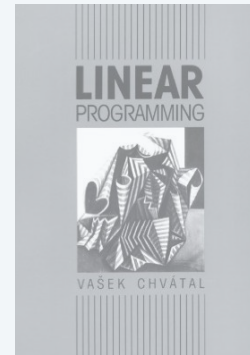


LINEAR PROGRAMMING II

- ▶ LP duality
- ▶ strong duality theorem
- ▶ bonus proof of LP duality
- ▶ applications

Lecture slides by Kevin Wayne

Last updated on 7/25/17 11:09 AM



LINEAR PROGRAMMING II

- ▶ LP duality
- ▶ Strong duality theorem
- ▶ Bonus proof of LP duality
- ▶ Applications

LP duality

Primal problem.

$$\begin{array}{ll} \text{(P) max} & 13A + 23B \\ \text{s. t.} & 5A + 15B \leq 480 \\ & 4A + 4B \leq 160 \\ & 35A + 20B \leq 1190 \\ & A, B \geq 0 \end{array}$$

Goal. Find a **lower bound** on optimal value.

Easy. Any feasible solution provides one.

Ex 1. $(A, B) = (34, 0) \Rightarrow z^* \geq 442$

Ex 2. $(A, B) = (0, 32) \Rightarrow z^* \geq 736$

Ex 3. $(A, B) = (7.5, 29.5) \Rightarrow z^* \geq 776$

Ex 4. $(A, B) = (12, 28) \Rightarrow z^* \geq 800$

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LP duality

Primal problem.

$$\begin{array}{ll} \text{(P) max} & 13A + 23B \\ \text{s. t.} & 5A + 15B \leq 480 \\ & 4A + 4B \leq 160 \\ & 35A + 20B \leq 1190 \\ & A, B \geq 0 \end{array}$$

Goal. Find an **upper bound** on optimal value.

Ex 1. Multiply 2nd inequality by 6: $24A + 24B \leq 960$.

$$\Rightarrow z^* = \underbrace{13A + 23B}_{\text{objective function}} \leq 24A + 24B \leq 960.$$

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LP duality

Primal problem.

$$\begin{array}{ll} \text{(P)} & \max \quad 13A + 23B \\ & \text{s. t.} \quad 5A + 15B \leq 480 \\ & \quad \quad 4A + 4B \leq 160 \\ & \quad \quad 35A + 20B \leq 1190 \\ & \quad \quad A, B \geq 0 \end{array}$$

Goal. Find an **upper bound** on optimal value.

Ex 2. Add 2 times 1st inequality to 2nd inequality:

$$\Rightarrow z^* = 13A + 23B \leq 14A + 34B \leq 1120.$$

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LP duality

Primal problem.

$$\begin{array}{ll} \text{(P)} & \max \quad 13A + 23B \\ & \text{s. t.} \quad 5A + 15B \leq 480 \\ & \quad \quad 4A + 4B \leq 160 \\ & \quad \quad 35A + 20B \leq 1190 \\ & \quad \quad A, B \geq 0 \end{array}$$

Goal. Find an **upper bound** on optimal value.

Ex 2. Add 1 times 1st inequality to 2 times 2nd inequality:

$$\Rightarrow z^* = 13A + 23B \leq 13A + 23B \leq 800.$$

Recall lower bound. $(A, B) = (34, 0) \Rightarrow z^* \geq 442$

Combine upper and lower bounds: **$z^* = 800$** .

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LP duality

Primal problem.

$$\begin{array}{ll} \text{(P)} & \max \quad 13A + 23B \\ & \text{s. t.} \quad 5A + 15B \leq 480 \\ & \quad \quad 4A + 4B \leq 160 \\ & \quad \quad 35A + 20B \leq 1190 \\ & \quad \quad A, B \geq 0 \end{array}$$

Idea. Add nonnegative combination (C, H, M) of the constraints s.t.

$$\begin{array}{l} 13A + 23B \leq (5C + 4H + 35M)A + (15C + 4H + 20M)B \\ \leq 480C + 160H + 1190M \end{array}$$

Dual problem. Find best such upper bound.

$$\begin{array}{ll} \text{(D)} & \min \quad 480C + 160H + 1190M \\ & \text{s. t.} \quad 5C + 4H + 35M \geq 13 \\ & \quad \quad 15C + 4H + 20M \geq 23 \\ & \quad \quad C, H, M \geq 0 \end{array}$$

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LP duality: economic interpretation

Brewer: find optimal mix of beer and ale to maximize profits.

$$\begin{array}{ll} \text{(P)} & \max \quad 13A + 23B \\ & \text{s. t.} \quad 5A + 15B \leq 480 \\ & \quad \quad 4A + 4B \leq 160 \\ & \quad \quad 35A + 20B \leq 1190 \\ & \quad \quad A, B \geq 0 \end{array}$$

Entrepreneur: buy individual resources from brewer at min cost.

- C, H, M = unit price for corn, hops, malt.
- Brewer won't agree to sell resources if $5C + 4H + 35M < 13$.

$$\begin{array}{ll} \text{(D)} & \min \quad 480C + 160H + 1190M \\ & \text{s. t.} \quad 5C + 4H + 35M \geq 13 \\ & \quad \quad 15C + 4H + 20M \geq 23 \\ & \quad \quad C, H, M \geq 0 \end{array}$$

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LP duals

Canonical form.

$$(P) \quad \max c^T x \\ \text{s. t. } Ax \leq b \\ x \geq 0$$

$$(D) \quad \min y^T b \\ \text{s. t. } A^T y \geq c \\ y \geq 0$$

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Double dual

Canonical form.

$$(P) \quad \max c^T x \\ \text{s. t. } Ax \leq b \\ x \geq 0$$

$$(D) \quad \min y^T b \\ \text{s. t. } A^T y \geq c \\ y \geq 0$$

Property. The dual of the dual is the primal.

Pf. Rewrite (D) as a maximization problem in canonical form; take dual.

$$(D') \quad \max -y^T b \\ \text{s. t. } -A^T y \leq -c \\ y \geq 0$$

$$(DD) \quad \min -c^T z \\ \text{s. t. } -(A^T)^T z \geq -b \\ z \geq 0$$

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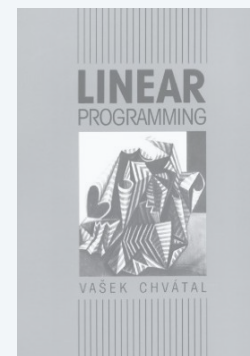
Taking duals

LP dual recipe.

Primal (P)	maximize	minimize	Dual (D)
constraints	$a x = b_i$ $a x \leq b$ $a x \geq b_i$	y_i unrestricted $y_i \geq 0$ $y_i \leq 0$	variables
variables	$x_j \geq 0$ $x_j \leq 0$ unrestricted	$a^T y \geq c_j$ $a^T y \leq c_j$ $a^T y = c_j$	constraints

Pf. Rewrite LP in standard form and take dual.

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LINEAR PROGRAMMING II

- ▶ LP duality
- ▶ strong duality theorem
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LP strong duality

Theorem. [Gale–Kuhn–Tucker 1951, Dantzig–von Neumann 1947]
 For $A \in \mathfrak{R}^{m \times n}, b \in \mathfrak{R}^m, c \in \mathfrak{R}^n$, if (P) and (D) are nonempty, then $\max = \min$.

$$\begin{array}{ll} \text{(P)} & \max c^T x \\ & \text{s. t. } Ax \leq b \\ & \quad x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{(D)} & \min y^T b \\ & \text{s. t. } A^T y \geq c \\ & \quad y \geq 0 \end{array}$$

Generalizes:

- Dilworth's theorem.
- König–Egervary theorem.
- Max-flow min-cut theorem.
- von Neumann's minimax theorem.
- ...

Pf. [ahead]

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LP weak duality

Theorem. For $A \in \mathfrak{R}^{m \times n}, b \in \mathfrak{R}^m, c \in \mathfrak{R}^n$, if (P) and (D) are nonempty, then $\max \leq \min$.

$$\begin{array}{ll} \text{(P)} & \max c^T x \\ & \text{s. t. } Ax \leq b \\ & \quad x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{(D)} & \min y^T b \\ & \text{s. t. } A^T y \geq c \\ & \quad y \geq 0 \end{array}$$

Pf. Suppose $x \in \mathfrak{R}^n$ is feasible for (P) and $y \in \mathfrak{R}^m$ is feasible for (D).

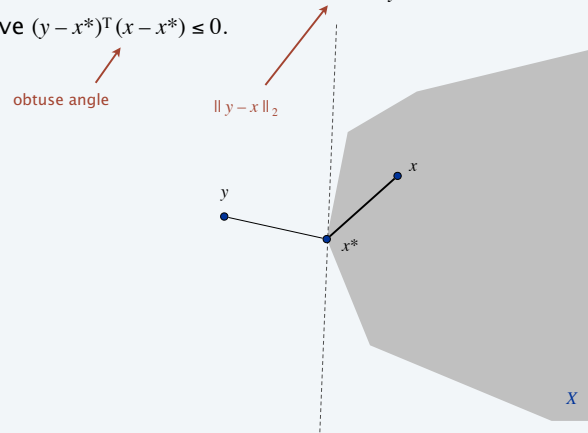
- $y \geq 0, Ax \leq b \implies y^T Ax \leq y^T b$
- $x \geq 0, A^T y \geq c \implies y^T Ax \geq c^T x$
- Combine: $c^T x \leq y^T Ax \leq y^T b$. ▀

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Projection lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X . Then $\min \{f(x) : x \in X\}$ exists.

Projection lemma. Let $X \subset \mathfrak{R}^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y . Moreover, for all $x \in X$ we have $(y - x^*)^T (x - x^*) \leq 0$.



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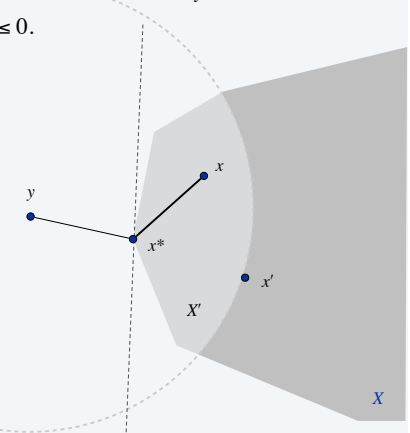
Projection lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X . Then $\min \{f(x) : x \in X\}$ exists.

Projection lemma. Let $X \subset \mathfrak{R}^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y . Moreover, for all $x \in X$ we have $(y - x^*)^T (x - x^*) \leq 0$.

Pf.

- Define $f(x) = \|y - x\|$.
- Want to apply Weierstrass, but X not necessarily bounded.
- $X \neq \emptyset \implies$ there exists $x' \in X$.
- Define $X' = \{x \in X : \|y - x\| \leq \|y - x'\|\}$ so that X' is closed, bounded, and $\min \{f(x) : x \in X\} = \min \{f(x) : x \in X'\}$.
- By Weierstrass, min exists.



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Projection lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X . Then $\min \{f(x) : x \in X\}$ exists.

Projection lemma. Let $X \subset \mathfrak{R}^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y . Moreover, for all $x \in X$ we have $(y - x^*)^T(x - x^*) \leq 0$.

Pf.

- x^* min distance $\Rightarrow \|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.
- By convexity: if $x \in X$, then $x^* + \varepsilon(x - x^*) \in X$ for all $0 < \varepsilon < 1$.
- $\|y - x^*\|^2 \leq \|y - x^* - \varepsilon(x - x^*)\|^2$
 $= \|y - x^*\|^2 + \varepsilon^2 \|x - x^*\|^2 - 2\varepsilon (y - x^*)^T(x - x^*)$
- Thus, $(y - x^*)^T(x - x^*) \leq \frac{1}{2}\varepsilon \|(x - x^*)\|^2$.
- Letting $\varepsilon \rightarrow 0^+$, we obtain the desired result. ■

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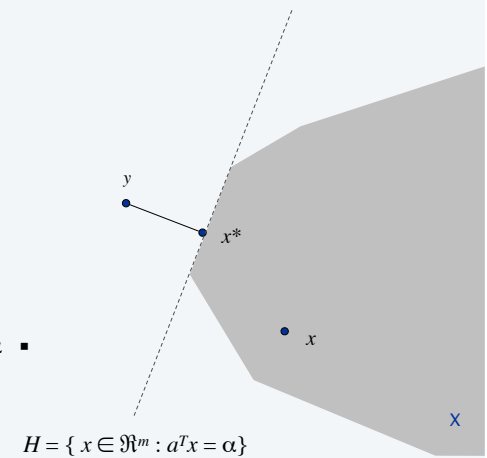
Separating hyperplane theorem

Theorem. Let $X \subset \mathfrak{R}^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists a hyperplane $H = \{x \in \mathfrak{R}^m : a^T x = \alpha\}$ where $a \in \mathfrak{R}^m$, $\alpha \in \mathfrak{R}$ that separates y from X .

$$\begin{aligned} a^T x &\geq \alpha \text{ for all } x \in X \\ a^T y &< \alpha \end{aligned}$$

Pf.

- Let x^* be closest point in X to y .
- By projection lemma,
 $(y - x^*)^T(x - x^*) \leq 0$ for all $x \in X$
- Choose $a = x^* - y \neq 0$ and $\alpha = a^T x^*$.
- If $x \in X$, then $a^T(x - x^*) \geq 0$;
 thus $\Rightarrow a^T x \geq a^T x^* = \alpha$.
- Also, $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$. ■



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Farkas' lemma

Theorem. For $A \in \mathfrak{R}^{m \times n}$, $b \in \mathfrak{R}^m$ exactly one of the following two systems holds:

$$\text{(I) } \begin{aligned} \exists x \in \mathfrak{R}^n \\ \text{s. t. } Ax &= b \\ x &\geq 0 \end{aligned}$$

$$\text{(II) } \begin{aligned} \exists y \in \mathfrak{R}^m \\ \text{s. t. } A^T y &\geq 0 \\ y^T b &< 0 \end{aligned}$$

Pf. [not both] Suppose x satisfies (I) and y satisfies (II). Then $0 > y^T b = y^T A x \geq 0$, a contradiction.

Pf. [at least one] Suppose (I) infeasible. We will show (II) feasible.

- Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.
- Let $y \in \mathfrak{R}^m, \alpha \in \mathfrak{R}$ be a hyperplane that separates b from S :
 $y^T b < \alpha, y^T s \geq \alpha$ for all $s \in S$.
- $0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$
- $y^T A x \geq \alpha$ for all $x \geq 0 \Rightarrow y^T A \geq 0$ since x can be arbitrarily large. ■

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Another theorem of the alternative

Corollary. For $A \in \mathfrak{R}^{m \times n}$, $b \in \mathfrak{R}^m$ exactly one of the following two systems holds:

$$\text{(I) } \begin{aligned} \exists x \in \mathfrak{R}^n \\ \text{s. t. } Ax &\leq b \\ x &\geq 0 \end{aligned}$$

$$\text{(II) } \begin{aligned} \exists y \in \mathfrak{R}^m \\ \text{s. t. } A^T y &\geq 0 \\ y^T b &< 0 \\ y &\geq 0 \end{aligned}$$

Pf. Apply Farkas' lemma to:

$$\text{(I')} \begin{aligned} \exists x \in \mathfrak{R}^n, s \in \mathfrak{R}^m \\ \text{s. t. } Ax + Is &= b \\ x, s &\geq 0 \end{aligned}$$

$$\text{(II')} \begin{aligned} \exists y \in \mathfrak{R}^m \\ \text{s. t. } A^T y &\geq 0 \\ Iy &\geq 0 \\ y^T b &< 0 \end{aligned}$$

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LP strong duality

Theorem. [strong duality] For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, if (P) and (D) are nonempty then $\max = \min$.

$$(P) \quad \max c^T x \\ \text{s. t. } Ax \leq b \\ x \geq 0$$

$$(D) \quad \min y^T b \\ \text{s. t. } A^T y \geq c \\ y \geq 0$$

Pf. [max \leq min] Weak LP duality.

Pf. [min \leq max] Suppose $\max < \alpha$. We show $\min < \alpha$.

$$(I) \quad \exists x \in \mathbb{R}^n \\ \text{s. t. } Ax \leq b \\ -c^T x \leq -\alpha \\ x \geq 0$$

$$(II) \quad \exists y \in \mathbb{R}^m, z \in \mathbb{R} \\ \text{s. t. } A^T y - cz \geq 0 \\ y^T b - \alpha z < 0 \\ y, z \geq 0$$

- By definition of α , (I) infeasible \Rightarrow (II) feasible by Farkas' corollary.

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LP strong duality

$$(II) \quad \exists y \in \mathbb{R}^m, z \in \mathbb{R} \\ \text{s. t. } A^T y - cz \geq 0 \\ y^T b - \alpha z < 0 \\ y, z \geq 0$$

Let y, z be a solution to (II).

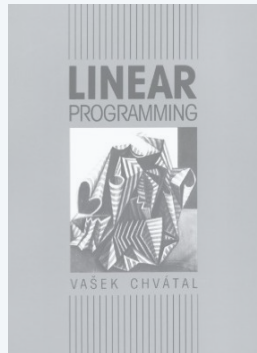
Case 1. [$z=0$]

- Then, $\{y \in \mathbb{R}^m : A^T y \geq 0, y^T b < 0, y \geq 0\}$ is feasible.
- Farkas Corollary $\Rightarrow \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ is infeasible.
- Contradiction since by assumption (P) is nonempty.

Case 2. [$z>0$]

- Scale y, z so that y satisfies (II) and $z = 1$.
- Resulting y feasible to (D) and $y^T b < \alpha$. ■

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LINEAR PROGRAMMING II

- ▶ LP duality
- ▶ strong duality theorem
- ▶ bonus proof of LP duality
- ▶ applications

Strong duality theorem

Theorem. For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, if (P) and (D) are nonempty, then $\max = \min$.

$$(P) \quad \max c^T x \\ \text{s. t. } Ax = b \\ x \geq 0$$

$$(D) \quad \min y^T b \\ \text{s. t. } A^T y \geq c$$

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Review: simplex tableaux

$$\begin{aligned} c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0 \end{aligned}$$

initial tableaux

subtract $c_B^T A_B^{-1}$ times constraints

$$\begin{aligned} (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ I x_B + A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0 \end{aligned}$$

tableaux corresponding to basis B

multiply by A_B^{-1}

Primal solution. $x_B = A_B^{-1} b \geq 0, x_N = 0$

Optimal basis. $c_N^T - c_B^T A_B^{-1} A_N \leq 0$

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Simplex tableaux: dual solution

$$\begin{aligned} c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0 \end{aligned}$$

initial tableaux

subtract $c_B^T A_B^{-1}$ times constraints

$$\begin{aligned} (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ I x_B + A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0 \end{aligned}$$

tableaux corresponding to basis B

multiply by A_B^{-1}

Primal solution. $x_B = A_B^{-1} b \geq 0, x_N = 0$

Optimal basis. $c_N^T - c_B^T A_B^{-1} A_N \leq 0$

Dual solution. $y^T = c_B^T A_B^{-1}$

$$\begin{aligned} y^T b &= c_B^T A_B^{-1} b \\ &= c_B^T x_B + c_B^T x_B \\ &= c^T x \end{aligned}$$

min \leq max

$$\begin{aligned} y^T A &= \begin{bmatrix} y^T A_B & y^T A_N \end{bmatrix} \\ &= \begin{bmatrix} c_B^T A_B^{-1} A_B & c_B^T A_B^{-1} A_N \end{bmatrix} \\ &= \begin{bmatrix} c_B^T & c_B^T A_B^{-1} A_N \end{bmatrix} \\ &\geq \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} \\ &= c^T \end{aligned}$$

dual feasible

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Simplex algorithm: LP duality

$$\begin{aligned} c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0 \end{aligned}$$

initial tableaux

subtract $c_B^T A_B^{-1}$ times constraints

$$\begin{aligned} (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ I x_B + A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0 \end{aligned}$$

tableaux corresponding to basis B

multiply by A_B^{-1}

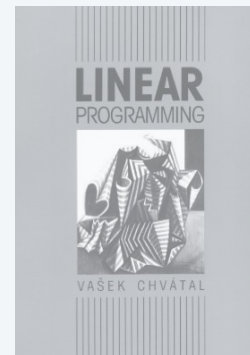
Primal solution. $x_B = A_B^{-1} b \geq 0, x_N = 0$

Optimal basis. $c_N^T - c_B^T A_B^{-1} A_N \leq 0$

Dual solution. $y^T = c_B^T A_B^{-1}$

Simplex algorithm yields **constructive** proof of LP duality.

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LINEAR PROGRAMMING II

- ▶ LP duality
- ▶ strong duality theorem
- ▶ alternate proof of LP duality
- ▶ applications

LP duality: economic interpretation

Brewer: find optimal mix of beer and ale to maximize profits.

$$\begin{array}{ll} \text{(P) max} & 13A + 23B \\ \text{s. t.} & 5A + 15B \leq 480 \\ & 4A + 4B \leq 160 \\ & 35A + 20B \leq 1190 \\ & A, B \geq 0 \end{array} \quad \begin{array}{l} A^* = 12 \\ B^* = 28 \\ OPT = 800 \end{array}$$

Entrepreneur: buy individual resources from brewer at min cost.

$$\begin{array}{ll} \text{(D) min} & 480C + 160H + 1190M \\ \text{s. t.} & 5C + 4H + 35M \geq 13 \\ & 15C + 4H + 20M \geq 23 \\ & C, H, M \geq 0 \end{array} \quad \begin{array}{l} C^* = 1 \\ H^* = 2 \\ M^* = 0 \\ OPT = 800 \end{array}$$

LP duality. Market clears.

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LP duality: sensitivity analysis

Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?

A. corn \$1, hops \$2, malt \$0.

Q. Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?

A. At least $2 (\$1) + 5 (\$2) + 24 (\$0) = \12 / barrel.

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LP is in $\text{NP} \cap \text{co-NP}$

LP. For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, does there exist $x \in \mathbb{R}^n$ such that: $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Theorem. LP is in $\text{NP} \cap \text{co-NP}$.

Pf.

- Already showed LP is in **NP**.
- If LP is infeasible, then apply Farkas' lemma to get certificate of infeasibility:

$$\begin{array}{ll} \text{(II) } \exists y \in \mathbb{R}^m, z \in \mathbb{R} \\ \text{s. t.} & A^T y \geq 0 \\ & y^T b - \alpha z < 0 \\ & z \geq 0 \end{array} \quad \begin{array}{l} \text{or equivalently,} \\ y^T b - \alpha z = -1 \end{array}$$

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