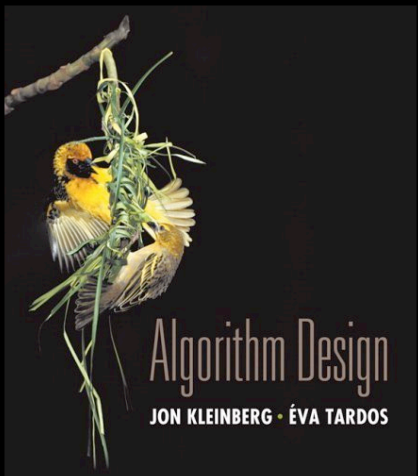



# Chapter 9

## PSPACE: A Class of Problems Beyond NP



## Geography Game

Geography. Alice names capital city c of country she is in. Bob names a capital city c' that starts with the letter on which c ends. Alice and Bob repeat this game until one player is unable to continue. Does Alice have a forced win?

Ex. Budapest → Tokyo → Ottawa → Ankara → Amsterdam → Moscow → Washington → Nairobi → …

Geography on graphs. Given a directed graph $G = (V, E)$ and a start node s, two players alternate turns by following, if possible, an edge out of the current node to an unvisited node. Can first player guarantee to make the last legal move?

Remark. Some problems (especially involving 2-player games and AI) defy classification according to P, EXPTIME, NP, and NP-complete.



# 9.1 PSPACE

## PSPACE

P. Decision problems solvable in polynomial time.

PSPACE. Decision problems solvable in polynomial space.

Observation. P $\subseteq$ PSPACE.

poly-time algorithm can consume only polynomial space

## PSPACE

Binary counter. Count from 0 to $2^n - 1$ in binary.
Algorithm. Use n bit odometer.

Claim. 3-SAT is in PSPACE.
Pf.
- Enumerate all $2^n$ possible truth assignments using counter.
- Check each assignment to see if it satisfies all clauses. ▪

Theorem. NP $\subseteq$ PSPACE.
Pf. Consider arbitrary problem Y in NP.
- Since $Y \leq_P$ 3-SAT, there exists algorithm that solves Y in poly-time plus polynomial number of calls to 3-SAT black box.
- Can implement black box in poly-space. ▪



# 9.3 Quantified Satisfiability

## Quantified Satisfiability

QSAT. Let $\Phi(x_1, \ldots, x_n)$ be a Boolean CNF formula. Is the following propositional formula true?

$$\exists x_1 \, \forall x_2 \, \exists x_3 \, \forall x_4 \ldots \forall x_{n-1} \, \exists x_n \; \Phi(x_1, \ldots, x_n)$$

assume n is odd

Intuition. Amy picks truth value for $x_1$, then Bob for $x_2$, then Amy for $x_3$, and so on. Can Amy satisfy $\Phi$ no matter what Bob does?

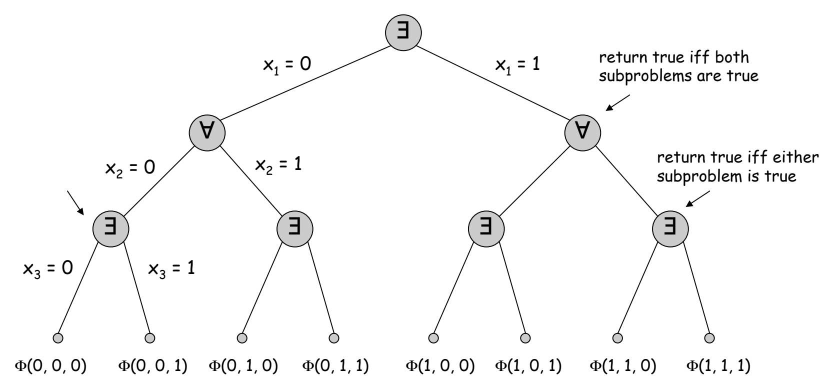
Ex. $(x_1 \vee x_2) \wedge (x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3)$
Yes. Amy sets $x_1$ true; Bob sets $x_2$; Amy sets $x_3$ to be same as $x_2$.

Ex. $(x_1 \vee x_2) \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3)$
No. If Amy sets $x_1$ false; Bob sets $x_2$ false; Amy loses;
if Amy sets $x_1$ true; Bob sets $x_2$ true; Amy loses.



## QSAT is in PSPACE

Theorem. QSAT $\in$ PSPACE.
Pf. Recursively try all possibilities.
- Only need one bit of information from each subproblem.
- Amount of space is proportional to depth of function call stack.

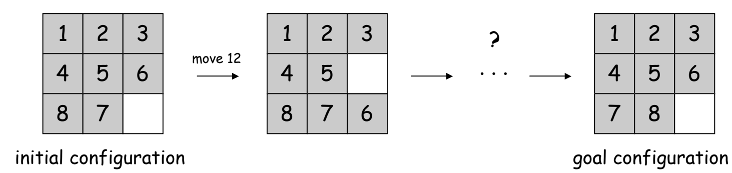


# 9.4 Planning Problem



8-puzzle, 15-puzzle. [Sam Loyd 1870s]

- Board: 3-by-3 grid of tiles labeled 1-8.
- Legal move: slide neighboring tile into blank (white) square.
- Find sequence of legal moves to transform initial configuration into goal configuration.








Conditions. Set $C = \{ C_1, \ldots, C_n \}$.
Initial configuration. Subset $c_0 \subseteq C$ of conditions initially satisfied.
Goal configuration. Subset $c^* \subseteq C$ of conditions we seek to satisfy.
Operators. Set $O = \{ O_1, \ldots, O_k \}$.

- To invoke operator $O_i$, must satisfy certain prereq conditions.
- After invoking $O_i$ certain conditions become true, and certain conditions become false.

PLANNING. Is it possible to apply sequence of operators to get from initial configuration to goal configuration?

Examples.

- 15-puzzle.
- Rubik's cube.
- Logistical operations to move people, equipment, and materials.





Planning example. Can we solve the 8-puzzle?

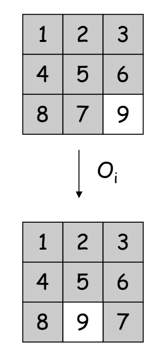
Conditions. $C_{ij}$, $1 \le i, j \le 9$. ← $C_{ij}$ means tile i is in square j

Initial state. $c_0 = \{C_{11}, C_{22}, \ldots, C_{66}, C_{78}, C_{87}, C_{99}\}$.

Goal state. $c^* = \{C_{11}, C_{22}, \ldots, C_{66}, C_{77}, C_{88}, C_{99}\}$.

Operators.

- Precondition to apply $O_i = \{C_{11}, C_{22}, \ldots, C_{66}, C_{78}, C_{87}, C_{99}\}$.
- After invoking $O_i$, conditions $C_{79}$ and $C_{97}$ become true.
- After invoking $O_i$, conditions $C_{78}$ and $C_{99}$ become false.




Solution. No solution to 8-puzzle or 15-puzzle!

## Diversion: Why is 8-Puzzle Unsolvable?

8-puzzle invariant. Any legal move preserves the parity of the number of pairs of pieces in reverse order (inversions).

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 4 | 5 | 6 |
| 8 | 7 | |

3 inversions
1-3, 2-3, 7-8

$\longrightarrow$

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 4 | 5 | 6 |
| 8 | | 7 |

3 inversions
1-3, 2-3, 7-8

$\longrightarrow$

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 4 | | 6 |
| 8 | 5 | 7 |

5 inversions
1-3, 2-3, 7-8, 5-8, 5-6

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | |

0 inversions

$\not\longrightarrow$

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 8 | 7 | |

1 inversion: 7-8



## Planning Problem: Binary Counter

Planning example. Can we increment an n-bit counter from the all-zeroes state to the all-ones state?

Conditions. $C_1, \ldots, C_n$. ← $C_i$ corresponds to bit i = 1
Initial state. $c_0 = \phi$. ← all 0s
Goal state. $c^* = \{C_1, \ldots, C_n\}$. ← all 1s
Operators. $O_1, \ldots, O_n$.

- To invoke operator $O_i$, must satisfy $C_1, \ldots, C_{i-1}$. ← i-1 least significant bits are 1
- After invoking $O_i$, condition $C_i$ becomes true. ← set bit i to 1
- After invoking $O_i$, conditions $C_1, \ldots, C_{i-1}$ become false. ← set i-1 least significant bits to 0

Solution. $\{\ \} \Rightarrow \{C_1\} \Rightarrow \{C_2\} \Rightarrow \{C_1, C_2\} \Rightarrow \{C_3\} \Rightarrow \{C_3, C_1\} \Rightarrow \ldots$

Observation. Any solution requires at least $2^n - 1$ steps.



## Planning Problem: In Exponential Space

Configuration graph G.

- Include node for each of $2^n$ possible configurations.
- Include an edge from configuration c' to configuration c'' if one of the operators can convert from c' to c''.

PLANNING. Is there a path from $c_0$ to $c^*$ in configuration graph?

Claim. PLANNING is in EXPTIME.
Pf. Run BFS to find path from $c_0$ to $c^*$ in configuration graph. ▪

Note. Configuration graph can have $2^n$ nodes, and shortest path can be of length = $2^n - 1$.
↑
binary counter



## Planning Problem: In Polynomial Space

Theorem. PLANNING is in PSPACE.
Pf.

- Suppose there is a path from $c_1$ to $c_2$ of length L.
- Path from $c_1$ to midpoint and from $c_2$ to midpoint are each $\le L/2$.
- Enumerate all possible midpoints.
- Apply recursively. Depth of recursion = $\log_2 L$. ▪

```
boolean hasPath(c1, c2, L) {
   if (L ≤ 1) return correct answer
                    enumerate using binary counter
   foreach configuration c' {
      boolean x = hasPath(c1, c', L/2)
      boolean y = hasPath(c2, c', L/2)
      if (x and y) return true
   }
   return false
}
```

# 9.5 PSPACE-Complete

## PSPACE-Complete

PSPACE. Decision problems solvable in polynomial space.

PSPACE-Complete. Problem Y is PSPACE-complete if (i) Y is in PSPACE and (ii) for every problem X in PSPACE, $X \leq_P Y$.

Theorem. [Stockmeyer-Meyer 1973] QSAT is PSPACE-complete.

Theorem. PSPACE $\subseteq$ EXPTIME.
Pf. Previous algorithm solves QSAT in exponential time, and QSAT is PSPACE-complete. ▪

Summary. P $\subseteq$ NP $\subseteq$ PSPACE $\subseteq$ EXPTIME.

it is known that P ≠ EXPTIME, but unknown which inclusion is strict;
conjectured that all are



## PSPACE-Complete Problems

More PSPACE-complete problems.

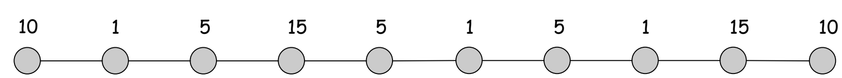
- Competitive facility location.
- Natural generalizations of games.
  - Othello, Hex, Geography, Rush-Hour, Instant Insanity
  - Shanghai, go-moku, Sokoban
- Given a memory restricted Turing machine, does it terminate in at most k steps?
- Do two regular expressions describe different languages?
- Is it possible to move and rotate complicated object with attachments through an irregularly shaped corridor?
- Is a deadlock state possible within a system of communicating processors?



## Competitive Facility Location

Input. Graph with positive edge weights, and target B.
Game. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors has been selected.

Competitive facility location. Can second player guarantee at least B units of profit?




Yes if B = 20; no if B = 25.

## Competitive Facility Location

Claim. COMPETITIVE-FACILITY is PSPACE-complete.

Pf.

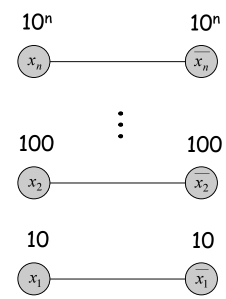
- To solve in poly-space, use recursion like QSAT, but at each step there are up to n choices instead of 2.
- To show that it's complete, we show that QSAT polynomial reduces to it. Given an instance of QSAT, we construct an instance of COMPETITIVE-FACILITY such that player 2 can force a win iff QSAT formula is true.



## Competitive Facility Location

Construction. Given instance $\Phi(x_1, \ldots, x_n) = C_1 \wedge C_1 \wedge \ldots C_k$ of QSAT. (assume n is odd)

- Include a node for each literal and its negation and connect them.
  - at most one of $x_i$ and its negation can be chosen
- Choose $c \geq k+2$, and put weight $c^i$ on literal $x^i$ and its negation; set $B = c^{n-1} + c^{n-3} + \ldots + c^4 + c^2 + 1$.
  - ensures variables are selected in order $x_n, x_{n-1}, \ldots, x_1$.
- As is, player 2 will lose by 1 unit: $c^{n-1} + c^{n-3} + \ldots + c^4 + c^2$.






## Competitive Facility Location

Construction. Given instance $\Phi(x_1, \ldots, x_n) = C_1 \wedge C_1 \wedge \ldots C_k$ of QSAT.

- Give player 2 one last move on which she can try to win.
- For each clause $C_j$, add node with value 1 and an edge to each of its literals.
- Player 2 can make last move iff truth assignment defined alternately by the players failed to satisfy some clause. ▪