Linear Programming
Lecture II

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COS 523
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Linear Programming II

- **LP duality**
- Strong duality theorem
- Bonus proof of LP duality
- Applications

LP Duality

Primal problem.

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & & 4A & + & 4B & \leq & 160 \\
 & & 35A & + & 20B & \leq & 1190 \\
 & & A & , & B & \geq & 0
\end{array}
$$

Goal. Find a lower bound on optimal value.

Easy. Any feasible solution provides one.

Ex 1. $(A, B) = (34, 0) \Rightarrow z^* \geq 442$
Ex 2. $(A, B) = (0, 32) \Rightarrow z^* \geq 736$
Ex 3. $(A, B) = (7.5, 29.5) \Rightarrow z^* \geq 776$
Ex 4. $(A, B) = (12, 28) \Rightarrow z^* \geq 800$

LP Duality

Primal problem.

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\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \leq & 480 \\
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 & & A & , & B & \geq & 0
\end{array}
$$

Goal. Find an upper bound on optimal value.

Ex 1. Multiply 2nd inequality by 6: $24A + 24B \leq 960$.

$$\Rightarrow \quad z^* = \underbrace{13A + 23B}_{\text{objective function}} \leq 24A + 24B \leq 960.$$

LP Duality

Primal problem.

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
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 & & A & , & B & \ge & 0
\end{array}
$$

Goal. Find an upper bound on optimal value.

Ex 2. Add 2 times 1st inequality to 2nd inequality:

$$\Rightarrow \quad z^* = 13A + 23B \le 14A + 34B \le 1120.$$

LP Duality

Primal problem.

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \le & 480 \\
 & & 4A & + & 4B & \le & 160 \\
 & & 35A & + & 20B & \le & 1190 \\
 & & A & , & B & \ge & 0
\end{array}
$$

Goal. Find an upper bound on optimal value.

Ex 2. Add 1 times 1st inequality to 2 times 2nd inequality:

$$\Rightarrow \quad z^* = 13A + 23B \le 13A + 23B \le 800.$$

Recall lower bound. $(A, B) = (34, 0) \Rightarrow z^* \ge 442$

Combine upper and lower bounds: $z^* = 800$.

LP Duality

Primal problem.

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \le & 480 \\
 & & 4A & + & 4B & \le & 160 \\
 & & 35A & + & 20B & \le & 1190 \\
 & & A & , & B & \ge & 0
\end{array}
$$

Idea. Add nonnegative combination (C, H, M) of the constraints s.t.

$$
\begin{array}{rcl}
13A + 23B & \le & (5C + 4H + 35M)\ A \ + \ (15C + 4H + 20M)\ B \\
 & \le & 480C + 160H + 1190M
\end{array}
$$

Dual problem. Find best such upper bound.

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
 & \text{s. t.} & 5C & + & 4H & + & 35M & \ge & 13 \\
 & & 15C & + & 4H & + & 20M & \ge & 23 \\
 & & C & , & H & , & M & \ge & 0
\end{array}
$$

LP Duality: Economic Interpretation

Brewer: find optimal mix of beer and ale to maximize profits.

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \le & 480 \\
 & & 4A & + & 4B & \le & 160 \\
 & & 35A & + & 20B & \le & 1190 \\
 & & A & , & B & \ge & 0
\end{array}
$$

Entrepreneur: buy individual resources from brewer at min cost.

- C, H, M = unit price for corn, hops, malt.
- Brewer won't agree to sell resources if $5C + 4H + 35M < 13$.

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
 & \text{s. t.} & 5C & + & 4H & + & 35M & \ge & 13 \\
 & & 15C & + & 4H & + & 20M & \ge & 23 \\
 & & C & , & H & , & M & \ge & 0
\end{array}
$$

LP Duals

Canonical form.

$$
\text{(P)} \quad \max\ c^T x \quad \text{s. t.} \quad Ax \le b, \quad x \ge 0
$$

$$
\text{(D)} \quad \min\ y^T b \quad \text{s. t.} \quad A^T y \ge c, \quad y \ge 0
$$

Double Dual

Canonical form.

$$
\text{(P)} \quad \max\ c^T x \quad \text{s. t.} \quad Ax \le b, \quad x \ge 0
$$

$$
\text{(D)} \quad \min\ y^T b \quad \text{s. t.} \quad A^T y \ge c, \quad y \ge 0
$$

Property. The dual of the dual is the primal.
Pf. Rewrite (D) as a maximization problem in canonical form; take dual.

$$
\text{(D')} \quad \max\ -y^T b \quad \text{s. t.} \quad -A^T y \le -c, \quad y \ge 0
$$

$$
\text{(DD)} \quad \min\ -c^T z \quad \text{s. t.} \quad -(A^T)^T z \ge -b, \quad z \ge 0
$$

Taking Duals

LP dual recipe.

Primal (P)	maximize	minimize	Dual (D)
constraints	$a x = b_i$ $a x \le b$ $a x \ge b_i$	y_i unrestricted $y_i \ge 0$ $y_i \le 0$	variables
variables	$x_j \ge 0$ $x_j \le 0$ unrestricted	$a^T y \ge c_j$ $a^T y \le c_j$ $a^T y = c_j$	constraints

Pf. Rewrite LP in standard form and take dual.

Linear Programming II

- LP duality
- **Strong duality theorem**
- Bonus proof of LP duality
- Applications

LP Strong Duality

Theorem. [Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947]
For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty, then max = min.

$$(P) \quad \max c^T x \quad \text{s. t.} \quad Ax \le b, \quad x \ge 0$$

$$(D) \quad \min y^T b \quad \text{s. t.} \quad A^T y \ge c, \quad y \ge 0$$

Generalizes:
- Dilworth's theorem.
- König-Egervary theorem.
- Max-flow min-cut theorem.
- von Neumann's minimax theorem.
- ...

Pf. [ahead]

LP Weak Duality

Theorem. For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty, then max ≤ min.

$$(P) \quad \max c^T x \quad \text{s. t.} \quad Ax \le b, \quad x \ge 0$$

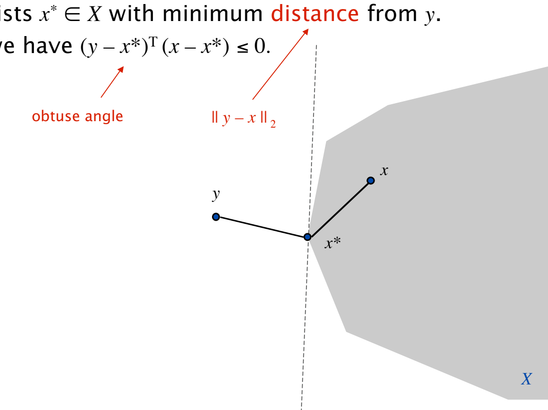
$$(D) \quad \min y^T b \quad \text{s. t.} \quad A^T y \ge c, \quad y \ge 0$$

Pf. Suppose $x \in \Re^m$ is feasible for (P) and $y \in \Re^n$ is feasible for (D).
- $y \ge 0,\ Ax \le b \quad \Rightarrow \quad y^T A x \le y^T b$
- $x \ge 0,\ A^T y \ge c \quad \Rightarrow \quad y^T A x \ge c^T x$
- Combine: $c^T x \le y^T A x \le y^T b$. ▪

Projection Lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X. Then min $\{ f(x) : x \in X \}$ exists.

Projection lemma. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y. Moreover, for all $x \in X$ we have $(y - x^*)^T (x - x^*) \le 0$.



Projection Lemma

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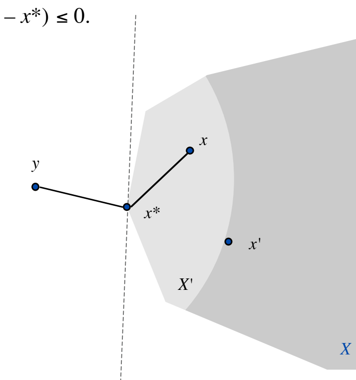
Pf.
- Define $f(x) = \| y - x \|$.
- Want to apply Weierstrass, but X not necessarily bounded.
- $X \ne \phi \quad \Rightarrow$ there exists $x' \in X$.
- Define $X' = \{ x \in X : \| y - x \| \le \| y - x' \| \}$ so that X' is closed, bounded, and min $\{ f(x) : x \in X \}$ = min $\{ f(x) : x \in X' \}$.
- By Weierstrass, min exists.



Projection Lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X. Then min $\{ f(x) : x \in X \}$ exists.

Projection lemma. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y.
Moreover, for all $x \in X$ we have $(y - x^*)^T (x - x^*) \le 0$.

Pf.

- x^* min distance $\Rightarrow$ $\|y - x^*\|^2 \le \|y - x\|^2$ for all $x \in X$.
- By convexity: if $x \in X$, then $x^* + \varepsilon\,(x - x^*) \in X$ for all $0 < \varepsilon < 1$.
- $\|y - x^*\|^2 \le \|y - x^* - \varepsilon(x - x^*)\|^2$
  $= \|y - x^*\|^2 + \varepsilon^2 \|(x - x^*)\|^2 - 2\,\varepsilon\,(y - x^*)^T (x - x^*)$
- Thus, $(y - x^*)^T (x - x^*) \le \frac{1}{2}\,\varepsilon\,\|(x - x^*)\|^2$.
- Letting $\varepsilon \to 0^+$, we obtain the desired result. ▪

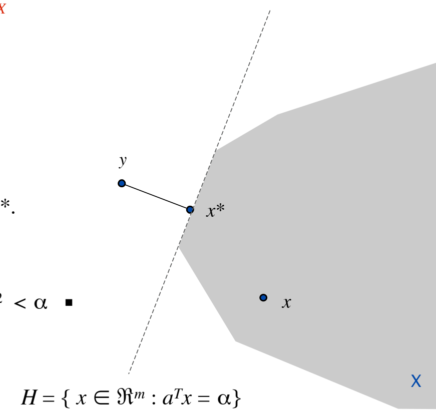
Separating Hyperplane Theorem

Theorem. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists a hyperplane $H = \{ x \in \Re^m : a^T x = \alpha \}$ where $a \in \Re^m$, $\alpha \in \Re$ that separates y from X.

$a^T x \ge \alpha$ for all $x \in X$
$a^T y < \alpha$

Pf.

- Let x^* be closest point in X to y.
- By projection lemma, $(y - x^*)^T (x - x^*) \le 0$ for all $x \in X$
- Choose $a = x^* - y \ne 0$ and $\alpha = a^T x^*$.
- If $x \in X$, then $a^T(x - x^*) \ge 0$; thus $\Rightarrow a^T x \ge a^T x^* = \alpha$.
- Also, $a^T y = a^T (x^* - a) = \alpha - \|a\|^2 < \alpha$ ▪



Farkas' Lemma

Theorem. For $A \in \Re^{m \times n}$, $b \in \Re^m$ exactly one of the following two systems holds:

(I) $\exists\, x \in \Re^n$ s. t. $Ax = b$, $x \ge 0$

(II) $\exists\, y \in \Re^m$ s. t. $A^T y \ge 0$, $y^T b < 0$

Pf. [not both] Suppose x satisfies (I) and y satisfies (II).
Then $0 > y^T b = y^T A x \ge 0$, a contradiction.

Pf. [at least one] Suppose (I) infeasible. We will show (II) feasible.

- Consider $S = \{ Ax : x \ge 0 \}$ so that S closed, convex, $b \notin S$.
- Let $y \in \Re^m, \alpha \in \Re$ be a hyperplane that separates b from S: $y^T b < \alpha$, $y^T s \ge \alpha$ for all $s \in S$.
- $0 \in S \Rightarrow \alpha \le 0 \Rightarrow y^T b < 0$
- $y^T A x \ge \alpha$ for all $x \ge 0 \Rightarrow y^T A \ge 0$ since x can be arbitrarily large. ▪

Another Theorem of the Alternative

Corollary. For $A \in \Re^{m \times n}$, $b \in \Re^m$ exactly one of the following two systems holds:

(I) $\exists\, x \in \Re^n$ s. t. $Ax \le b$, $x \ge 0$

(II) $\exists\, y \in \Re^m$ s. t. $A^T y \ge 0$, $y^T b < 0$, $y \ge 0$

Pf. Apply Farkas' lemma to:

(I') $\exists\, x \in \Re^n, s \in \Re^m$ s. t. $Ax + Is = b$, $x, s \ge 0$

(II') $\exists\, y \in \Re^m$ s. t. $A^T y \ge 0$, $Iy \ge 0$, $y^T b < 0$

LP Strong Duality

Theorem. [strong duality] For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty then max = min.

$$\text{(P)} \quad \max\ c^T x \quad \text{s.t.} \quad Ax \le b,\ x \ge 0$$

$$\text{(D)} \quad \min\ y^T b \quad \text{s.t.} \quad A^T y \ge c,\ y \ge 0$$

Pf. [max ≤ min] Weak LP duality.
Pf. [min ≤ max] Suppose max < α. We show min < α.

$$\text{(I)} \quad \exists\, x \in \Re^n \quad \text{s.t.} \quad Ax \le b,\ -c^T x \le -\alpha,\ x \ge 0$$

$$\text{(II)} \quad \exists\, y \in \Re^m,\ z \in \Re \quad \text{s.t.} \quad A^T y - c\, z \ge 0,\ y^T b - \alpha\, z < 0,\ y, z \ge 0$$

- By definition of α, (I) infeasible $\Rightarrow$ (II) feasible by Farkas' Corollary.

LP Strong Duality

$$\text{(II)} \quad \exists\, y \in \Re^m,\ z \in \Re \quad \text{s.t.} \quad A^T y - cz \ge 0,\ y^T b - \alpha z < 0,\ y, z \ge 0$$

Let y, z be a solution to (II).

Case 1. $[z = 0]$
- Then, $\{ y \in \Re^m : A^T y \ge 0,\ y^T b < 0,\ y \ge 0 \}$ is feasible.
- Farkas Corollary $\Rightarrow$ $\{ x \in \Re^n : Ax \le b, x \ge 0 \}$ is infeasible.
- Contradiction since by assumption (P) is nonempty.

Case 2. $[z > 0]$
- Scale y, z so that y satisfies (II) and $z = 1$.
- Resulting y feasible to (D) and $y^T b < \alpha$. ▪

Linear Programming II

- ➢ LP duality
- ➢ Strong duality theorem
- ➢ **Bonus proof of LP duality**
- ➢ Applications

Simplex Algorithm: Dual Solution

Observation. Final simplex tableaux reveals dual solution!

max Z subject to

$$\begin{array}{rrrrrrrrrrrrrr}
 & & & - & 1\, S_C & - & 2\, S_H & & & - & Z & = & -800 \\
 & & B & + & \frac{1}{10}\, S_C & + & \frac{1}{8}\, S_H & & & & & = & 28 \\
A & & & - & \frac{1}{10}\, S_C & + & \frac{3}{8}\, S_H & & & & & = & 12 \\
 & & & - & \frac{25}{6}\, S_C & - & \frac{85}{8}\, S_H & + & S_M & & & = & 110 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \ge & 0
\end{array}$$

$Z = 800$
$A^* = 12,\ B^* = 28$
$C^* = 1,\ H^* = 2,\ M^* = 0$

Review: Simplex Tableaux

subtract $c_B^T A_B^{-1}$ times constraints

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

initial tableaux

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N)\, x_N & = & Z - c_B^T A_B^{-1} b \\ I\, x_B & + & A_B^{-1} A_N\, x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

tableaux corresponding to basis B

multiply by A_B^{-1}

Primal solution. $x_B = A_B^{-1} b \geq 0,\ x_N = 0$

Optimal basis. $c_N^T - c_B^T A_B^{-1} A_N \leq 0$

Simplex Tableaux: Dual Solution

subtract $c_B^T A_B^{-1}$ times constraints

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

initial tableaux

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N)\, x_N & = & Z - c_B^T A_B^{-1} b \\ I\, x_B & + & A_B^{-1} A_N\, x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

tableaux corresponding to basis B

multiply by A_B^{-1}

Primal solution. $x_B = A_B^{-1} b \geq 0,\ x_N = 0$

Optimal basis. $c_N^T - c_B^T A_B^{-1} A_N \leq 0$

Dual solution. $y^T = c^T_B A_B^{-1}$

$$\begin{aligned} y^T b &= c_B^T A_B^{-1} b \\ &= c_B^T x_B + c_B^T x_B \\ &= c^T x \end{aligned}$$

max = min

$$\begin{aligned} y^T A &= \begin{bmatrix} y^T A_B & y^T A_N \end{bmatrix} \\ &= \begin{bmatrix} c_B^T A_B^{-1} A_B & c_B^T A_B^{-1} A_N \end{bmatrix} \\ &= \begin{bmatrix} c_B^T & c_B^T A_B^{-1} A_N \end{bmatrix} \\ &\geq \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} \\ &= c^T \end{aligned}$$

dual feasible

Simplex Algorithm: LP Duality

subtract $c_B^T A_B^{-1}$ times constraints

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

initial tableaux

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N)\, x_N & = & Z - c_B^T A_B^{-1} b \\ I\, x_B & + & A_B^{-1} A_N\, x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

tableaux corresponding to basis B

multiply by A_B^{-1}

Primal solution. $x_B = A_B^{-1} b \geq 0,\ x_N = 0$

Optimal basis. $c_N^T - c_B^T A_B^{-1} A_N \leq 0$

Dual solution. $y^T = c^T_B A_B^{-1}$

Simplex algorithm yields constructive proof of LP duality.

Linear Programming II

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- Alternate proof of LP duality
- Applications

LP Duality: Economic Interpretation

Brewer: find optimal mix of beer and ale to maximize profits.

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& & A & , & B & \ge & 0
\end{array}
$$

$A^* = 12$
$B^* = 28$
$OPT = 800$

Entrepreneur: buy individual resources from brewer at min cost.

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
& \text{s. t.} & 5C & + & 4H & + & 35M & \ge & 13 \\
& & 15C & + & 4H & + & 20M & \ge & 23 \\
& & C & , & H & , & M & \ge & 0
\end{array}
$$

$C^* = 1$
$H^* = 2$
$M^* = 0$
$OPT = 800$

LP duality. Market clears.

LP Duality: Sensitivity Analysis

Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?

A. corn \$1, hops \$2, malt \$0.

Q. Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?

A. At least 2 (\$1) + 5 (\$2) + 24 (\$0) = \$12 / barrel.

LP is in NP ∩ co-NP

LP. For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, $\alpha \in \Re$, does there exist $x \in \Re^n$ such that: $Ax = b$, $x \ge 0$, $c^T x \ge \alpha$?

Theorem. LP is in NP ∩ co-NP.

Pf.

- Already showed LP is in NP.
- If LP is infeasible, then apply Farkas' Lemma to get certificate of infeasibility:

$$
\begin{array}{llrcl}
\text{(II)} & \exists\, y \in \Re^m,\ z \in \Re & & & \\
& \text{s. t.} & A^T y & \ge & 0 \\
& & y^T b - \alpha z & < & 0 \\
& & z & \ge & 0
\end{array}
$$

or equivalently, $y^T b - \alpha z = -1$