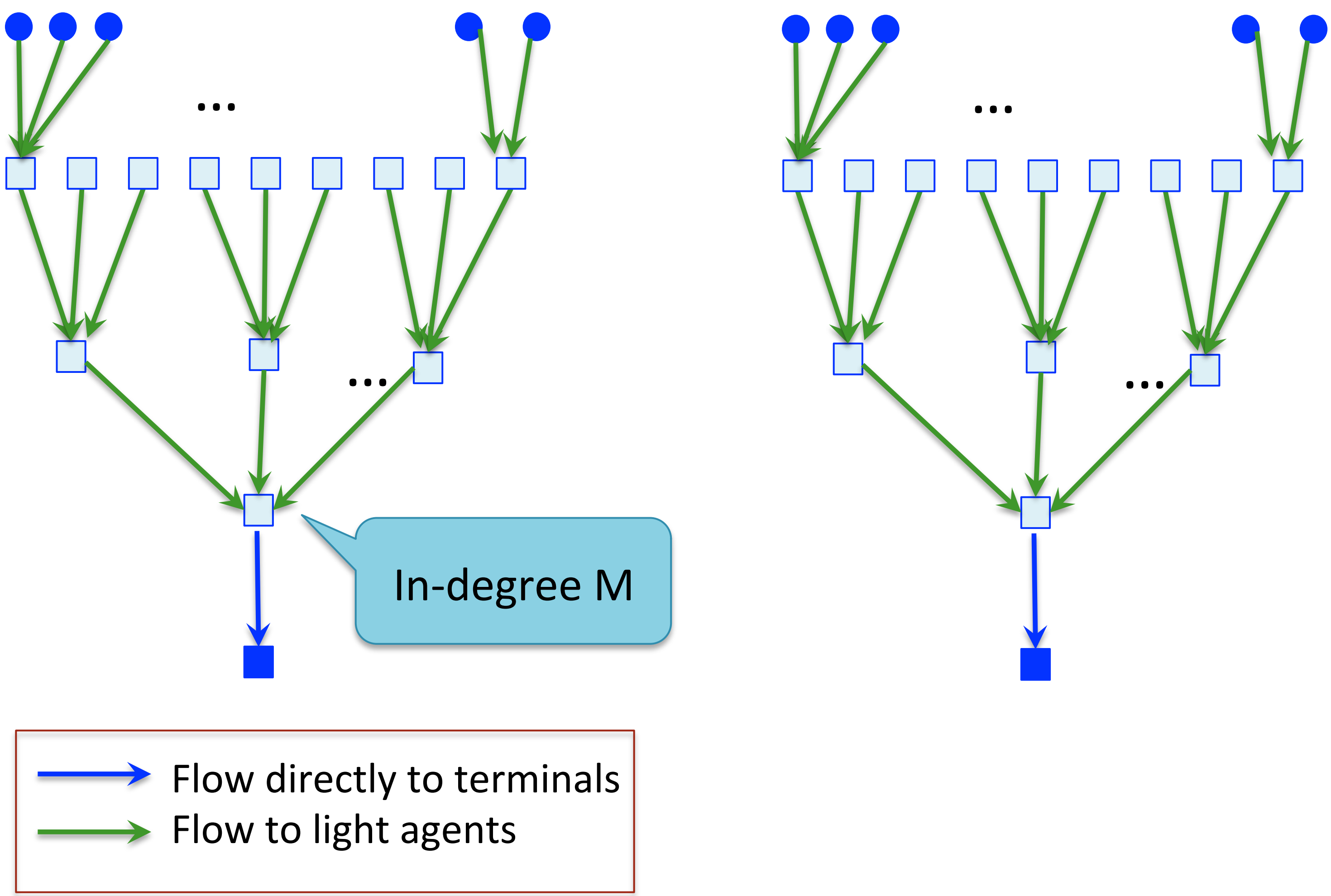
LP-rounding

- Can write LP relaxation of flow constraints and try LP-rounding.
  - Easy to see that such an LP is too weak.

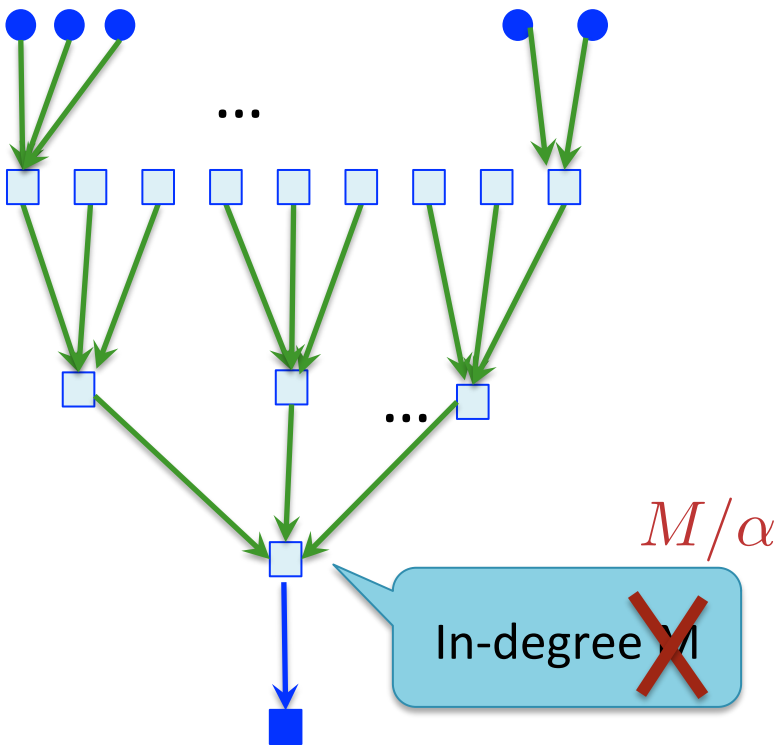


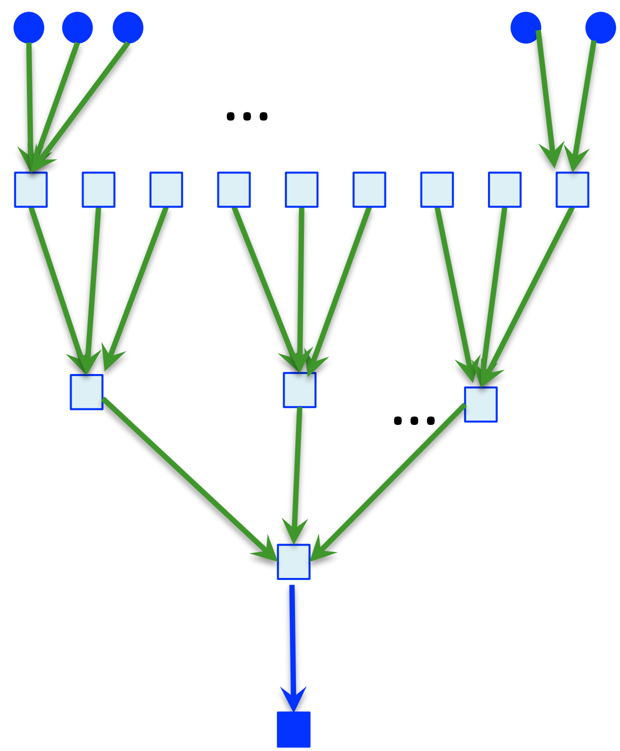
•We write a stronger LP.
•LP-variable for every h-tuple of light agents.
•LP-size: $n^{O(h)}$
•Integrality gap: $\Omega(\sqrt{m})$
•LP-rounding gives poly(log n)-approximate almost-feasible solutions!

Almost Feasible Solutions



Almost Feasible Solutions





Flow directly to terminals
Flow to light agents

An item may appear on one blue and one green path.

Rest of the Algorithm

- LP and its rounding
- Use the LP-rounding as a sub-routine to get final solution.

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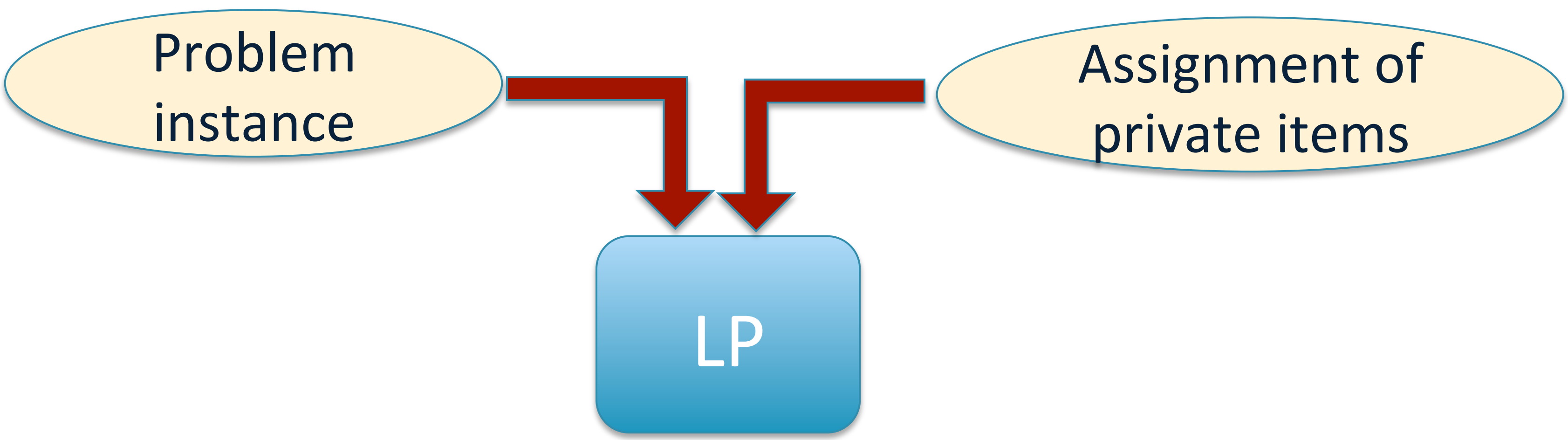
Getting around the Integrality Gap

Integrality gap of the LP is $\Omega(\sqrt{m})$

⇒For some inputs to LP the gap is large



Can we find a better assignment of private items, to make the gap go down?

Lower the Integrality Gap?

- The integrality gap is $\Omega(\sqrt{m})$
- But it is no more than the number of terminals
- If we assign private items so that we have few terminals, the gap will go down!

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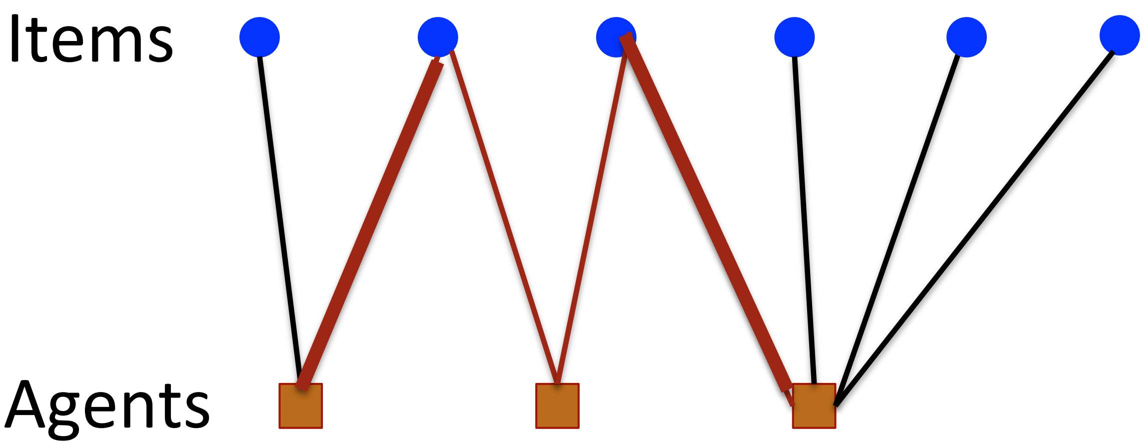

Maximum matching gives smallest possible number of terminals

Lower the Integrality Gap?

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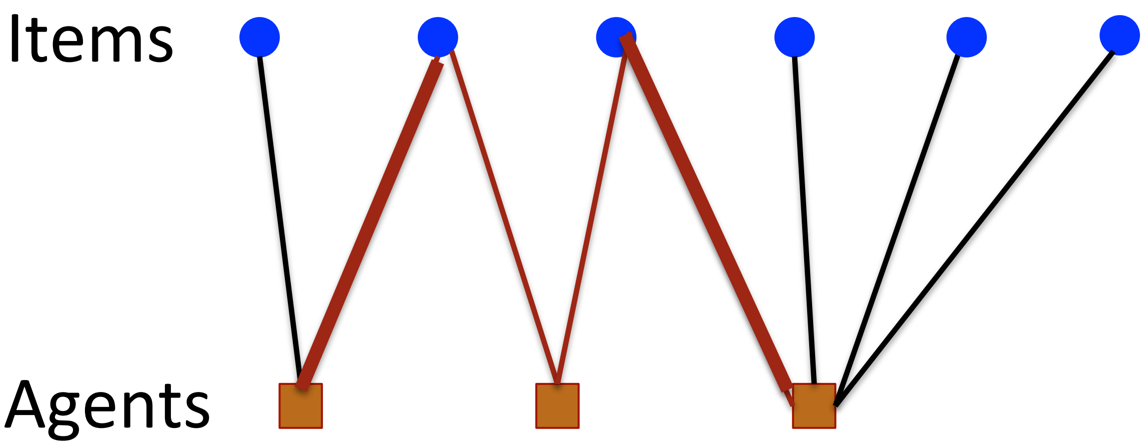


Maximum matching gives smallest possible number of terminals

A Revised Plan

- Compute a good assignment of items to some subset A’ of agents
- Remove agents of A’ from the instance
- Give their private items to other agents!

Number of terminals goes down
$\Rightarrow$ integrality gap improves!

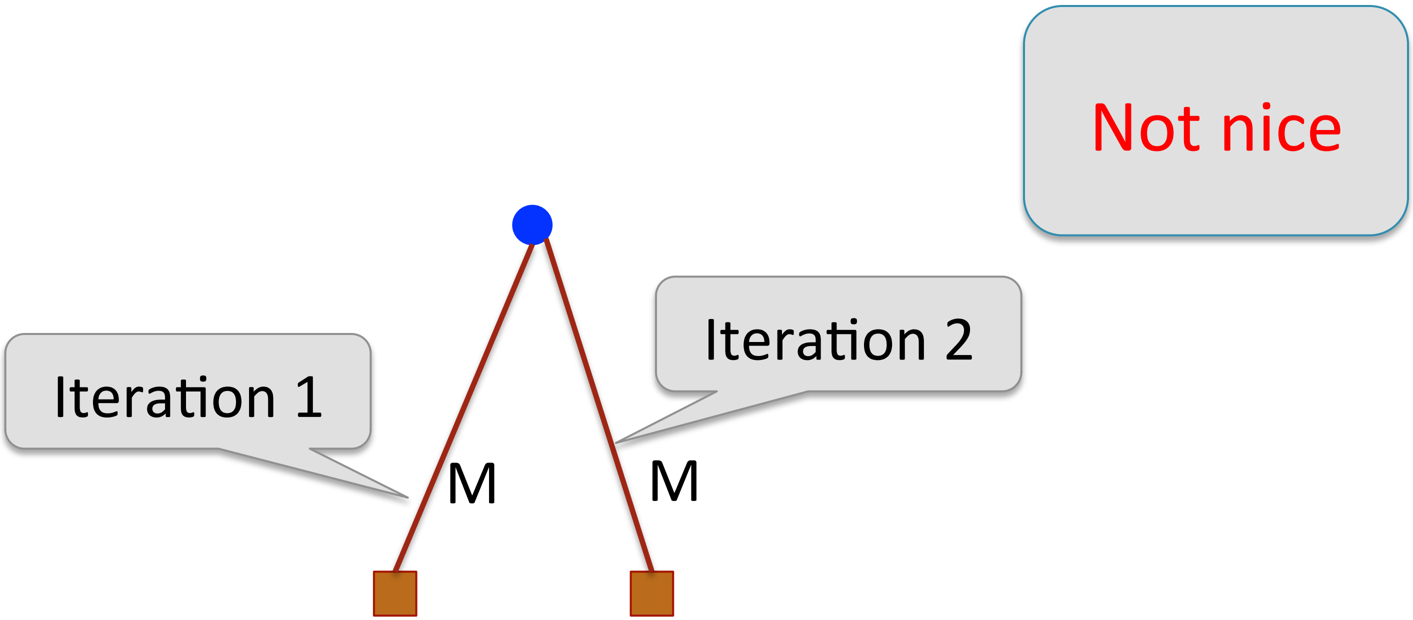
A Revised Plan

- Compute a good assignment of items to some subset A’ of agents
- Remove agents of A’ from the instance
- Give their private items to other agents!

Problem:
Items that are assigned to agents in A’ may be later assigned to other agents.

Nice assignments!

On Nice Partial Assignments



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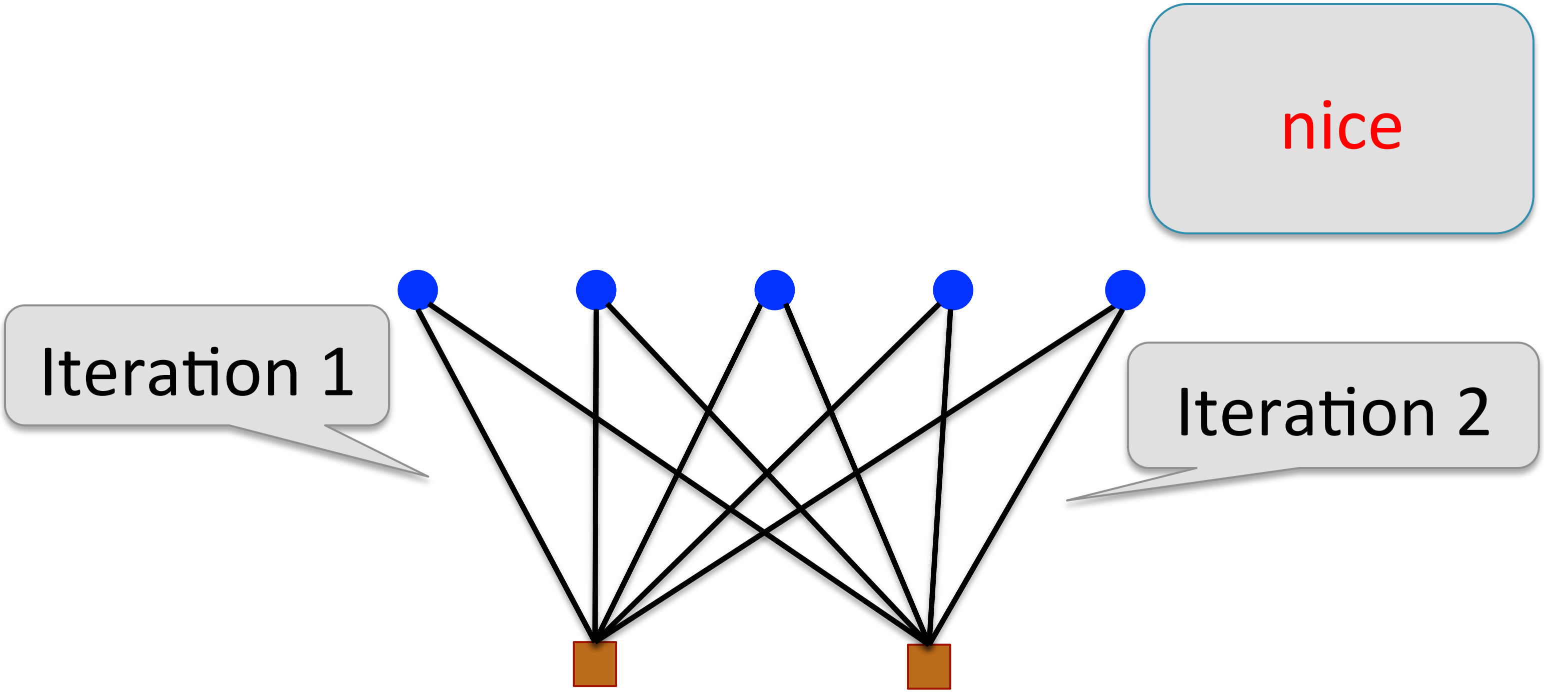
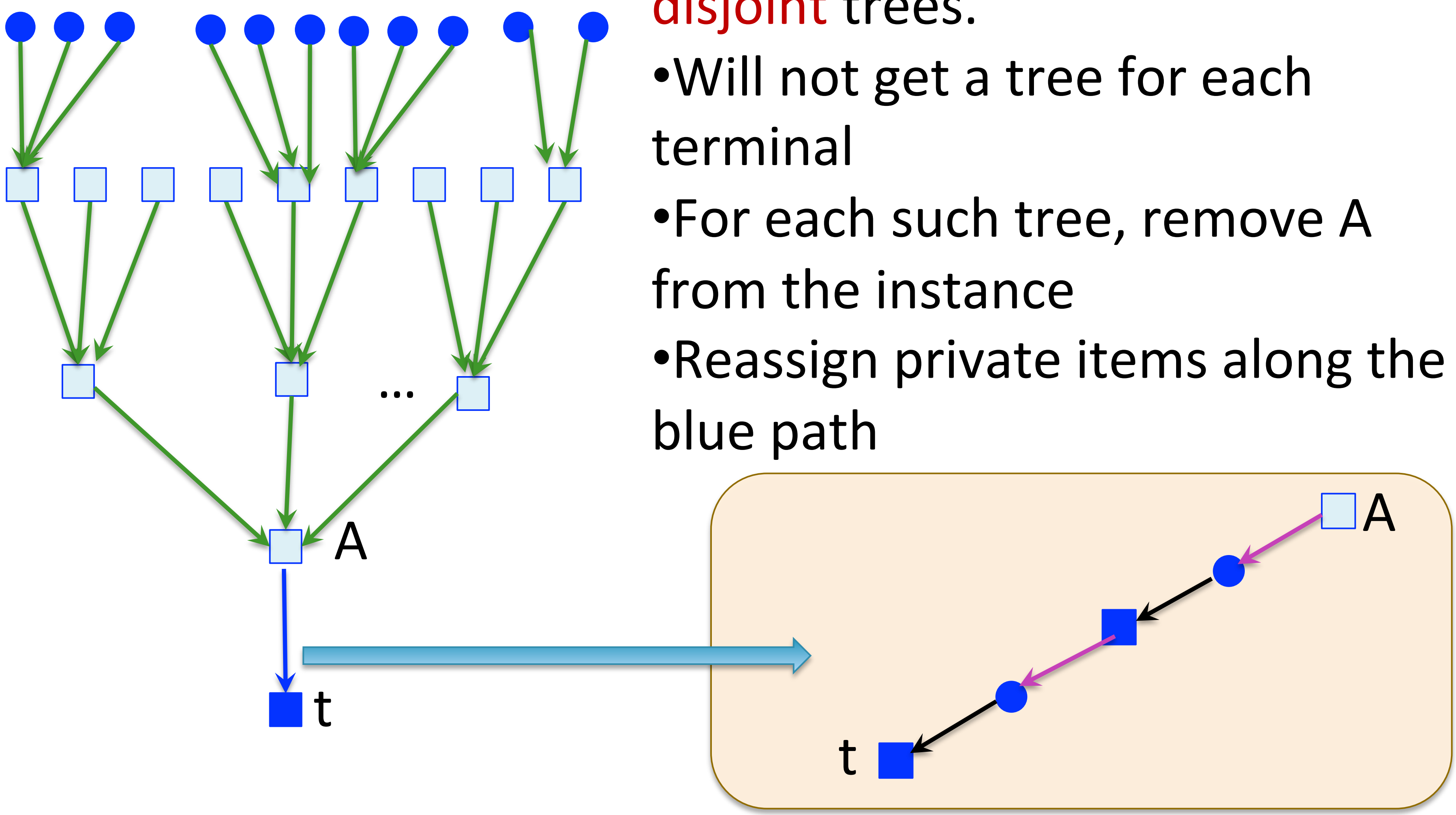


Our Nice Partial Assignments

Find a collection of completely disjoint trees.

- Will not get a tree for each terminal
- For each such tree, remove A from the instance
- Reassign private items along the blue path



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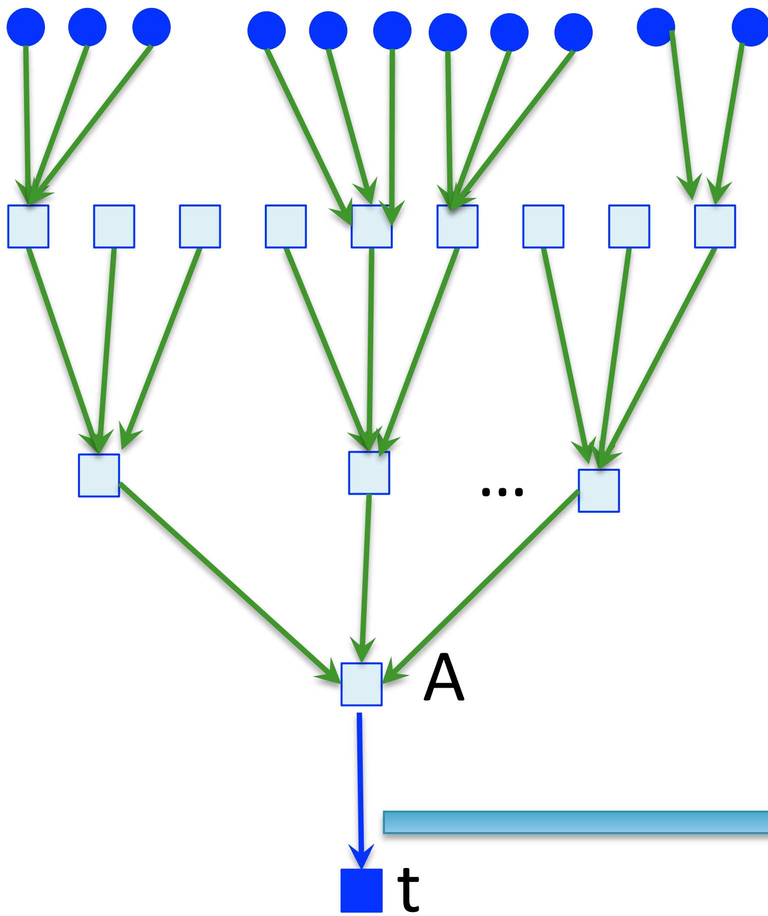

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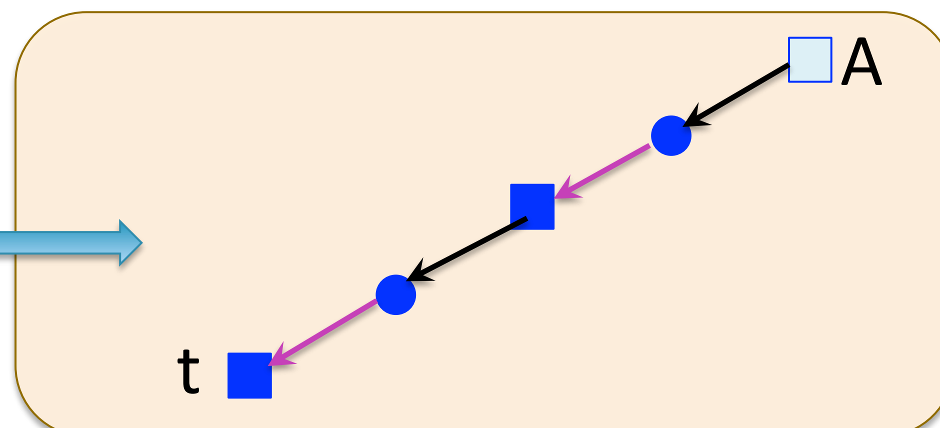

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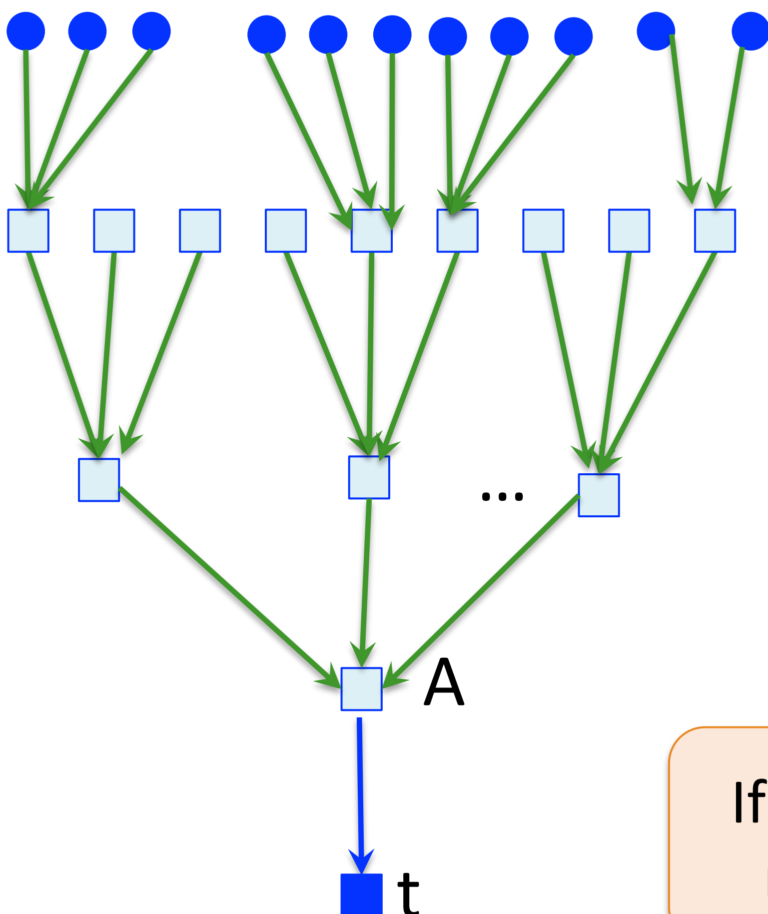


Find a collection of completely disjoint trees.

- Will not get a tree for each terminal
- For each such tree, remove A from the instance
- Reassign private items along the blue path
- t is not a terminal anymore!

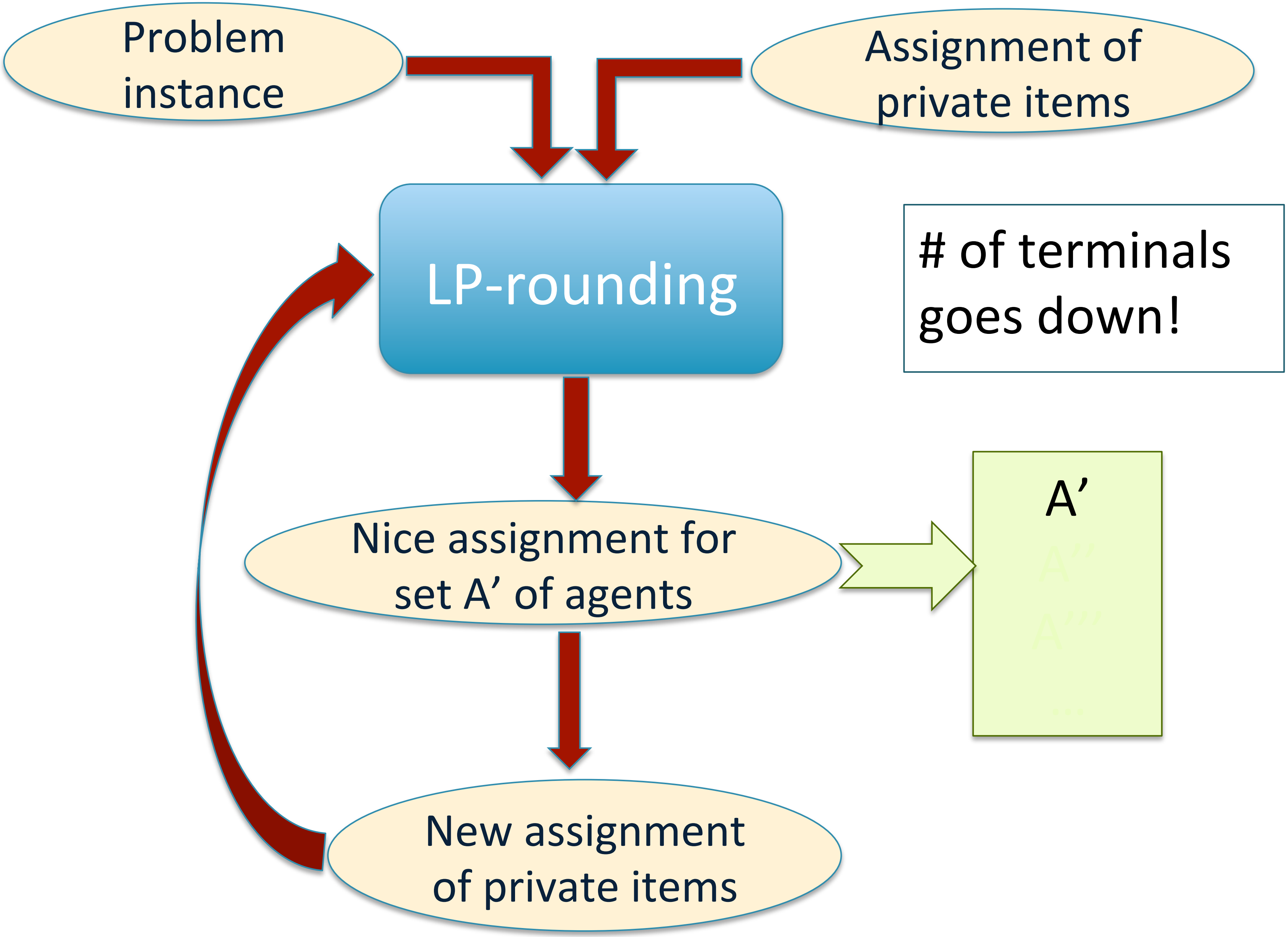
If almost every terminal gets a tree, the number of terminals goes down fast!

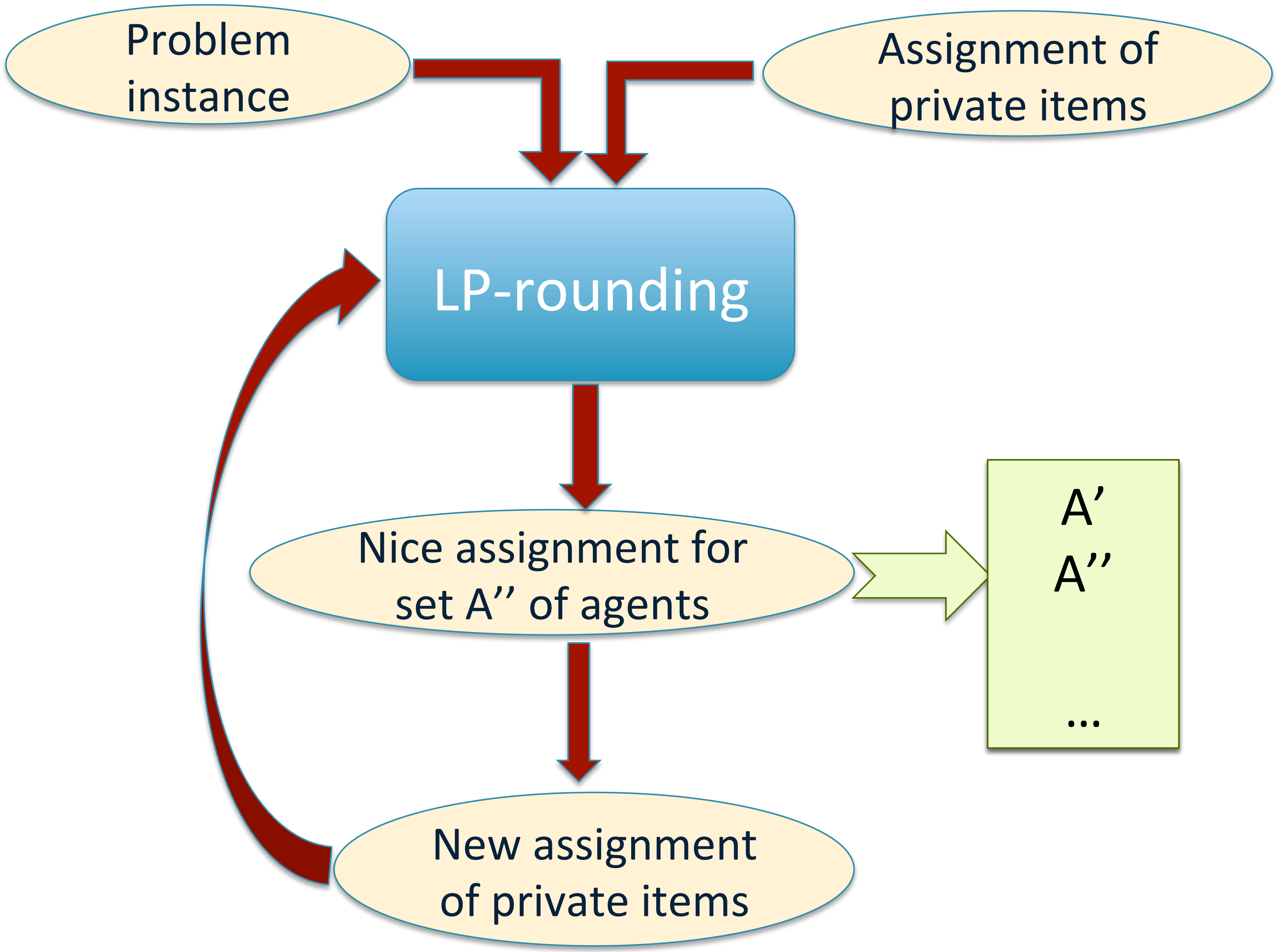
A Revised Plan

- Compute a nice assignment of items to some subset A’ of agents
- Remove agents of A’ from the instance
- Give their private items to other agents

Question:
How do we find this nice assignment?

By LP-rounding!


Problem instance
Assignment of private items
LP-rounding
# of terminals goes down!
Nice assignment for set A’ of agents
A’
New assignment of private items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LP-rounding
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A'
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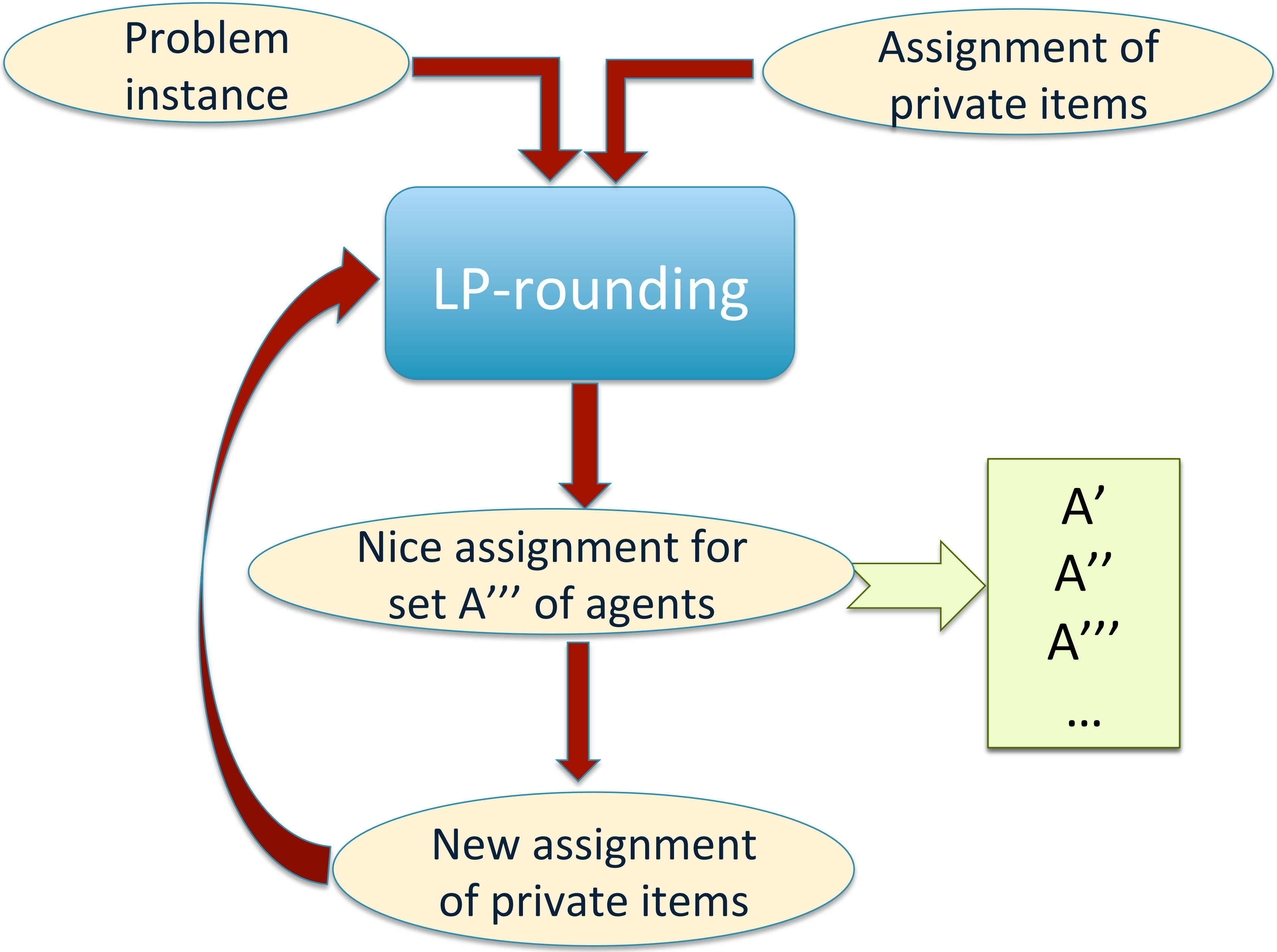
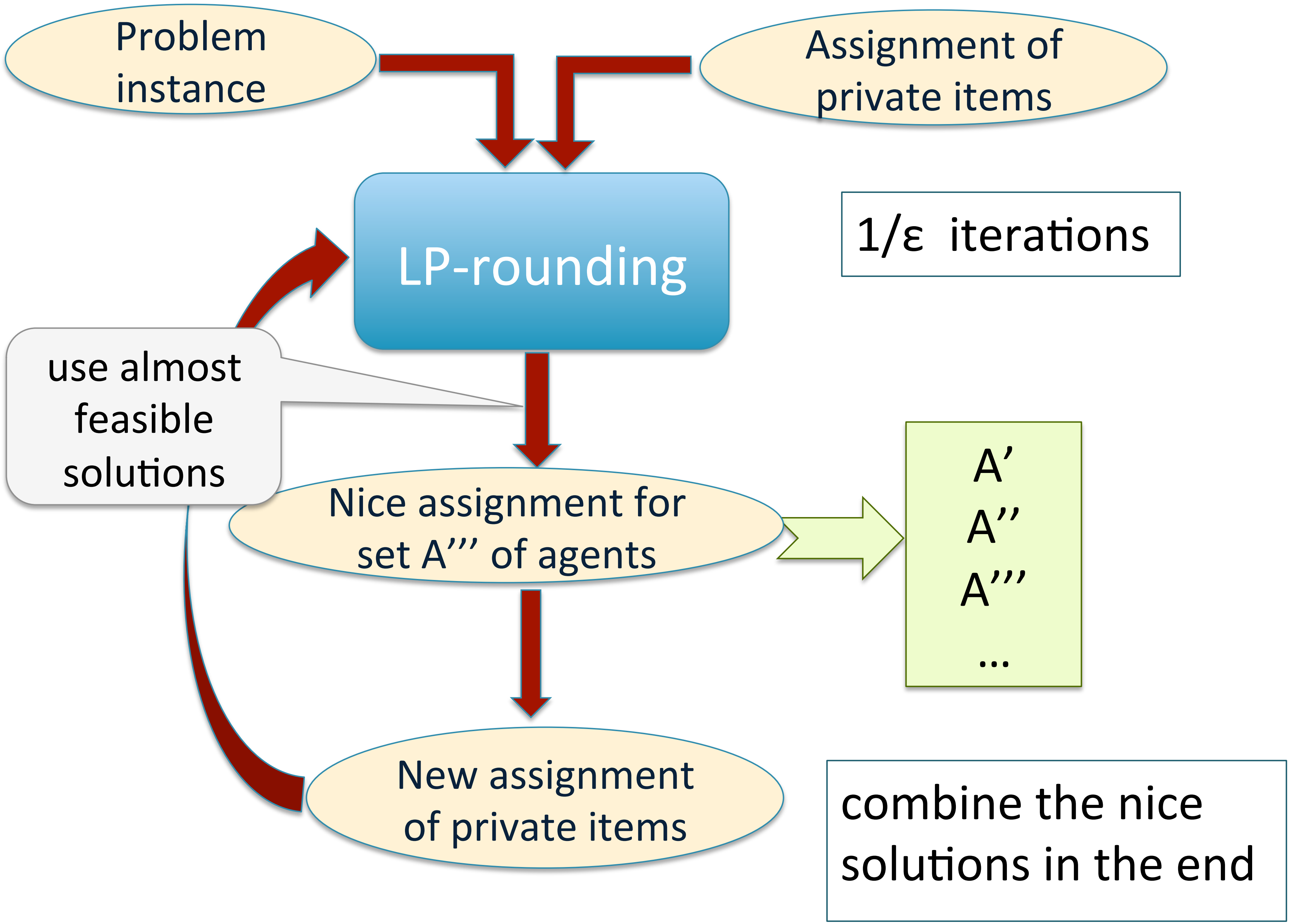


Problem instance
Assignment of private items
LP-rounding
Nice assignment for set A’’’ of agents
A’
A’’
A’’’
…
New assignment of private items


Problem instance
Assignment of private items
LP-rounding
1/ε iterations
use almost feasible solutions
Nice assignment for set A''' of agents
A'
A''
A'''
...
New assignment of private items
combine the nice solutions in the end

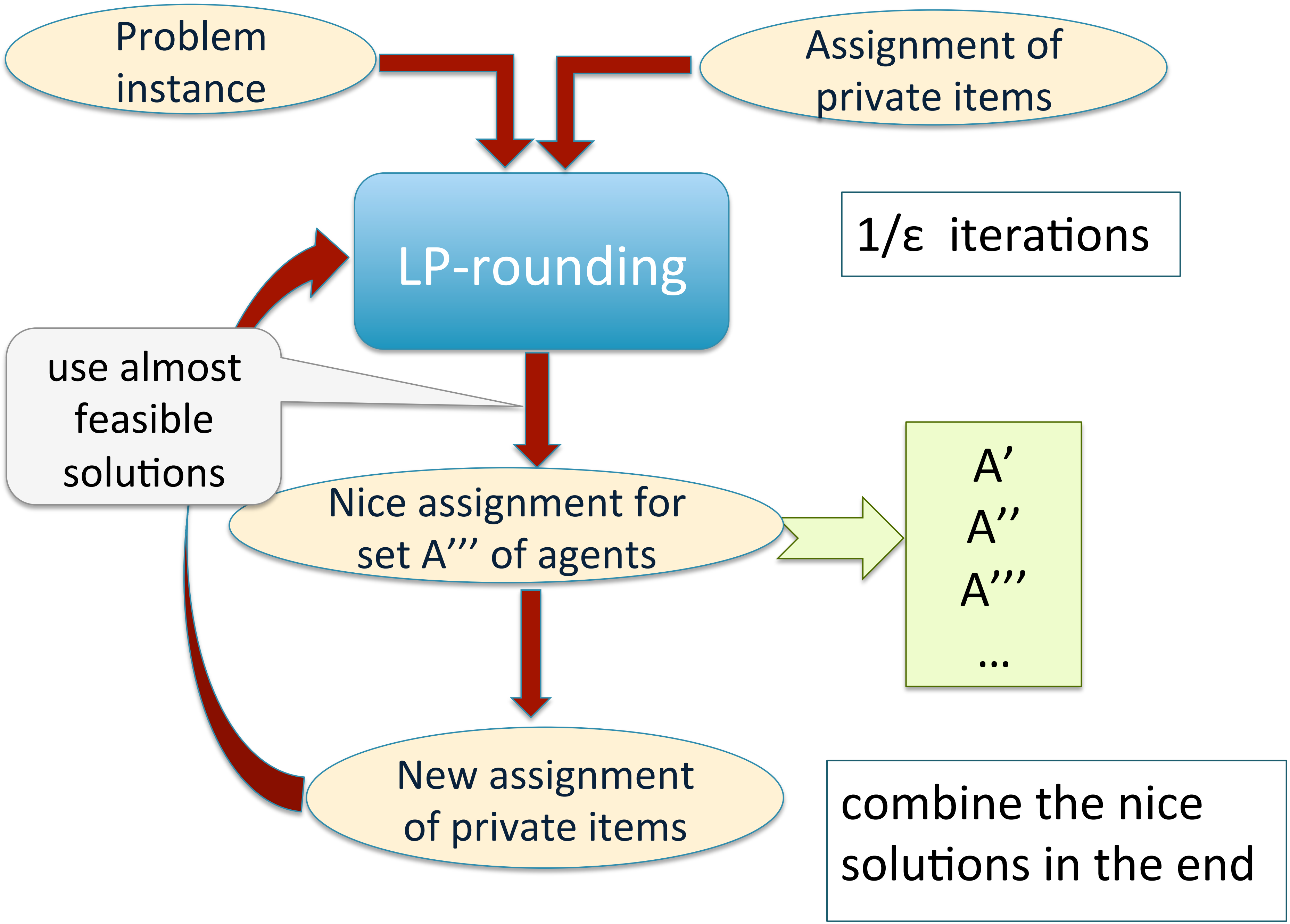
From Almost Feasible Solutions to Nice Assignments

Input: Almost feasible solution

- A tree for every terminal
- Green and blue paths share vertices

Output: Nice partial solution

- A tree for almost every terminal
- The trees are completely disjoint



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Summary

- We have shown $\tilde{O}\left(n^{\epsilon}\right)$ -approximation for Max Min Allocation, in $n^{O(1/\epsilon)}$ running time
  - poly-logarithmic approximation in quasi-polynomial time
- Best current hardness of approximation is 2.
- Can we use similar LP-rounding technique for other problems?

Thank you!