

Complex Multiplication

Complex multiplication.  $(a + bi)(c + di) = x + yi$ .

Grade-school.  $x = ac - bd, y = bc + ad$ .

4 multiplications, 2 additions

Q. Is it possible to do with fewer multiplications?

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Q. Is it possible to do with fewer multiplications?

A. Yes. [Gauss]  $x = ac - bd, y = (a + b)(c + d) - ac - bd$ .

3 multiplications, 5 additions

Remark. Improvement if no hardware multiply.

5.5 Integer Multiplication



## Karatsuba Multiplication

To multiply two  $n$ -bit integers  $a$  and  $b$ :

- Add two  $\frac{1}{2}n$  bit integers.
- Multiply **three**  $\frac{1}{2}n$ -bit integers, recursively.
- Add, subtract, and shift to obtain result.

$$\begin{aligned}
 a &= 2^{n/2} \cdot a_1 + a_0 \\
 b &= 2^{n/2} \cdot b_1 + b_0 \\
 ab &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot (a_1 b_0 + a_0 b_1) + a_0 b_0 \\
 &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0) + a_0 b_0
 \end{aligned}$$

①
②
①
③
③

## Karatsuba Multiplication

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 &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0) + a_0 b_0
 \end{aligned}$$

①
②
①
③
③

**Theorem.** [Karatsuba-Ofman 1962] Can multiply two  $n$ -bit integers in  $O(n^{1.585})$  bit operations.

$$T(n) \leq \underbrace{T(\lfloor n/2 \rfloor) + T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}} \Rightarrow T(n) = O(n^{\lg 3}) = O(n^{1.585})$$

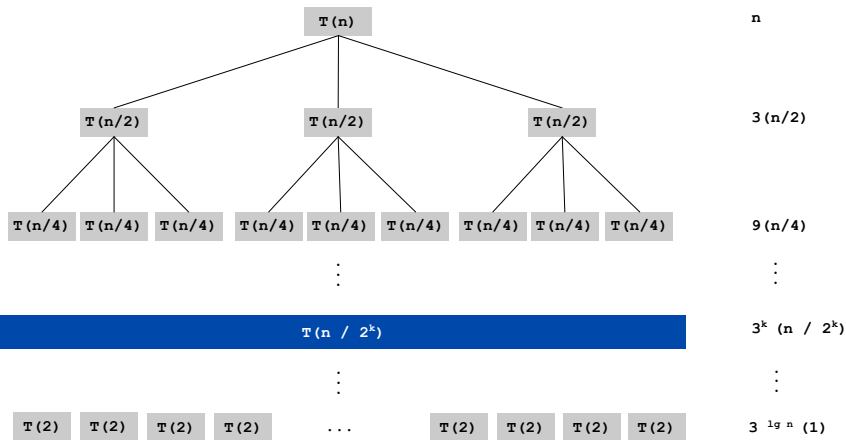
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## Karatsuba: Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

$$T(n) = \sum_{k=0}^{\lg n} n \left(\frac{3}{2}\right)^k = n \left( \frac{\left(\frac{3}{2}\right)^{1+\lg n} - 1}{\frac{3}{2} - 1} \right) = 3n^{\lg 3} - 2n$$



## Fast Integer Division Too (!)

**Integer division.** Given two  $n$ -bit (or less) integers  $s$  and  $t$ , compute quotient  $q = s / t$  and remainder  $r = s \bmod t$ .

**Fact.** Complexity of integer division is same as integer multiplication.

To compute quotient  $q$ :

- Approximate  $x = 1/t$  using Newton's method:  $x_{i+1} = 2x_i - tx_i^2$
- After  $\log n$  iterations, either  $q = \lfloor sx \rfloor$  or  $q = \lceil sx \rceil$ .

using fast multiplication

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# Matrix Multiplication

**Dot product.** Given two length  $n$  vectors  $a$  and  $b$ , compute  $c = a \cdot b$ .

**Grade-school.**  $\Theta(n)$  arithmetic operations.

$$a \cdot b = \sum_{i=1}^n a_i b_i$$

$$a = [.70 \ .20 \ .10]$$

$$b = [.30 \ .40 \ .30]$$

$$a \cdot b = (.70 \times .30) + (.20 \times .40) + (.10 \times .30) = .32$$

**Remark.** Grade-school dot product algorithm is optimal.

## Matrix Multiplication

**Matrix multiplication.** Given two  $n$ -by- $n$  matrices  $A$  and  $B$ , compute  $C = AB$ .

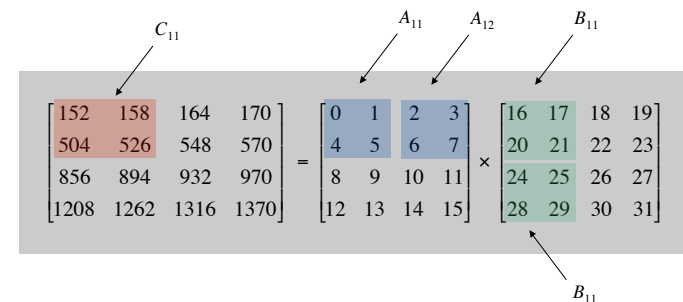
**Grade-school.**  $\Theta(n^3)$  arithmetic operations.

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

## Block Matrix Multiplication



$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

**Q.** Is grade-school matrix multiplication algorithm optimal?

## Matrix Multiplication: Warmup

To multiply two  $n$ -by- $n$  matrices  $A$  and  $B$ :

- Divide: partition  $A$  and  $B$  into  $\frac{1}{2}n$ -by- $\frac{1}{2}n$  blocks.
- Conquer: multiply 8 pairs of  $\frac{1}{2}n$ -by- $\frac{1}{2}n$  matrices, recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

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## Fast Matrix Multiplication

Key idea. multiply 2-by-2 blocks with only 7 multiplications.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &= A_{11} \times (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) \times B_{22} \\ P_3 &= (A_{21} + A_{22}) \times B_{11} \\ P_4 &= A_{22} \times (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

- 7 multiplications.
- 18 = 8 + 10 additions and subtractions.

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## Fast Matrix Multiplication

To multiply two  $n$ -by- $n$  matrices  $A$  and  $B$ : [Strassen 1969]

- Divide: partition  $A$  and  $B$  into  $\frac{1}{2}n$ -by- $\frac{1}{2}n$  blocks.
- Compute: 14  $\frac{1}{2}n$ -by- $\frac{1}{2}n$  matrices via 10 matrix additions.
- Conquer: multiply 7 pairs of  $\frac{1}{2}n$ -by- $\frac{1}{2}n$  matrices, recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

Analysis.

- Assume  $n$  is a power of 2.
- $T(n) = \#$  arithmetic operations.

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

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## Fast Matrix Multiplication: Practice

Implementation issues.

- Sparsity.
- Caching effects.
- Numerical stability.
- Odd matrix dimensions.
- Crossover to classical algorithm around  $n = 128$ .

Common misperception. "Strassen is only a theoretical curiosity."

- Apple reports 8x speedup on G4 Velocity Engine when  $n \approx 2,500$ .
- Range of instances where it's useful is a subject of controversy.

Remark. Can "Strassenize"  $Ax = b$ , determinant, eigenvalues, SVD, ...

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Q. Multiply two 2-by-2 matrices with 7 scalar multiplications?

A. Yes! [Strassen 1969]  $\Theta(n^{\log_2 7}) = O(n^{2.807})$

Q. Multiply two 2-by-2 matrices with 6 scalar multiplications?

A. Impossible. [Hopcroft and Kerr 1971]  $\Theta(n^{\log_2 6}) = O(n^{2.59})$

Q. Two 3-by-3 matrices with 21 scalar multiplications?

A. Also impossible.  $\Theta(n^{\log_3 21}) = O(n^{2.77})$

Begun, the decimal wars have. [Pan, Bini et al, Schönhage, ...]

- Two 20-by-20 matrices with 4,460 scalar multiplications.  $O(n^{2.805})$
- Two 48-by-48 matrices with 47,217 scalar multiplications.  $O(n^{2.7801})$
- A year later.  $O(n^{2.7799})$
- December, 1979.  $O(n^{2.521813})$
- January, 1980.  $O(n^{2.521801})$

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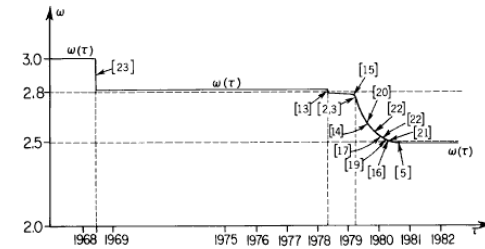


FIG. 1.  $\omega(t)$  is the best exponent announced by time  $\tau$ .

Best known.  $O(n^{2.376})$  [Coppersmith-Winograd, 1987]

Conjecture.  $O(n^{2+\epsilon})$  for any  $\epsilon > 0$ .

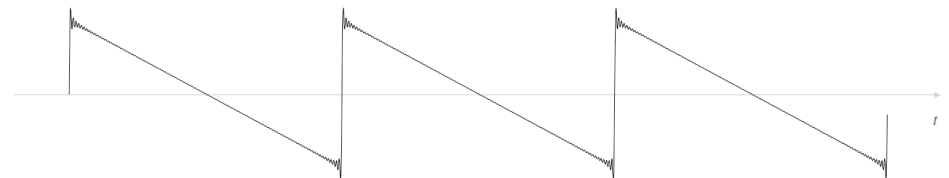
Caveat. Theoretical improvements to Strassen are progressively less practical.

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## 5.6 Convolution and FFT

### Fourier Analysis

Fourier theorem. [Fourier, Dirichlet, Riemann] Any periodic function can be expressed as the sum of a series of sinusoids. ↖ sufficiently smooth



$$y(t) = \frac{2}{\pi} \sum_{k=1}^N \frac{\sin kt}{k} \quad N = 100$$

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## Euler's Identity

**Sinusoids.** Sum of sine and cosines.

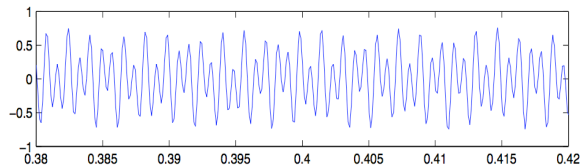
$$e^{ix} = \cos x + i \sin x$$

*Euler's identity*

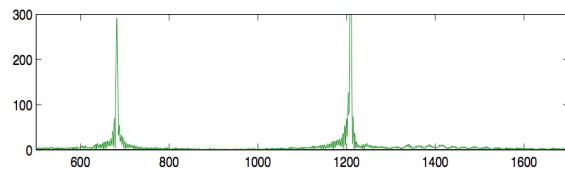
**Sinusoids.** Sum of complex exponentials.

## Time Domain vs. Frequency Domain

**Signal.** [recording, 8192 samples per second]



**Magnitude of discrete Fourier transform.**



Reference: Cleve Moler, Numerical Computing with MATLAB

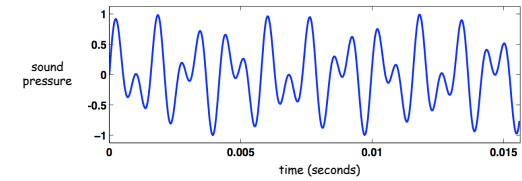
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## Time Domain vs. Frequency Domain

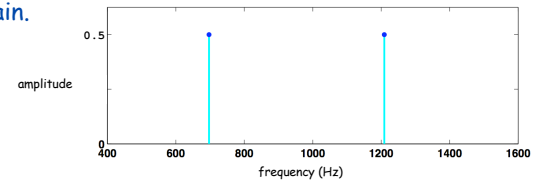
**Signal.** [touch tone button 1]  $y(t) = \frac{1}{2} \sin(2\pi \cdot 697 t) + \frac{1}{2} \sin(2\pi \cdot 1209 t)$



**Time domain.**



**Frequency domain.**



Reference: Cleve Moler, Numerical Computing with MATLAB

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## Fast Fourier Transform

**FFT.** Fast way to convert between time-domain and frequency-domain.

**Alternate viewpoint.** Fast way to multiply and evaluate **polynomials**.

we take this approach

If you speed up any nontrivial algorithm by a factor of a million or so the world will beat a path towards finding useful applications for it. -Numerical Recipes

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## Applications.

- Optics, acoustics, quantum physics, telecommunications, radar, control systems, signal processing, speech recognition, data compression, image processing, seismology, mass spectrometry...
- Digital media. [DVD, JPEG, MP3, H.264]
- Medical diagnostics. [MRI, CT, PET scans, ultrasound]
- Numerical solutions to Poisson's equation.
- Shor's quantum factoring algorithm.
- ...

The FFT is one of the truly great computational developments of [the 20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. -Charles van Loan

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## Polynomials: Coefficient Representation

**Polynomial.** [coefficient representation]

$$A(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$$

$$B(x) = b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1}$$

**Add.**  $O(n)$  arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_{n-1} + b_{n-1})x^{n-1}$$

**Evaluate.**  $O(n)$  using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \dots + x(a_{n-2} + x(a_{n-1}))))))$$

**Multiply (convolve).**  $O(n^2)$  using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \text{ where } c_i = \sum_{j=0}^i a_j b_{i-j}$$

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**Gauss (1805, 1866).** Analyzed periodic motion of asteroid Ceres.

**Runge-König (1924).** Laid theoretical groundwork.

**Danielson-Lanczos (1942).** Efficient algorithm, x-ray crystallography.

**Cooley-Tukey (1965).** Monitoring nuclear tests in Soviet Union and tracking submarines. Rediscovered and popularized FFT.

**Importance** not fully realized until advent of digital computers.

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## A Modest PhD Dissertation Title

"New Proof of the Theorem That Every Algebraic Rational Integral Function In One Variable can be Resolved into Real Factors of the First or the Second Degree."

- PhD dissertation, 1799 the University of Helmstedt

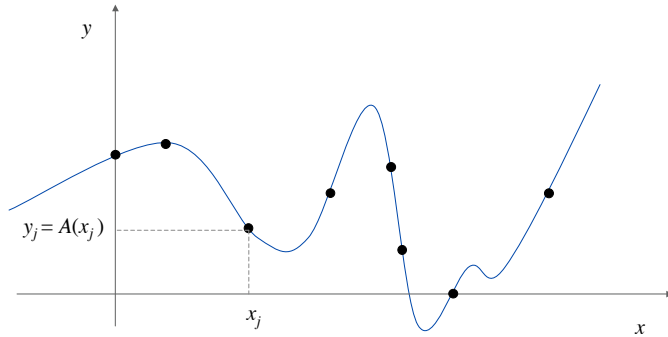


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**Fundamental theorem of algebra.** [Gauss, PhD thesis] A degree  $n$  polynomial with complex coefficients has exactly  $n$  complex roots.

**Corollary.** A degree  $n-1$  polynomial  $A(x)$  is uniquely specified by its evaluation at  $n$  distinct values of  $x$ .



**Polynomial.** [point-value representation]

$$A(x): (x_0, y_0), \dots, (x_{n-1}, y_{n-1})$$

$$B(x): (x_0, z_0), \dots, (x_{n-1}, z_{n-1})$$

**Add.**  $O(n)$  arithmetic operations.

$$A(x) + B(x): (x_0, y_0 + z_0), \dots, (x_{n-1}, y_{n-1} + z_{n-1})$$

**Multiply (convolve).**  $O(n)$ , but need  $2n-1$  points.

$$A(x) \times B(x): (x_0, y_0 \times z_0), \dots, (x_{2n-1}, y_{2n-1} \times z_{2n-1})$$

**Evaluate.**  $O(n^2)$  using Lagrange's formula.

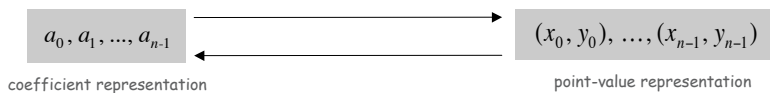
$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Converting Between Two Polynomial Representations

**Tradeoff.** Fast evaluation **or** fast multiplication. We want both!

representation	multiply	evaluate
coefficient	$O(n^2)$	$O(n)$
point-value	$O(n)$	$O(n^2)$

**Goal.** Efficient conversion between two representations  $\Rightarrow$  all ops fast.



Converting Between Two Representations: Brute Force

**Coefficient  $\Rightarrow$  point-value.** Given a polynomial  $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ , evaluate it at  $n$  distinct points  $x_0, \dots, x_{n-1}$ .

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

**Running time.**  $O(n^2)$  for matrix-vector multiply (or  $n$  Horner's).

## Converting Between Two Representations: Brute Force

**Point-value  $\Rightarrow$  coefficient.** Given  $n$  distinct points  $x_0, \dots, x_{n-1}$  and values  $y_0, \dots, y_{n-1}$ , find unique polynomial  $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ , that has given values at given points.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Vandermonde matrix is invertible iff  $x_i$  distinct

**Running time.**  $O(n^3)$  for Gaussian elimination.

or  $O(n^{2.376})$  via fast matrix multiplication

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## Divide-and-Conquer

**Decimation in frequency.** Break up polynomial into low and high powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$ .
- $A_{low}(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ .
- $A_{high}(x) = a_4 + a_5x + a_6x^2 + a_7x^3$ .
- $A(x) = A_{low}(x) + x^4 A_{high}(x)$ .

**Decimation in time.** Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$ .
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$ .
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$ .
- $A(x) = A_{even}(x^2) + x A_{odd}(x^2)$ .

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## Coefficient to Point-Value Representation: Intuition

**Coefficient  $\Rightarrow$  point-value.** Given a polynomial  $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ , evaluate it at  $n$  distinct points  $x_0, \dots, x_{n-1}$ .

we get to choose which ones!

**Divide.** Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$ .
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$ .
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$ .
- $A(x) = A_{even}(x^2) + x A_{odd}(x^2)$ .
- $A(-x) = A_{even}(x^2) - x A_{odd}(x^2)$ .

**Intuition.** Choose two points to be  $\pm 1$ .

- $A(1) = A_{even}(1) + 1 A_{odd}(1)$ .
- $A(-1) = A_{even}(1) - 1 A_{odd}(1)$ .

Can evaluate polynomial of degree  $\leq n$  at 2 points by evaluating two polynomials of degree  $\leq \frac{1}{2}n$  at 1 point.

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## Coefficient to Point-Value Representation: Intuition

**Coefficient  $\Rightarrow$  point-value.** Given a polynomial  $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ , evaluate it at  $n$  distinct points  $x_0, \dots, x_{n-1}$ .

we get to choose which ones!

**Divide.** Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$ .
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$ .
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$ .
- $A(x) = A_{even}(x^2) + x A_{odd}(x^2)$ .
- $A(-x) = A_{even}(x^2) - x A_{odd}(x^2)$ .

**Intuition.** Choose four **complex** points to be  $\pm 1, \pm i$ .

- $A(1) = A_{even}(1) + 1 A_{odd}(1)$ .
- $A(-1) = A_{even}(1) - 1 A_{odd}(1)$ .
- $A(i) = A_{even}(-1) + i A_{odd}(-1)$ .
- $A(-i) = A_{even}(-1) - i A_{odd}(-1)$ .

Can evaluate polynomial of degree  $\leq n$  at 4 points by evaluating two polynomials of degree  $\leq \frac{1}{2}n$  at 2 points.

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## Discrete Fourier Transform

**Coefficient  $\Rightarrow$  point-value.** Given a polynomial  $a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ , evaluate it at  $n$  distinct points  $x_0, \dots, x_{n-1}$ .

**Key idea.** Choose  $x_k = \omega^k$  where  $\omega$  is principal  $n^{\text{th}}$  root of unity.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

↑ DFT                      ↑ Fourier matrix  $F_n$

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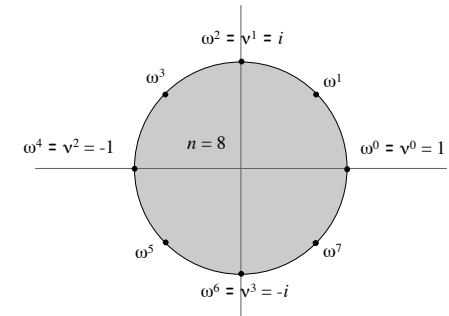
## Roots of Unity

**Def.** An  $n^{\text{th}}$  root of unity is a complex number  $x$  such that  $x^n = 1$ .

**Fact.** The  $n^{\text{th}}$  roots of unity are:  $\omega^0, \omega^1, \dots, \omega^{n-1}$  where  $\omega = e^{2\pi i/n}$ .

**Pf.**  $(\omega^k)^n = (e^{2\pi i k/n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$ .

**Fact.** The  $\frac{1}{2}n^{\text{th}}$  roots of unity are:  $v^0, v^1, \dots, v^{n/2-1}$  where  $v = \omega^2 = e^{4\pi i/n}$ .



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## Fast Fourier Transform

**Goal.** Evaluate a degree  $n-1$  polynomial  $A(x) = a_0 + \dots + a_{n-1}x^{n-1}$  at its  $n^{\text{th}}$  roots of unity:  $\omega^0, \omega^1, \dots, \omega^{n-1}$ .

**Divide.** Break up polynomial into even and odd powers.

- $A_{\text{even}}(x) = a_0 + a_2x + a_4x^2 + \dots + a_{n-2}x^{n/2-1}$ .
- $A_{\text{odd}}(x) = a_1 + a_3x + a_5x^2 + \dots + a_{n-1}x^{n/2-1}$ .
- $A(x) = A_{\text{even}}(x^2) + x A_{\text{odd}}(x^2)$ .

**Conquer.** Evaluate  $A_{\text{even}}(x)$  and  $A_{\text{odd}}(x)$  at the  $\frac{1}{2}n^{\text{th}}$  roots of unity:  $v^0, v^1, \dots, v^{n/2-1}$ .

**Combine.**

- $A(\omega^k) = A_{\text{even}}(v^k) + \omega^k A_{\text{odd}}(v^k), \quad 0 \leq k < n/2$
  - $A(\omega^{k+\frac{1}{2}n}) = A_{\text{even}}(v^k) - \omega^k A_{\text{odd}}(v^k), \quad 0 \leq k < n/2$
- $v^k = (\omega^k)^2$   
 $v^k = (\omega^{k+\frac{1}{2}n})^2$        $\omega^{k+\frac{1}{2}n} = -\omega^k$

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## FFT Algorithm

```
fft(n, a₀, a₁, ..., aₙ₋₁) {
  if (n == 1) return a₀

  (e₀, e₁, ..., eₙ/₂₋₁) ← FFT(n/2, a₀, a₂, a₄, ..., aₙ₋₂)
  (d₀, d₁, ..., dₙ/₂₋₁) ← FFT(n/2, a₁, a₃, a₅, ..., aₙ₋₁)

  for k = 0 to n/2 - 1 {
    ωᵏ ← e^{2πik/n}
    yₖ ← eₖ + ωᵏ dₖ
    yₖ₊ₙ/₂ ← eₖ - ωᵏ dₖ
  }

  return (y₀, y₁, ..., yₙ₋₁)
}
```

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## Inverse FFT: Proof of Correctness

**Claim.**  $F_n$  and  $G_n$  are inverses.

**Pf.**

$$(F_n G_n)_{kk'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{kj} \omega^{-jk'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{(k-k')j} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma

**Summation lemma.** Let  $\omega$  be a principal  $n^{\text{th}}$  root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{kj} = \begin{cases} n & \text{if } k \equiv 0 \pmod n \\ 0 & \text{otherwise} \end{cases}$$

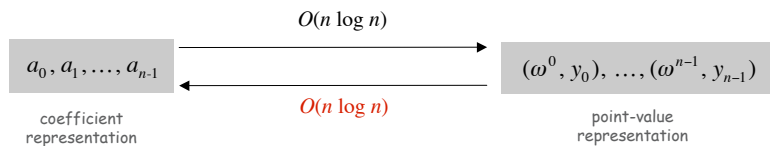
**Pf.**

- If  $k$  is a multiple of  $n$  then  $\omega^k = 1 \Rightarrow$  series sums to  $n$ .
- Each  $n^{\text{th}}$  root of unity  $\omega^k$  is a root of  $x^n - 1 = (x - 1)(1 + x + x^2 + \dots + x^{n-1})$ .
- if  $\omega^k \neq 1$  we have:  $1 + \omega^k + \omega^{k(2)} + \dots + \omega^{k(n-1)} = 0 \Rightarrow$  series sums to 0. ▀

## Inverse FFT Summary

**Theorem.** Inverse FFT algorithm interpolates a degree  $n-1$  polynomial given values at each of the  $n^{\text{th}}$  roots of unity in  $O(n \log n)$  steps.

assumes  $n$  is a power of 2



## Inverse FFT: Algorithm

```

ifft(n, a_0, a_1, ..., a_{n-1}) {
  if (n == 1) return a_0

  (e_0, e_1, ..., e_{n/2-1}) ← FFT(n/2, a_0, a_2, a_4, ..., a_{n-2})
  (d_0, d_1, ..., d_{n/2-1}) ← FFT(n/2, a_1, a_3, a_5, ..., a_{n-1})

  for k = 0 to n/2 - 1 {
    omega^k ← e^{-2πik/n}
    y_{k+n/2} ← (e_k + omega^k d_k) / n
    y_{k+n/2} ← (e_k - omega^k d_k) / n
  }

  return (y_0, y_1, ..., y_{n-1})
}
    
```

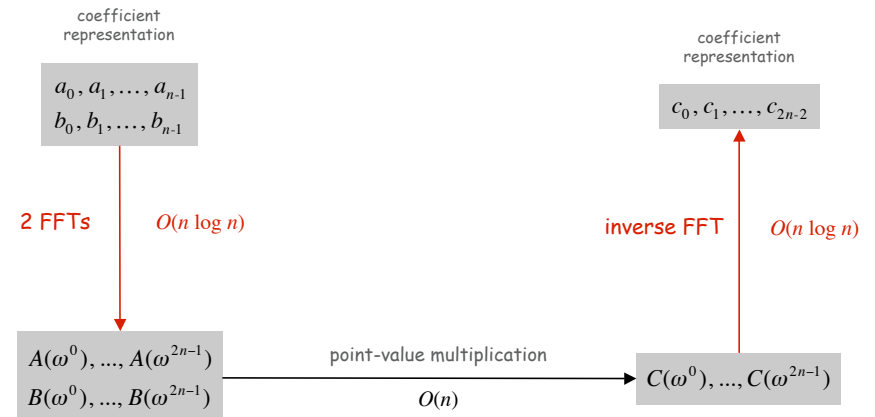
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## Polynomial Multiplication

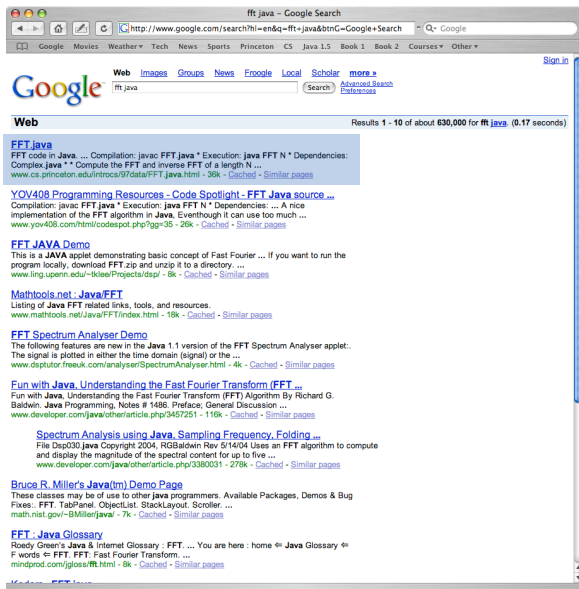
**Theorem.** Can multiply two degree  $n-1$  polynomials in  $O(n \log n)$  steps.

pad with 0s to make  $n$  a power of 2



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Fastest Fourier transform in the West. [Frigo and Johnson]

- Optimized C library.
- Features: DFT, DCT, real, complex, any size, any dimension.
- Won 1999 Wilkinson Prize for Numerical Software.
- Portable, competitive with vendor-tuned code.

Implementation details.

- Instead of executing predetermined algorithm, it evaluates your hardware and uses a special-purpose compiler to generate an optimized algorithm catered to "shape" of the problem.
- Core algorithm is nonrecursive version of Cooley-Tukey.
- $O(n \log n)$ , even for prime sizes.

Reference: <http://www.fftw.org>

Integer Multiplication, Redux

Integer multiplication. Given two  $n$  bit integers  $a = a_{n-1} \dots a_1 a_0$  and  $b = b_{n-1} \dots b_1 b_0$ , compute their product  $a \cdot b$ .

Convolution algorithm.

- Form two polynomials.  $A(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$
- Note:  $a = A(2)$ ,  $b = B(2)$ .  $B(x) = b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1}$
- Compute  $C(x) = A(x) \cdot B(x)$ .
- Evaluate  $C(2) = a \cdot b$ .
- Running time:  $O(n \log n)$  complex arithmetic operations.

Theory. [Schönhage-Strassen 1971]  $O(n \log n \log \log n)$  bit operations.

Theory. [Fürer 2007]  $O(n \log n 2^{O(\log^* n)})$  bit operations.

Integer Multiplication, Redux

Integer multiplication. Given two  $n$  bit integers  $a = a_{n-1} \dots a_1 a_0$  and  $b = b_{n-1} \dots b_1 b_0$ , compute their product  $a \cdot b$ .

"the fastest bignum library on the planet"

Practice. [GNU Multiple Precision Arithmetic Library]

It uses brute force, Karatsuba, and FFT, depending on the size of  $n$ .

## Integer Arithmetic

**Fundamental open question.** What is complexity of arithmetic?

Operation	Upper Bound	Lower Bound
addition	$O(n)$	$\Omega(n)$
multiplication	$O(n \log n 2^{O(\log^* n)})$	$\Omega(n)$
division	$O(n \log n 2^{O(\log^* n)})$	$\Omega(n)$

## Factoring

**Factoring.** Given an  $n$ -bit integer, find its prime factorization.

$$2773 = 47 \times 59$$

$$2^{67} - 1 = 147573952589676412927 = 193707721 \times 761838257287$$

a disproof of Mersenne's conjecture that  $2^{67} - 1$  is prime

```
740375634795617128280467960974295731425931888892312890849
362326389727650340282662768919964196251178439958943305021
275853701189680982867331732731089309005525051168770632990
72396380786710086096962537934650563796359
```

RSA-704  
(\$30,000 prize if you can factor)

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## Factoring and RSA

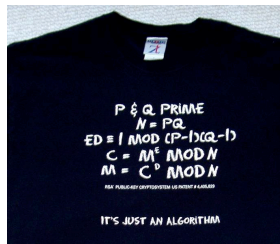
**Primality.** Given an  $n$ -bit integer, is it prime?

**Factoring.** Given an  $n$ -bit integer, find its prime factorization.

**Significance.** Efficient primality testing  $\Rightarrow$  can implement RSA.

**Significance.** Efficient factoring  $\Rightarrow$  can break RSA.

**Theorem.** [AKS 2002] Poly-time algorithm for primality testing.



## Shor's Algorithm

**Shor's algorithm.** Can factor an  $n$ -bit integer in  $O(n^3)$  time on a quantum computer.

algorithm uses quantum QFT!

**Ramification.** At least one of the following is wrong:

- RSA is secure.
- Textbook quantum mechanics.
- Extending Church-Turing thesis.



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# Shor's Factoring Algorithm

Period finding.

$2^i$	1	2	4	8	16	32	64	128	...
$2^i \bmod 15$	1	2	4	8	1	2	4	8	...
$2^i \bmod 21$	1	2	4	8	16	11	1	2	...

↖ period = 4  
↖ period = 6

**Theorem.** [Euler] Let  $p$  and  $q$  be prime, and let  $N = p q$ . Then, the following sequence repeats with a period divisible by  $(p-1)(q-1)$ :

$$x \bmod N, x^2 \bmod N, x^3 \bmod N, x^4 \bmod N, \dots$$

**Consequence.** If we can learn something about the period of the sequence, we can learn something about the divisors of  $(p-1)(q-1)$ .

↖ by using random values of  $x$ , we get the divisors of  $(p-1)(q-1)$ , and from this, can get the divisors of  $N = p q$

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## Extra Slides

### Fourier Matrix Decomposition

$$F_n = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}$$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_4 = \begin{bmatrix} \omega^0 & 0 & 0 & 0 \\ 0 & \omega^1 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{bmatrix}$$

$$a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$y = F_n a = \begin{bmatrix} I_{n/2} & D_{n/2} \\ I_{n/2} & -D_{n/2} \end{bmatrix} \begin{bmatrix} F_{n/2} a_{\text{even}} \\ F_{n/2} a_{\text{odd}} \end{bmatrix}$$

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