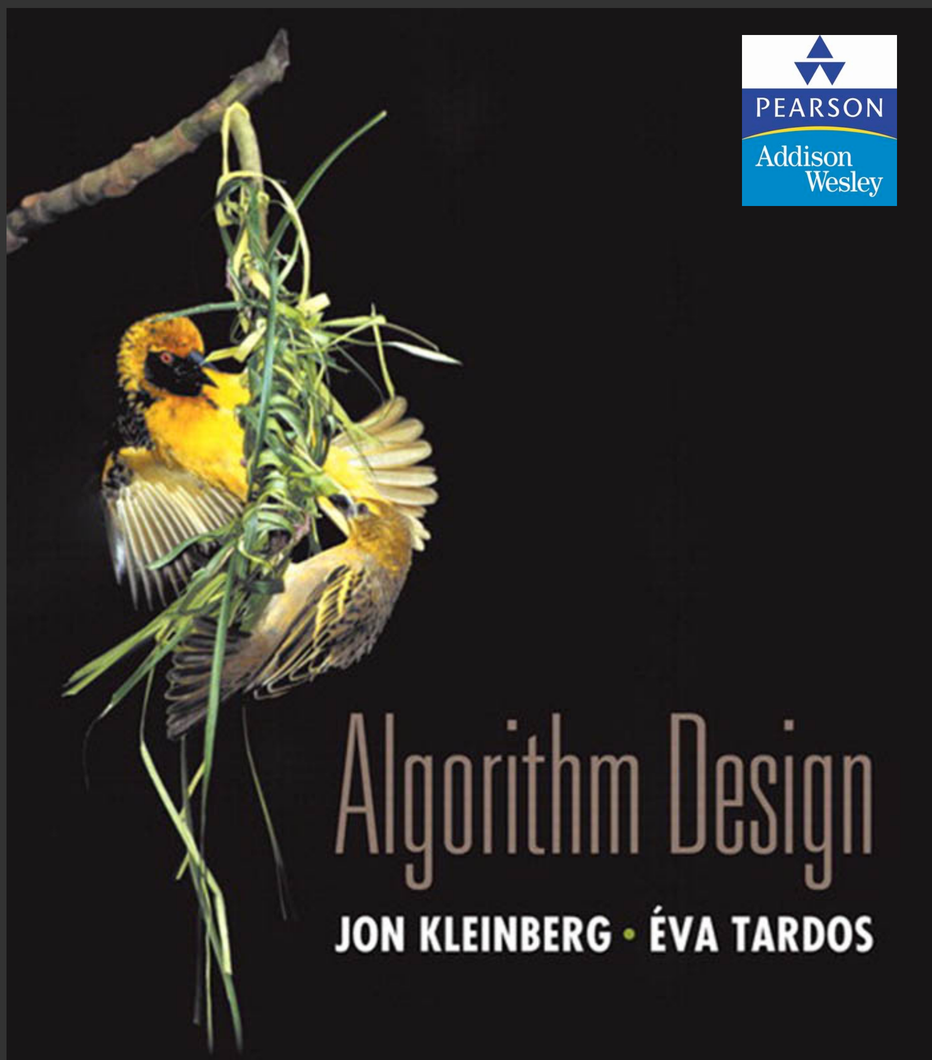
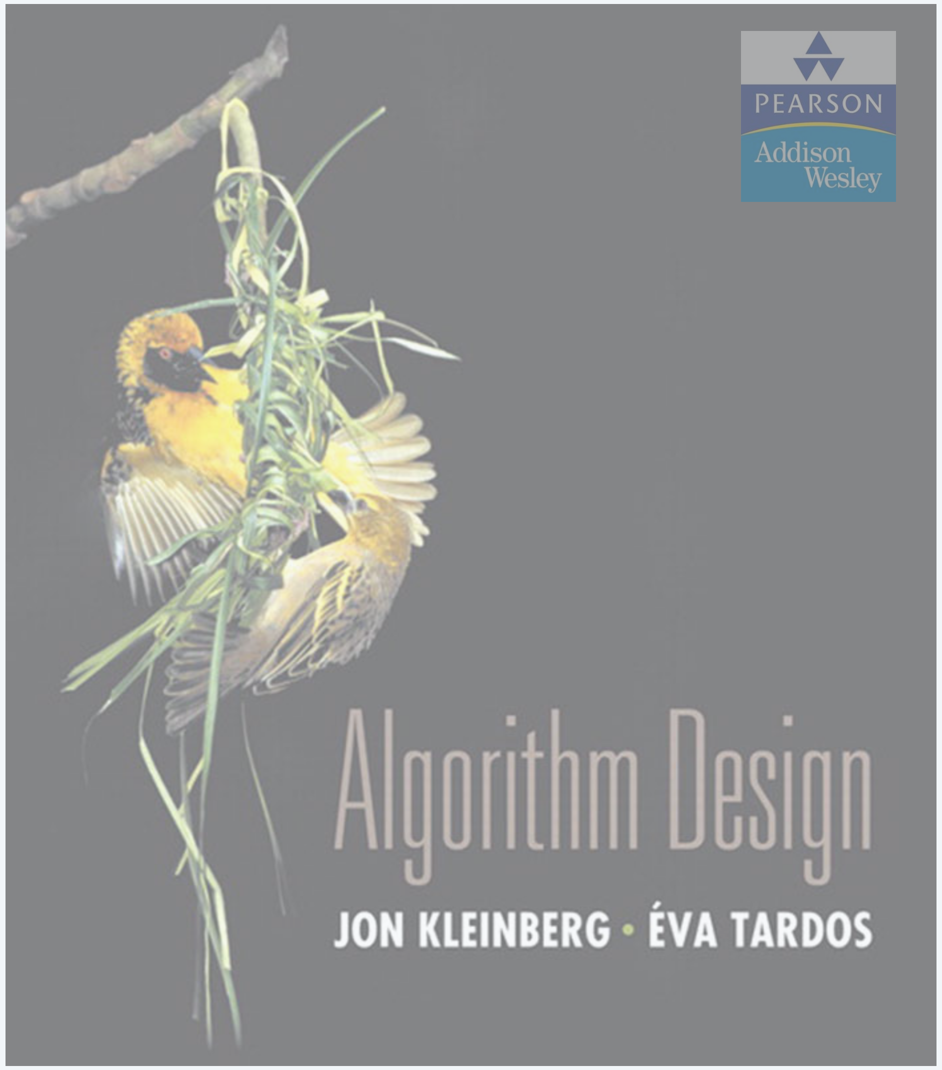


Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

8. Intractability I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*



SECTION 8.1

8. INTRACTABILITY I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*

Algorithm design patterns and antipatterns

Algorithm design patterns.

- Greedy.
- Divide and conquer.
- Dynamic programming.
- Duality.
- Reductions.
- Local search.
- Randomization.

Algorithm design antipatterns.

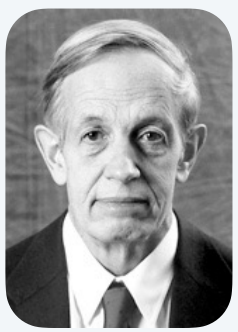
- **NP**-completeness. $O(n^k)$ algorithm unlikely.
- **PSPACE**-completeness. $O(n^k)$ certification algorithm unlikely.
- Undecidability. No algorithm possible.

Classify problems according to computational requirements

Q. Which problems will we be able to solve in practice?

A working definition. Those with poly-time algorithms.

von Neumann
(1953)

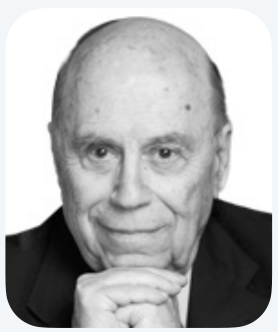


Nash
(1955)

Gödel
(1956)

Cobham
(1964)

Edmonds
(1965)



Rabin
(1966)

Turing machine, word RAM, uniform circuits, …

Theory. Definition is broad and robust.

constants tend to be small, e.g., $3n^2$

Practice. Poly-time algorithms scale to huge problems.

Classify problems according to computational requirements

Q. Which problems will we be able to solve in practice?

A working definition. Those with poly-time algorithms.

yes	probably no
shortest path	longest path
min cut	max cut
2-satisfiability	3-satisfiability
planar 4-colorability	planar 3-colorability
bipartite vertex cover	vertex cover
matching	3d-matching
primality testing	factoring
linear programming	integer linear programming

Classify problems

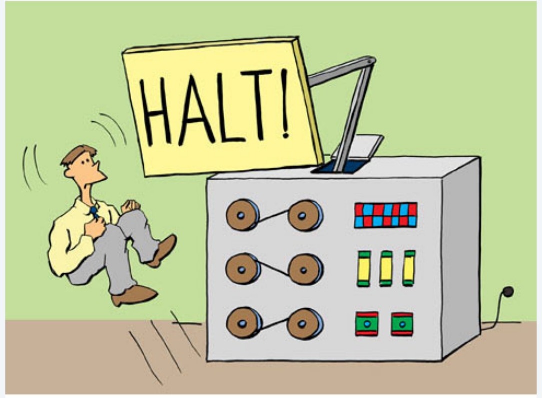
Desiderata. Classify problems according to those that can be solved in polynomial time and those that cannot.

Provably requires exponential time.

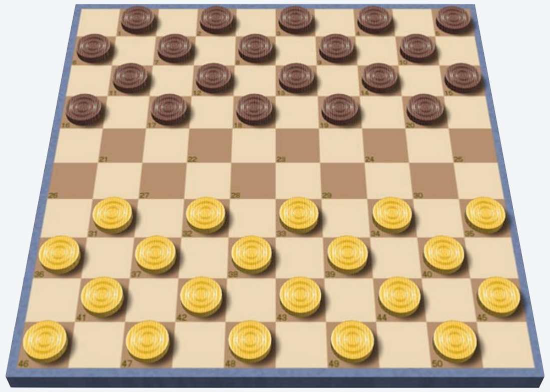
input size = $c + \log k$

- Given a constant-size program, does it halt in at most k steps?
- Given a board position in an n-by-n generalization of checkers, can black guarantee a win?

using forced capture rule



Alan designed the perfect computer



Frustrating news. Huge number of fundamental problems have defied classification for decades.

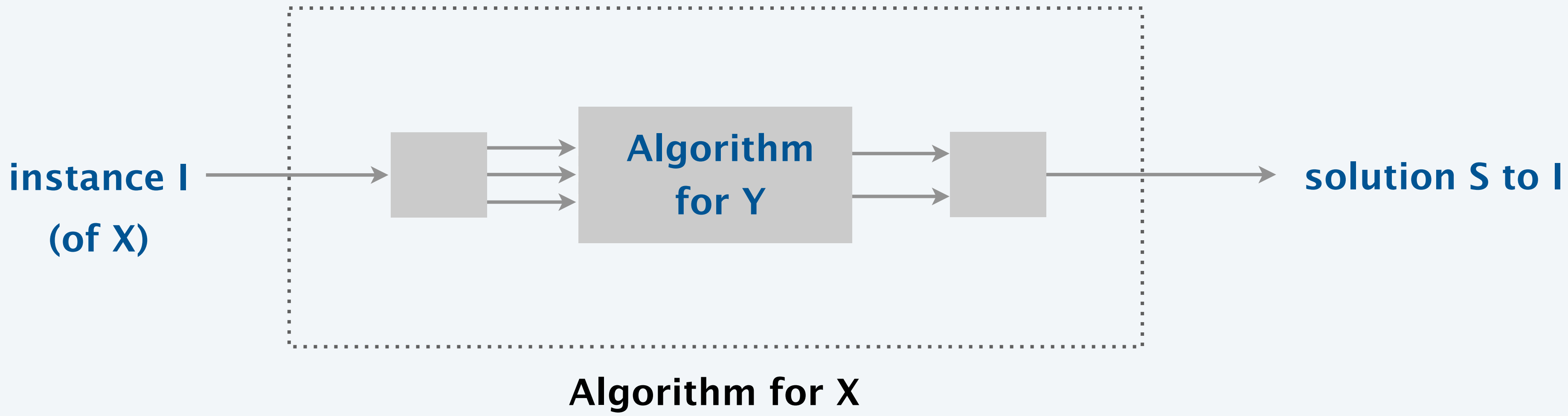
Poly-time reductions

Desiderata′. Suppose we could solve problem Y in polynomial time. What else could we solve in polynomial time?

Reduction. Problem X polynomial-time (Cook) reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

computational model supplemented by special piece of hardware that solves instances of Y in a single step



Poly-time reductions

Desiderata′. Suppose we could solve problem Y in polynomial time. What else could we solve in polynomial time?

Reduction. Problem X polynomial-time (Cook) reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Notation. $X \leq_P Y$.

Note. We pay for time to write down instances of Y sent to oracle $\Rightarrow$ instances of Y must be of polynomial size.

Novice mistake. Confusing $X \leq_P Y$ with $Y \leq_P X$.

Intractability: quiz 1

Suppose that $X \le_P Y$. Which of the following can we infer?

A. If X can be solved in polynomial time, then so can Y.

B. X can be solved in poly time iff Y can be solved in poly time.

C. If X cannot be solved in polynomial time, then neither can Y.

D. If Y cannot be solved in polynomial time, then neither can X.

Which of the following poly-time reductions are known?

A. FIND-MAX-FLOW $\le_P$ FIND-MIN-CUT.

B. FIND-MIN-CUT $\le_P$ FIND-MAX-FLOW.

C. Both A and B.

D. Neither A nor B.

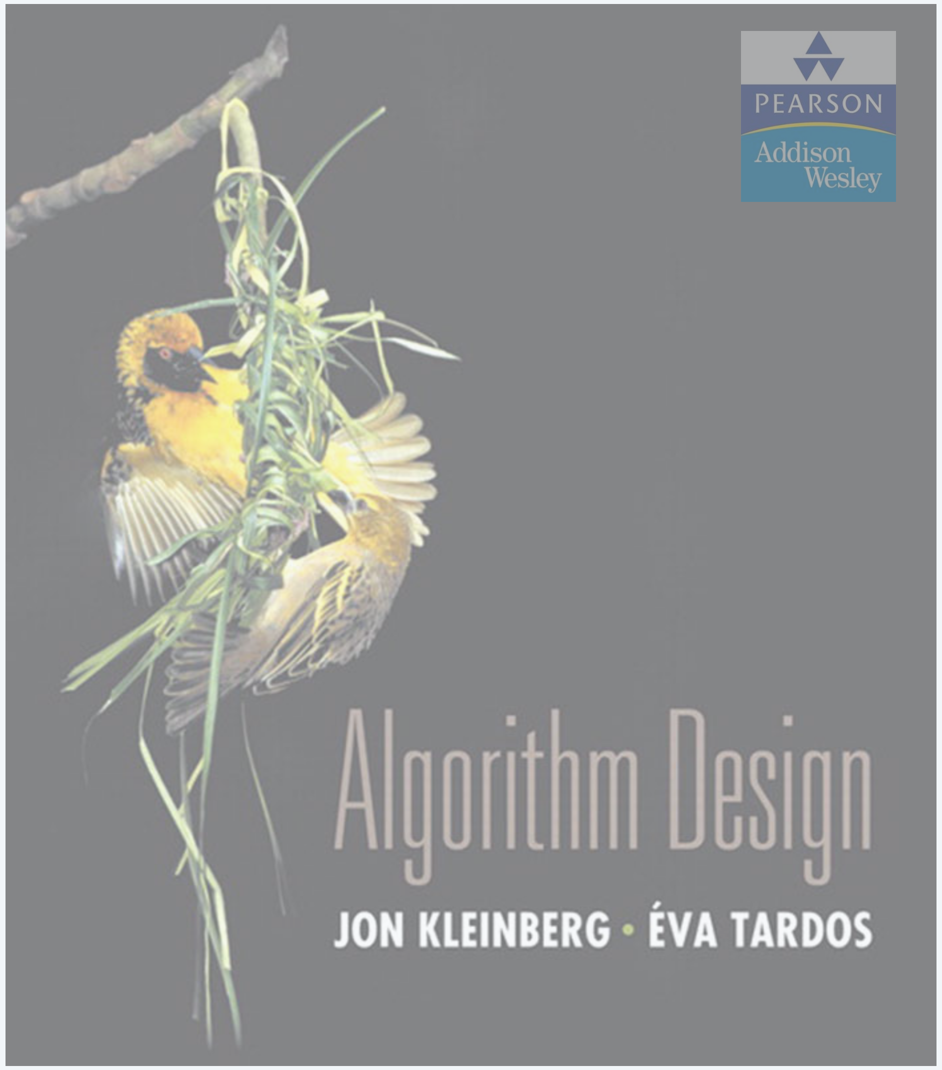
Poly-time reductions

Design algorithms. If $X \leq_P Y$ and Y can be solved in polynomial time, then X can be solved in polynomial time.

Establish intractability. If $X \leq_P Y$ and X cannot be solved in polynomial time, then Y cannot be solved in polynomial time.

Establish equivalence. If both $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$. In this case, X can be solved in polynomial time iff Y can be.

Bottom line. Reductions classify problems according to relative difficulty.



SECTION 8.1

8. INTRACTABILITY I

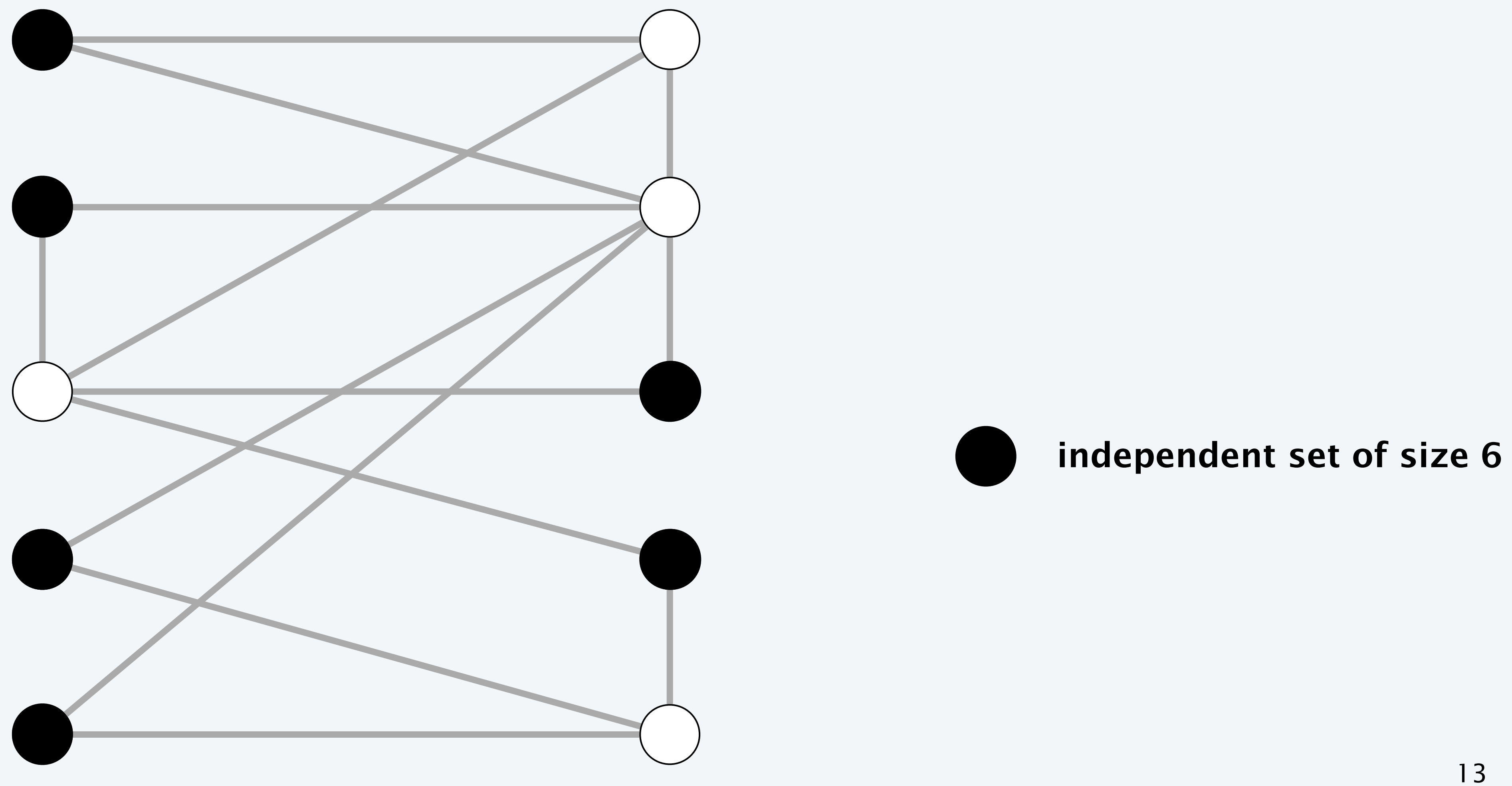
- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*

Independent set

INDEPENDENT-SET. Given a graph $G = (V, E)$ and an integer k, is there a subset of k (or more) vertices such that no two are adjacent?

Ex. Is there an independent set of size ≥ 6 ?

Ex. Is there an independent set of size ≥ 7 ?

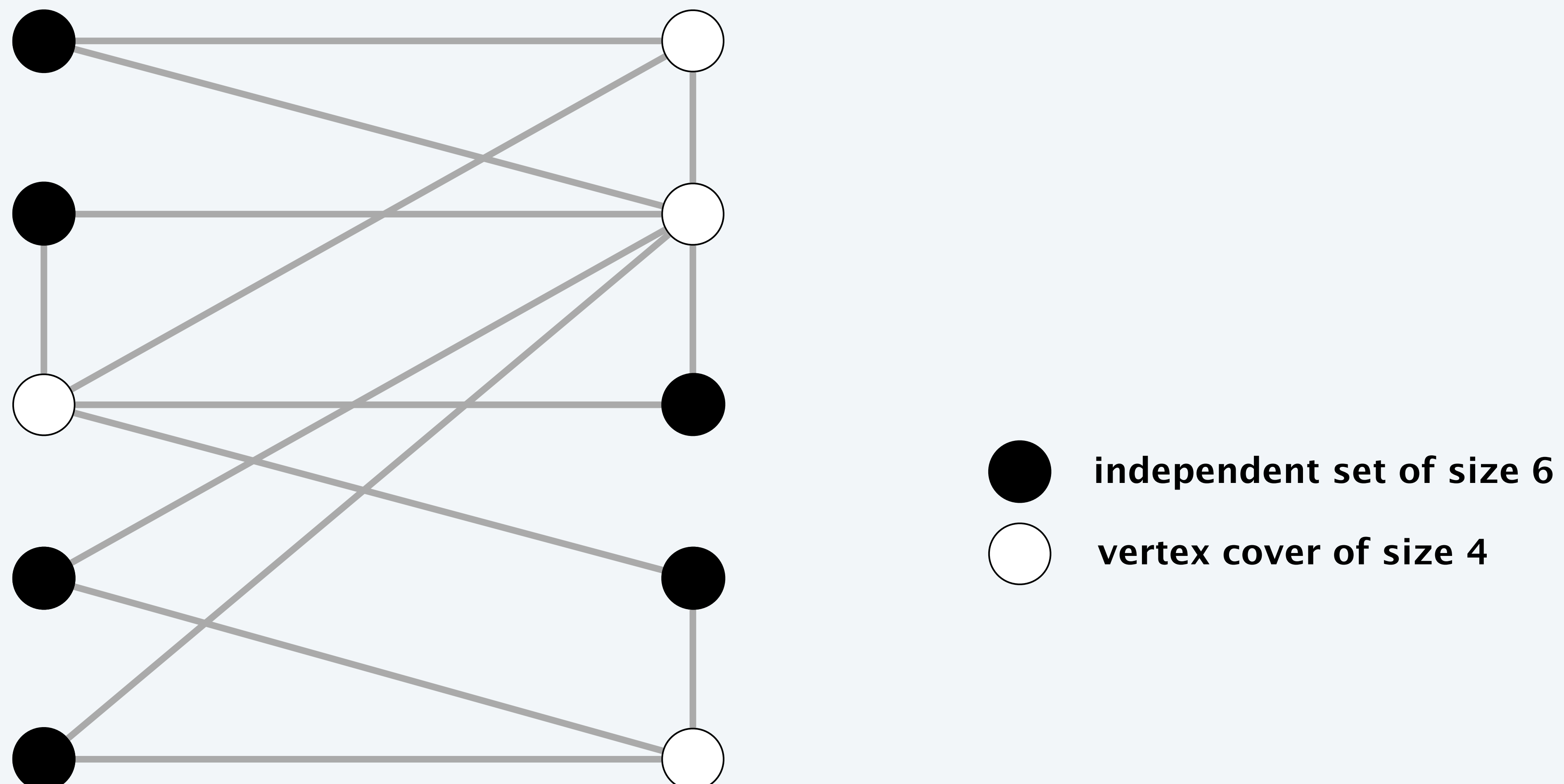


Vertex cover

VERTEX-COVER. Given a graph $G = (V, E)$ and an integer k, is there a subset of k (or fewer) vertices such that each edge is incident to at least one vertex in the subset?

Ex. Is there a vertex cover of size ≤ 4 ?

Ex. Is there a vertex cover of size ≤ 3 ?



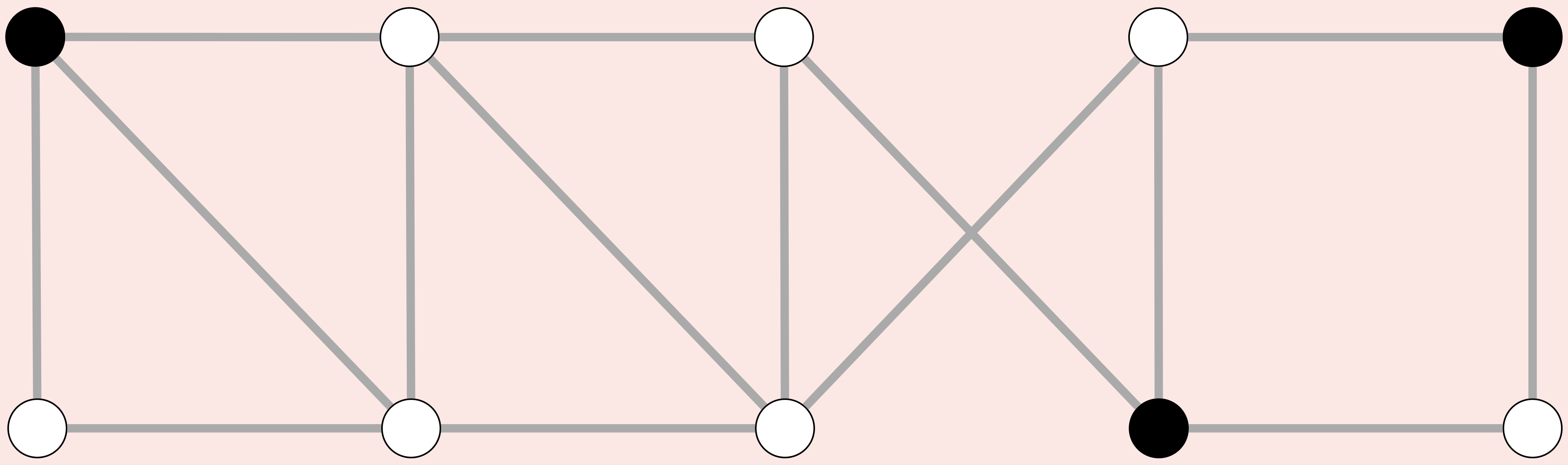
Consider the following graph G. Which are true?

A. The white vertices are a vertex cover of size 7.

B. The black vertices are an independent set of size 3.

C. Both A and B.

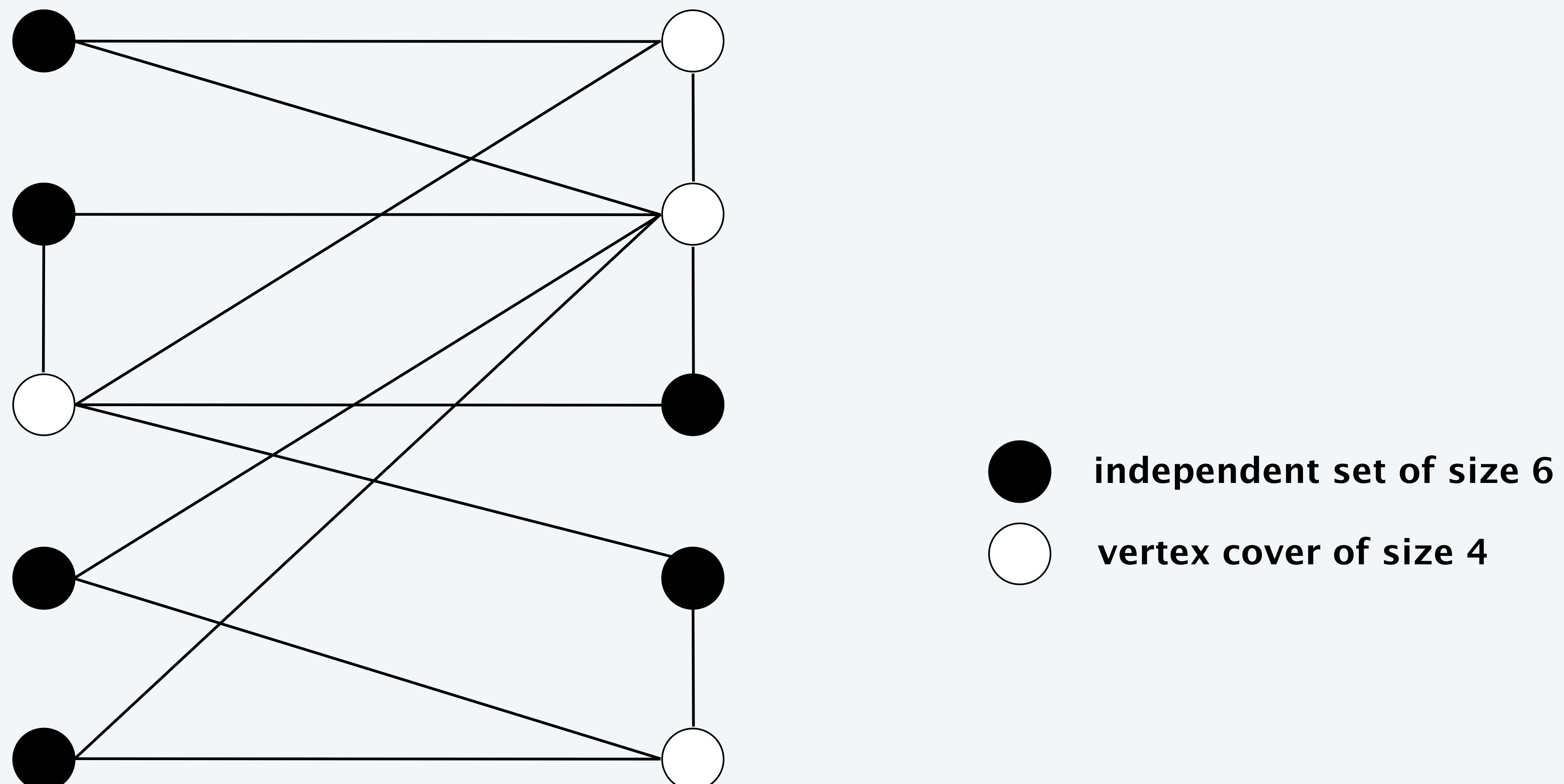
D. Neither A nor B.



Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.



Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.

$\Rightarrow$

- Let S be any independent set of size k.
- $V - S$ is of size $n - k$.
- Consider an arbitrary edge $(u, v) \in E$.
- S independent $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
  $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
- Thus, $V - S$ covers (u, v). ▪

Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.

$\Leftarrow$

- Let $V - S$ be any vertex cover of size $n - k$.
- S is of size k.
- Consider an arbitrary edge $(u, v) \in E$.
- $V - S$ is a vertex cover $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
  $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
- Thus, S is an independent set. ▪

Set cover

SET-COVER. Given a set U of elements, a collection S of subsets of U, and an integer k, are there $\leq k$ of these subsets whose union is equal to U?

Sample application.

- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The i^{th} piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad S_f = \{ 1, 2, 6, 7 \}$

$k = 2$

a set cover instance

Given the universe U = { 1, 2, 3, 4, 5, 6, 7 } and the following sets, which is the minimum size of a set cover?

- **A.** 1
- **B.** 2
- **C.** 3
- **D.** None of the above.

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 1, 4, 6 \}$ $S_b = \{ 1, 6, 7 \}$

$S_c = \{ 1, 2, 3, 6 \}$ $S_d = \{ 1, 3, 5, 7 \}$

$S_e = \{ 2, 6, 7 \}$ $S_f = \{ 3, 4, 5 \}$

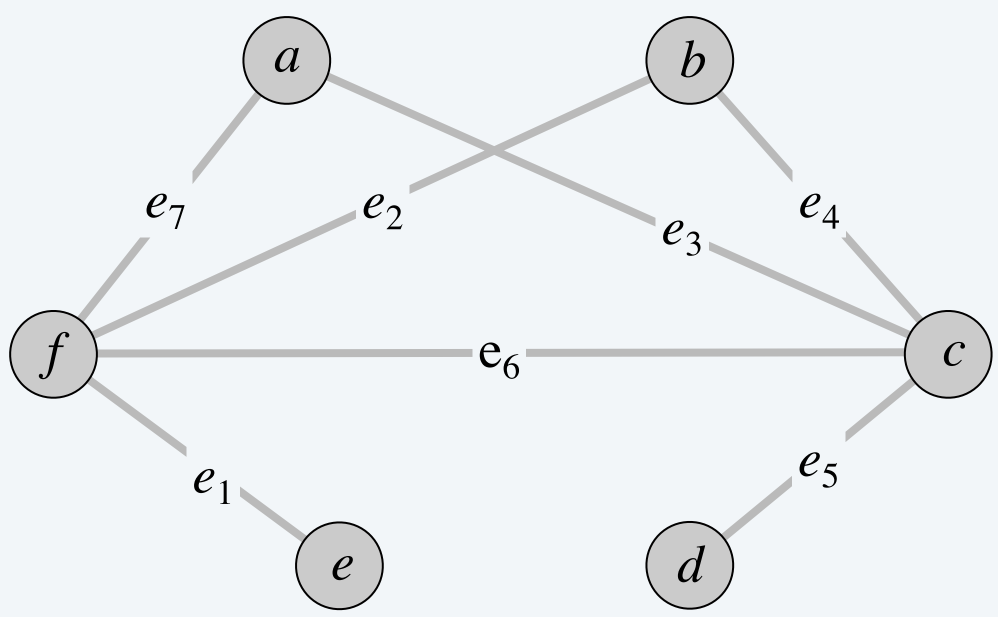
Vertex cover reduces to set cover

Theorem. VERTEX-COVER $\le_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance $G = (V, E)$ and k, we construct a SET-COVER instance (U, S, k) that has a set cover of size k iff G has a vertex cover of size k.

Construction.

- Universe $U = E$.
- Include one subset for each node $v \in V : S_v = \{e \in E : e \text{ incident to } v\}$.



vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

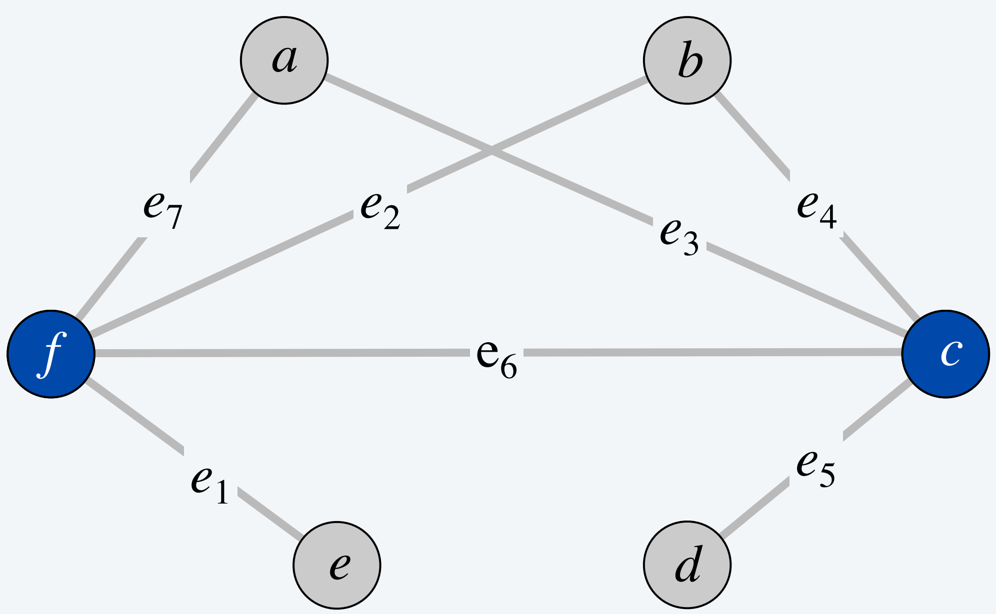
Vertex cover reduces to set cover

Lemma. $G = (V, E)$ contains a vertex cover of size k iff (U, S, k) contains a set cover of size k.

Pf. $\Rightarrow$ Let $X \subseteq V$ be a vertex cover of size k in G.

- Then $Y = \{ S_v : v \in X \}$ is a set cover of size k. ▪

"yes" instances of VERTEX-COVER are solved correctly



vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

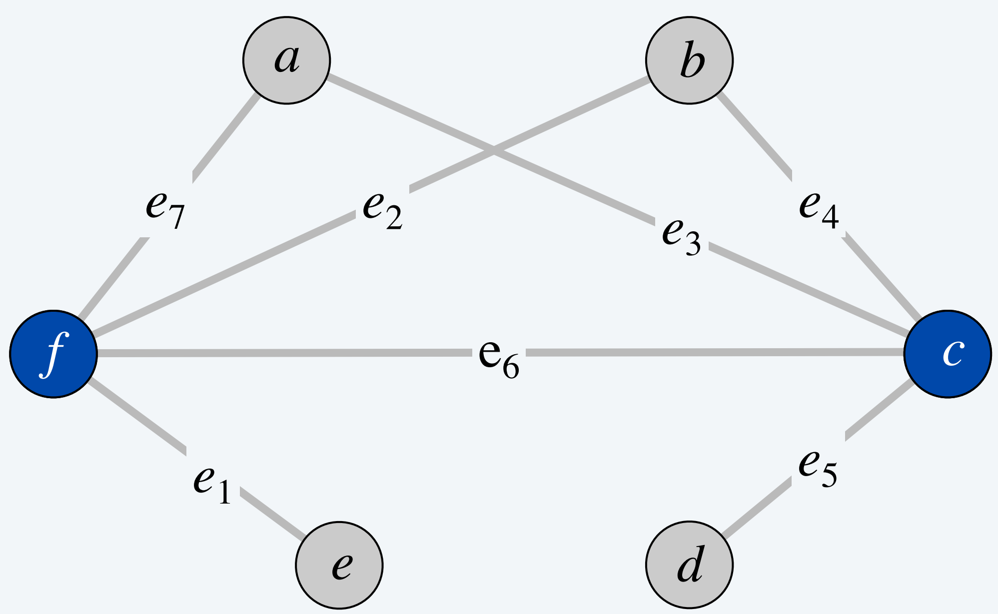
Vertex cover reduces to set cover

Lemma. $G = (V, E)$ contains a vertex cover of size k iff (U, S, k) contains a set cover of size k.

Pf. $\Leftarrow$ Let $Y \subseteq S$ be a set cover of size k in (U, S, k).

- Then $X = \{ v : S_v \in Y \}$ is a vertex cover of size k in G. ▪

"no" instances of VERTEX-COVER are solved correctly



vertex cover instance
(k = 2)

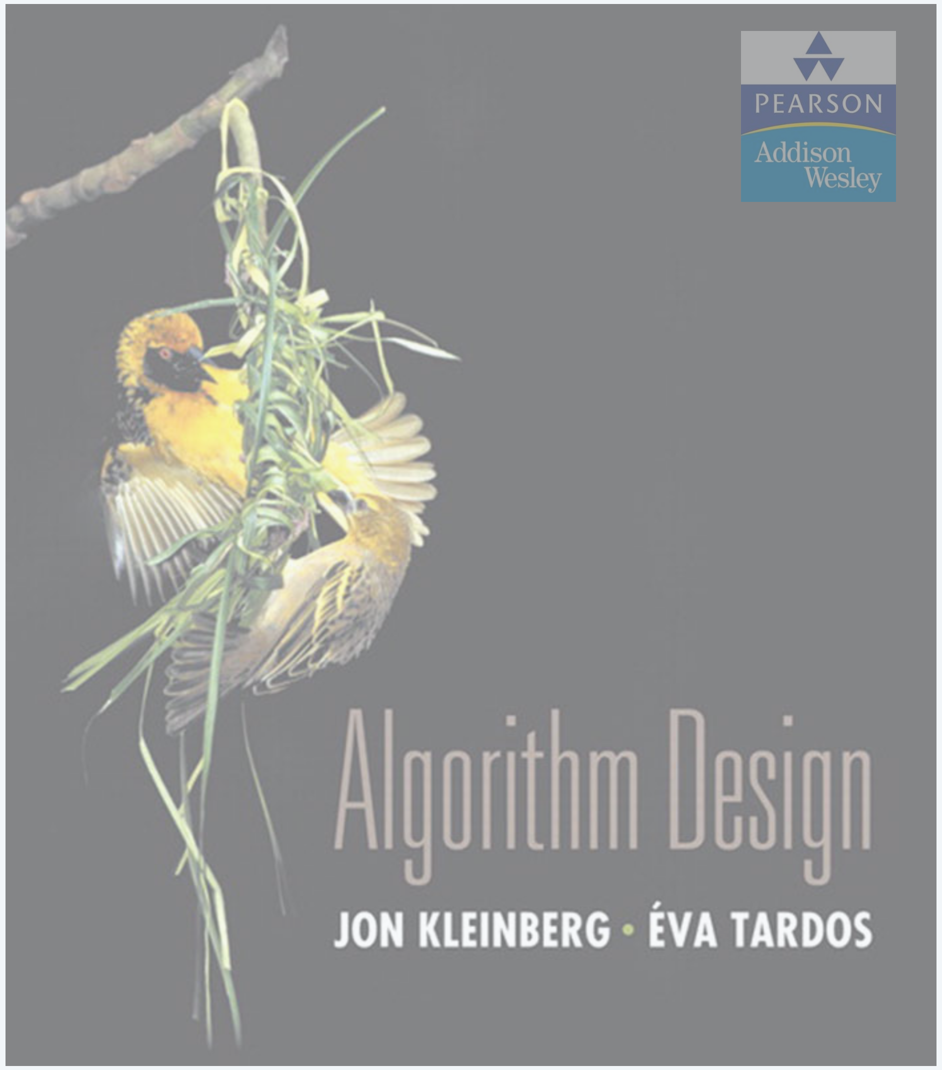
$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)



SECTION 8.2

8. INTRACTABILITY I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*

Satisfiability

Literal. A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause. A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form (CNF). A propositional formula Φ that is a conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT. Given a CNF formula Φ, does it have a satisfying truth assignment?
3-SAT. SAT where each clause contains exactly 3 literals (and each literal corresponds to a different variable).

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

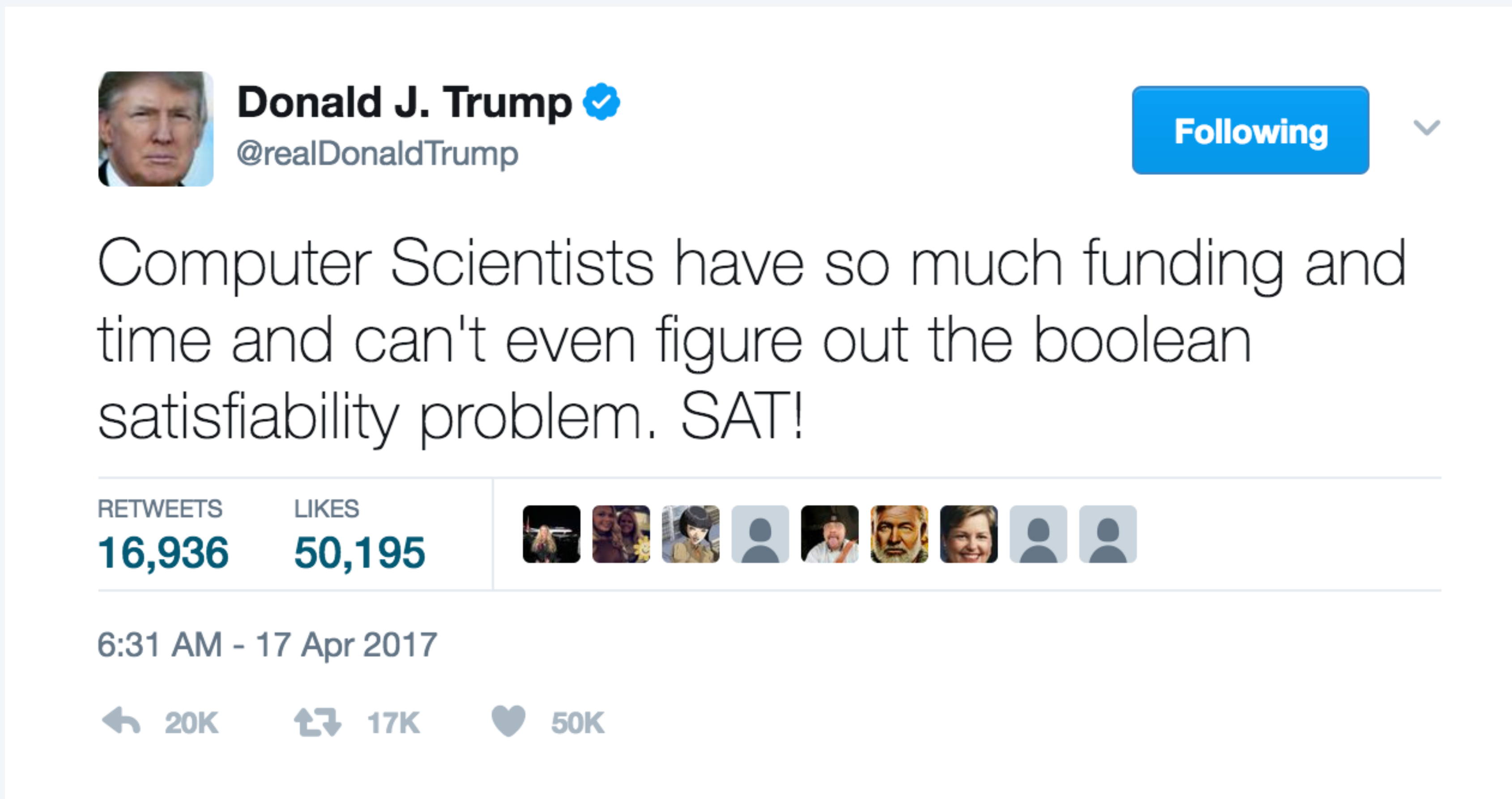
yes instance: x_1 = true, x_2 = true, x_3 = false, x_4 = false

Key application. Electronic design automation (EDA).

Satisfiability is hard

Scientific hypothesis. There does not exist a poly-time algorithm for 3-SAT.

P vs. NP. This hypothesis is equivalent to **P ≠ NP** conjecture.



https://www.facebook.com/pg/npcompleteteens

3-satisfiability reduces to independent set

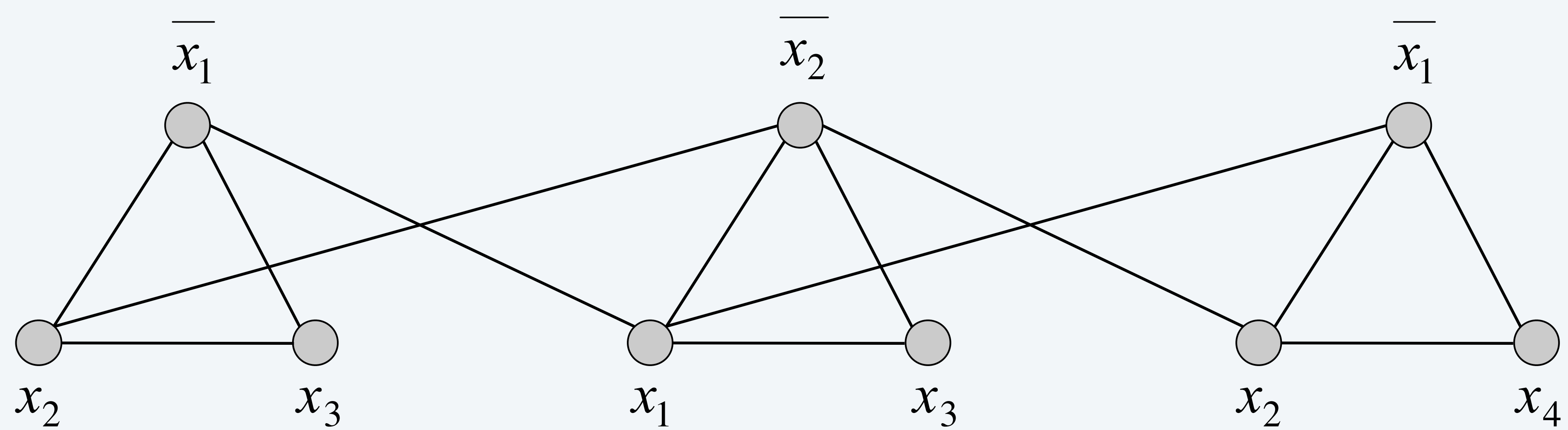
Theorem. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Construction.

- G contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3-satisfiability reduces to independent set

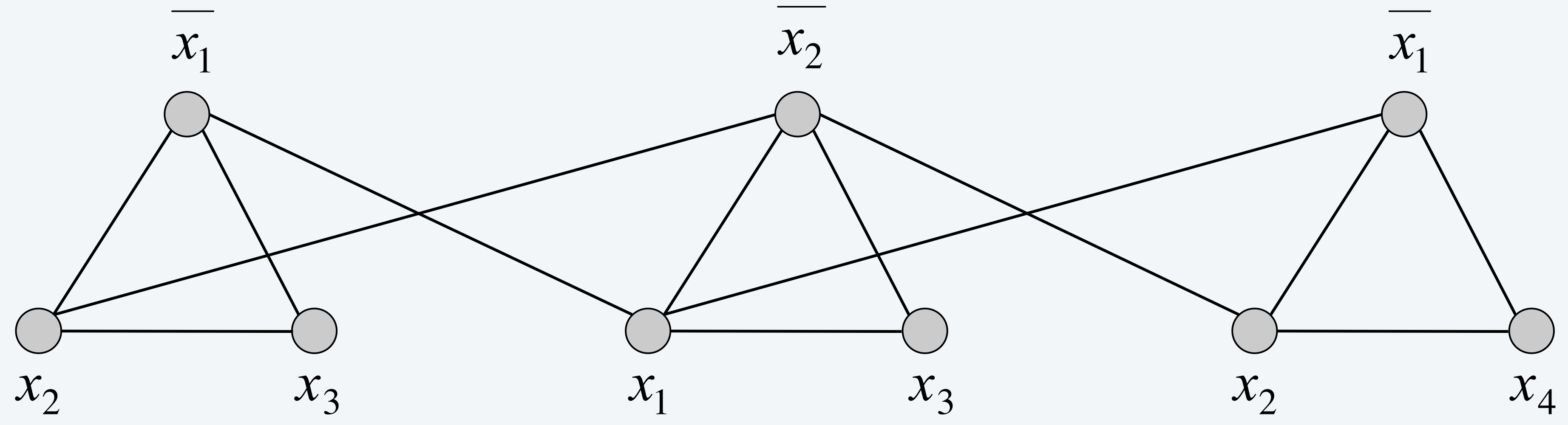
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Rightarrow$ Consider any satisfying assignment for Φ.

- Select one true literal from each clause/triangle.
- This is an independent set of size $k = |\Phi|$. ▪

"yes" instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3-satisfiability reduces to independent set

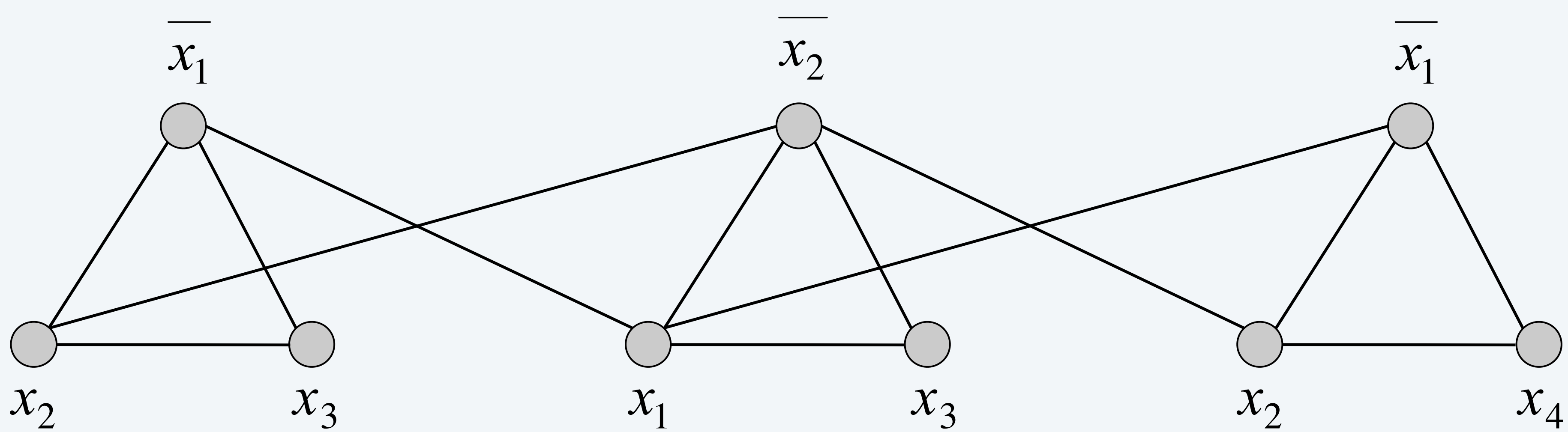
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Leftarrow$ Let S be independent set of size k.

- S must contain exactly one node in each triangle.
- Set these literals to *true* (and remaining literals consistently).
- All clauses in Φ are satisfied. ▪

"no" instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

Review

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.
Pf idea. Compose the two algorithms.

Ex. 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

DECISION, SEARCH, AND OPTIMIZATION PROBLEMS

Decision problem. Does there exist a vertex cover of size ≤ k?

Search problem. Find a vertex cover of size ≤ k.

Optimization problem. Find a vertex cover of minimum size.

Goal. Show that all three problems poly-time reduce to one another.

Search problems vs. decision problems

VERTEX-COVER. Does there exist a vertex cover of size $\leq k$?

FIND-VERTEX-COVER. Find a vertex cover of size $\leq k$.

Theorem. VERTEX-COVER $\equiv_P$ FIND-VERTEX-COVER.

Pf. $\leq_P$ Decision problem is a special case of search problem. ▪

Pf. $\geq_P$

To find a vertex cover of size $\leq k$:

- Determine if there exists a vertex cover of size $\leq k$.
- Find a vertex v such that $G - \{ v \}$ has a vertex cover of size $\leq k - 1$.
  (any vertex in any vertex cover of size $\leq k$ will have this property)
- Include v in the vertex cover.
- Recursively find a vertex cover of size $\leq k - 1$ in $G - \{ v \}$. ▪

delete v and all incident edges

OPTIMIZATION PROBLEMS VS. SEARCH PROBLEMS

FIND-VERTEX-COVER. Find a vertex cover of size ≤ k.

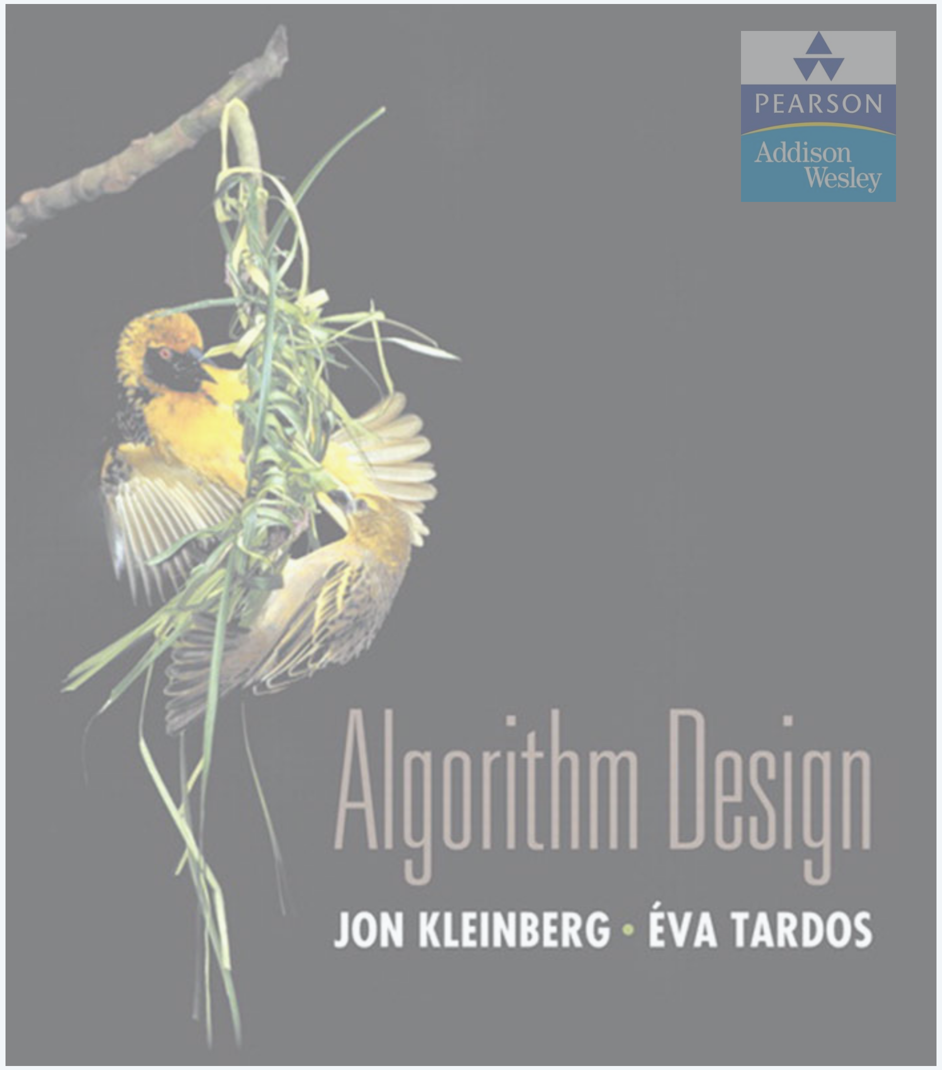
FIND-MIN-VERTEX-COVER. Find a vertex cover of minimum size.

Theorem. FIND-VERTEX-COVER $\equiv_P$ FIND-MIN-VERTEX-COVER.

Pf. $\leq_P$ Search problem is a special case of optimization problem. ▪

Pf. $\geq_P$ To find vertex cover of minimum size:

- Binary search (or linear search) for size k^* of min vertex cover.
- Solve search problem for given k^*. ▪



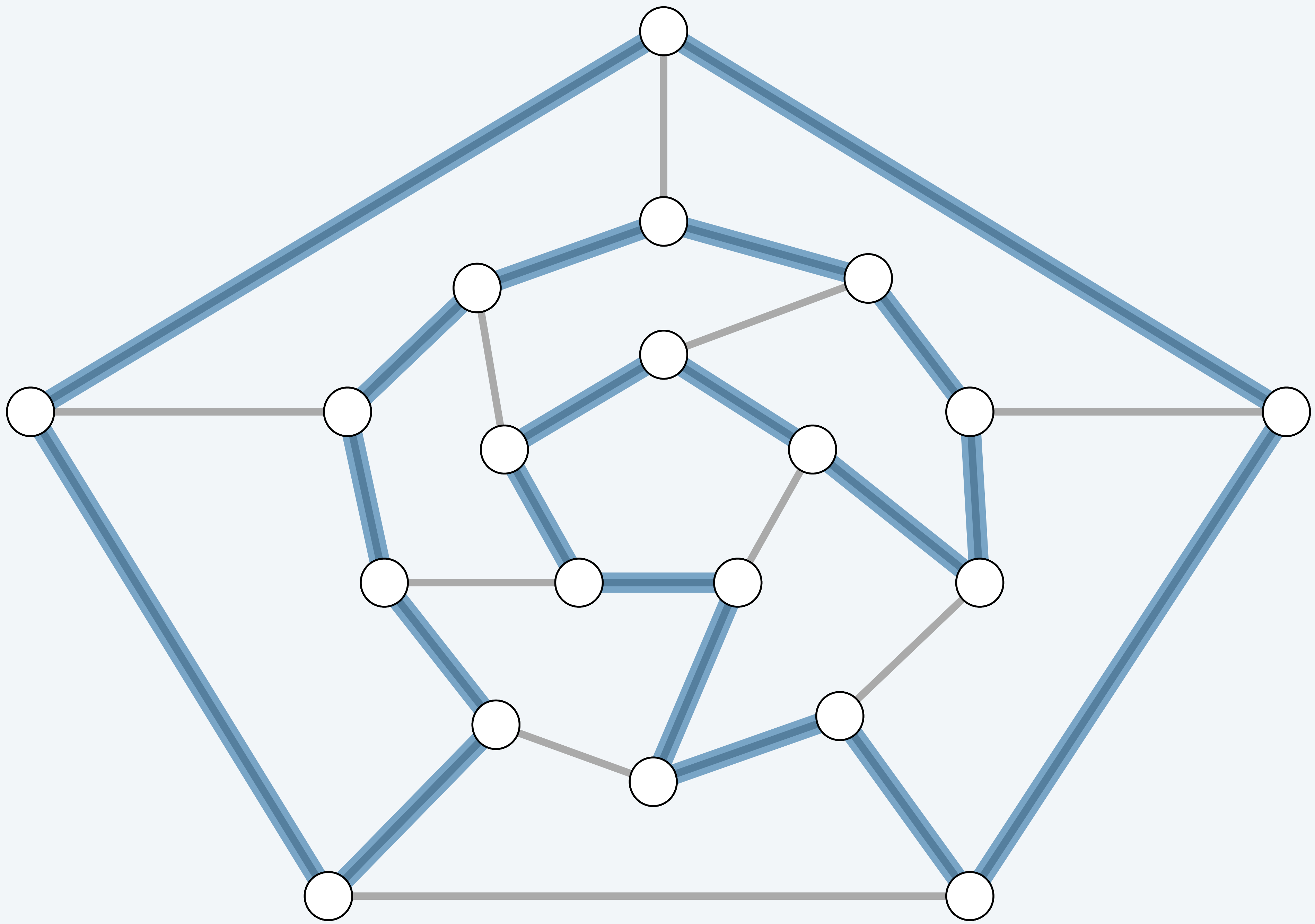
SECTION 8.5

8. INTRACTABILITY I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*

Hamilton cycle

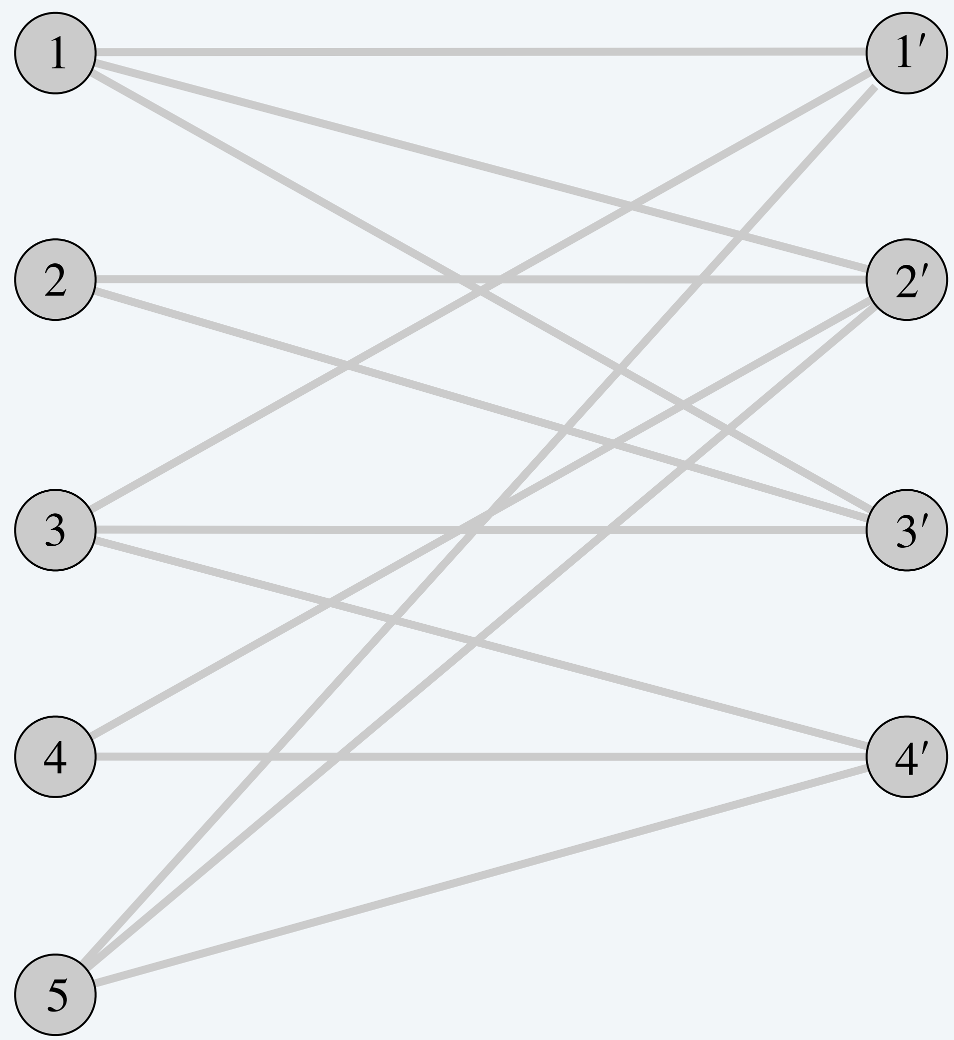
HAMILTON-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a cycle Γ that visits every node exactly once?



yes

Hamilton cycle

HAMILTON-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a cycle Γ that visits every node exactly once?



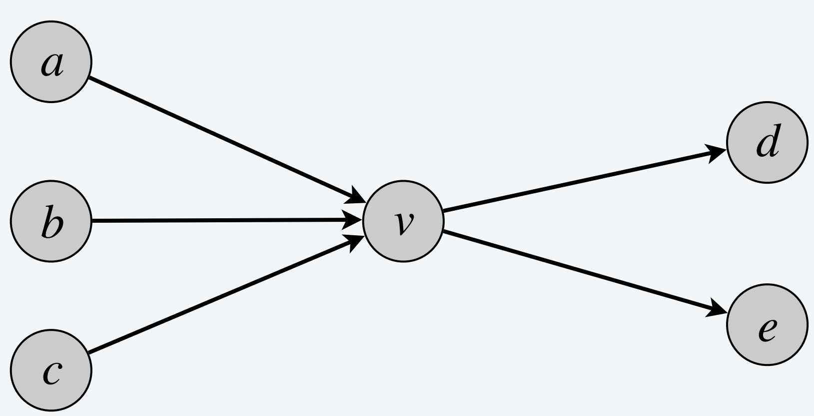
no

Directed Hamilton cycle reduces to Hamilton cycle

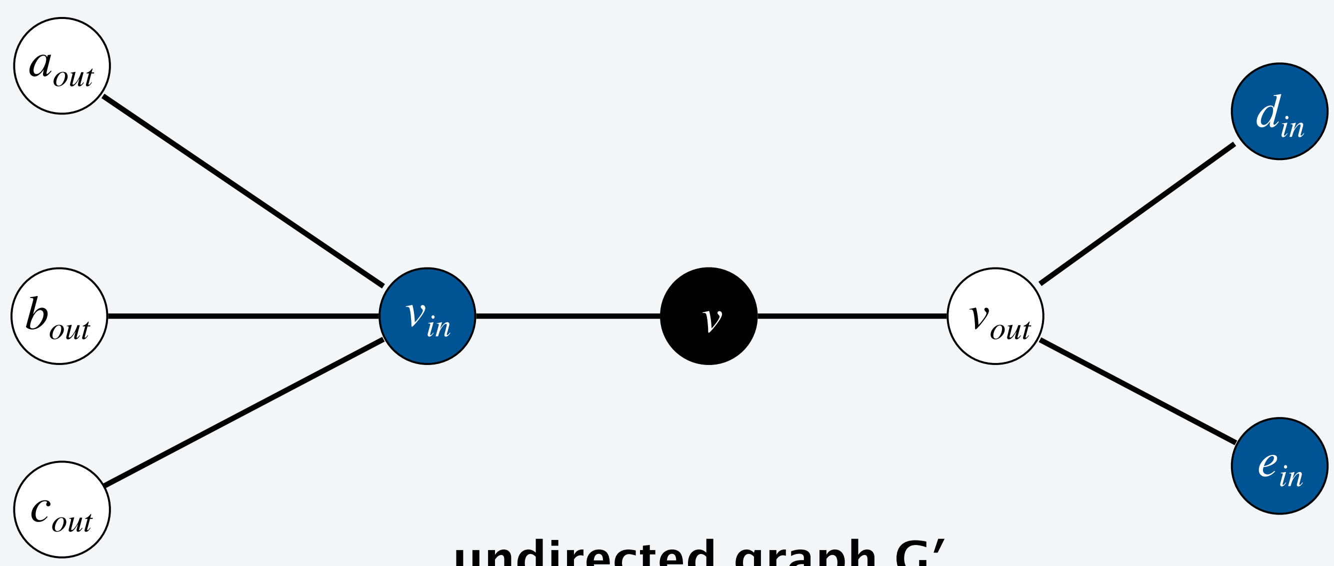
DIRECTED-HAMILTON-CYCLE. Given a directed graph $G = (V, E)$, does there exist a directed cycle Γ that visits every node exactly once?

Theorem. DIRECTED-HAMILTON-CYCLE $\le_P$ HAMILTON-CYCLE.

Pf. Given a directed graph $G = (V, E)$, construct a graph G' with $3n$ nodes.



directed graph G



undirected graph G′

Directed Hamilton cycle reduces to Hamilton cycle

Lemma. G has a directed Hamilton cycle iff G' has a Hamilton cycle.

Pf. $\Rightarrow$

- Suppose G has a directed Hamilton cycle Γ.
- Then G' has an undirected Hamilton cycle (same order). ▪

Pf. $\Leftarrow$

- Suppose G' has an undirected Hamilton cycle Γ'.
- Γ' must visit nodes in G' using one of following two orders:

  $\ldots, black, white, blue, black, white, blue, black, white, blue, \ldots$

  $\ldots, black, blue, white, black, blue, white, black, blue, white, \ldots$
- Black nodes in Γ' comprise either a directed Hamilton cycle Γ in G, or reverse of one. ▪

3-satisfiability reduces to directed Hamilton cycle

Theorem. 3-SAT $\le_P$ DIRECTED-HAMILTON-CYCLE.

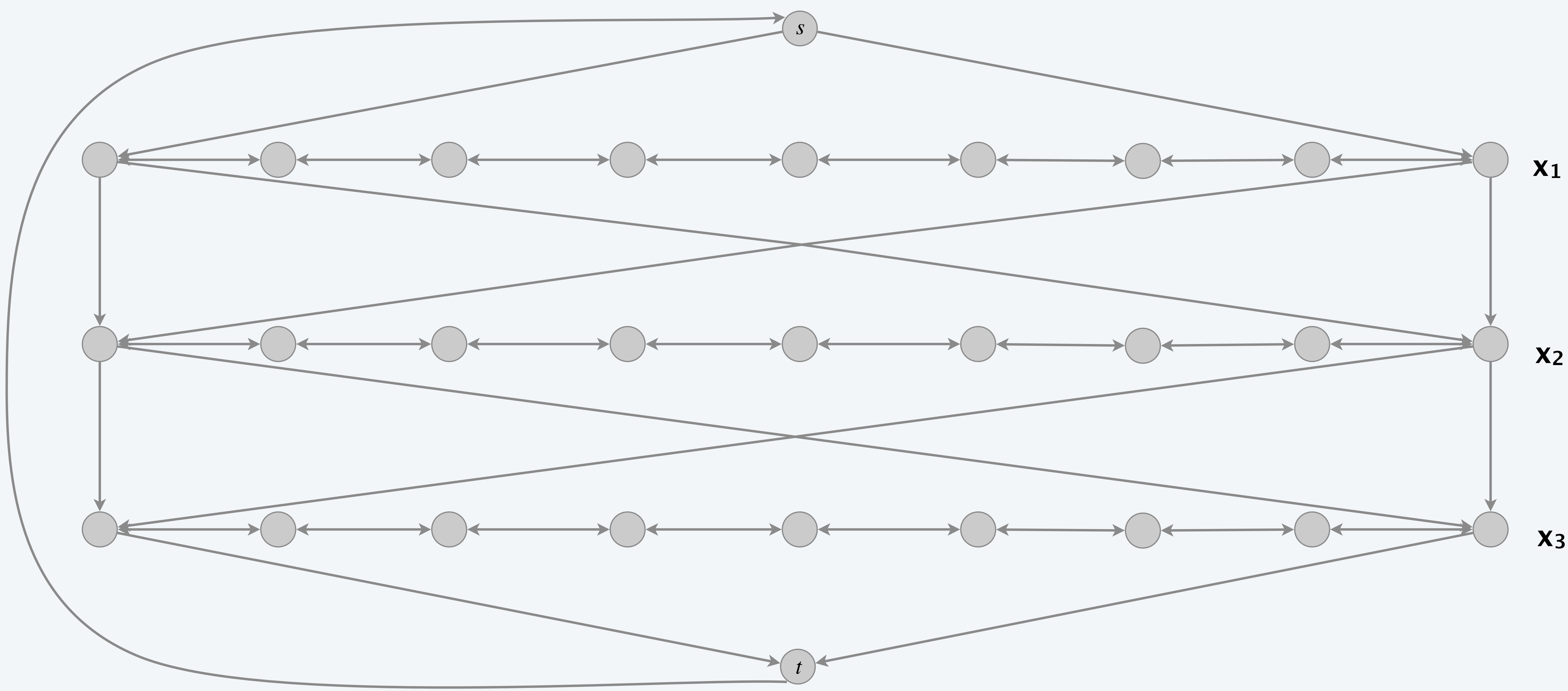
Pf. Given an instance Φ of 3-SAT, we construct an instance G of DIRECTED-HAMILTON-CYCLE that has a Hamilton cycle iff Φ is satisfiable.

Construction overview. Let n denote the number of variables in Φ. We will construct a graph G that has 2^n Hamilton cycles, with each cycle corresponding to one of the 2^n possible truth assignments.

3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

- Construct G to have 2^n Hamilton cycles.
- Intuition: traverse path i from left to right $\Leftrightarrow$ set variable $x_i = true$.



Intractability: quiz 5

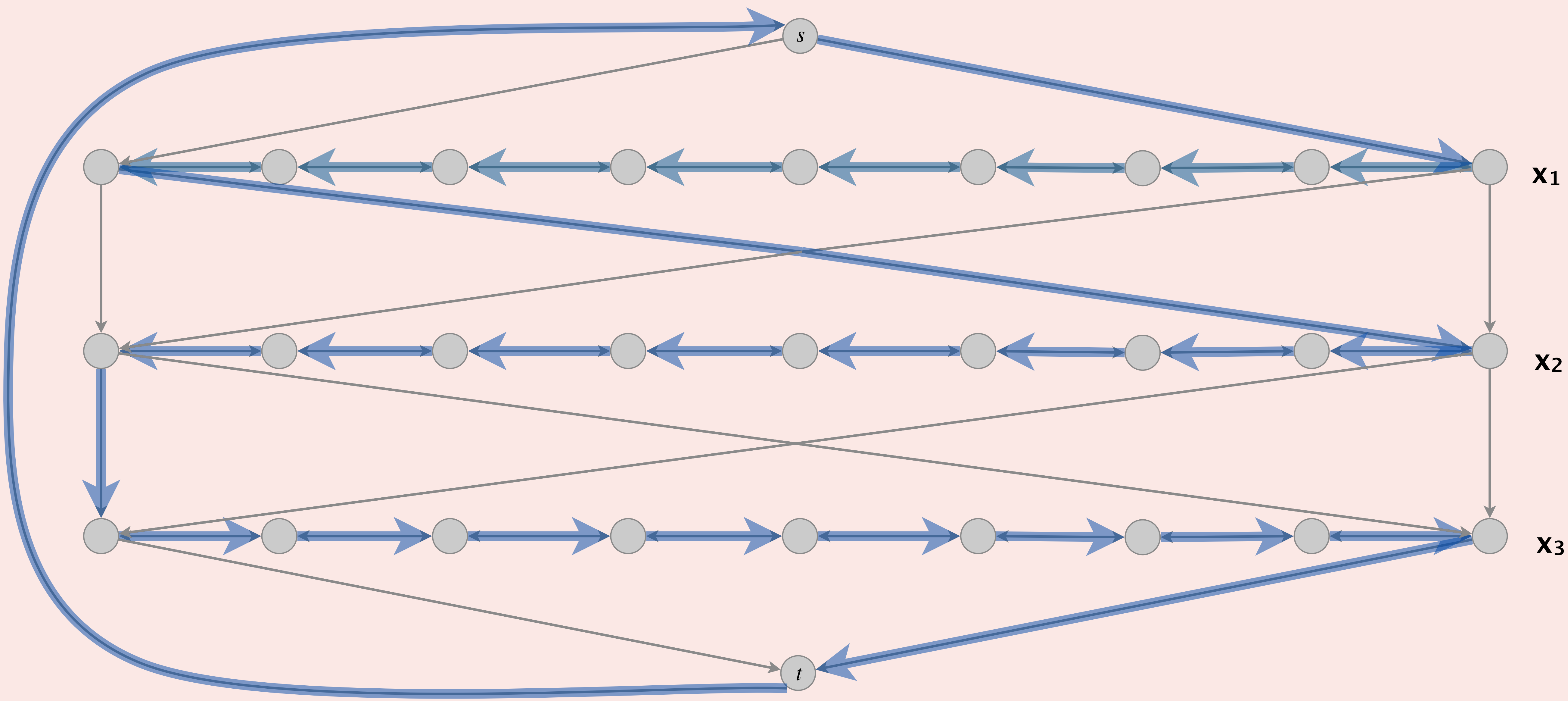
Which is truth assignment corresponding to Hamilton cycle below?

A. $x_1 = true, x_2 = true, x_3 = true$

B. $x_1 = true, x_2 = true, x_3 = false$

C. $x_1 = false, x_2 = false, x_3 = true$

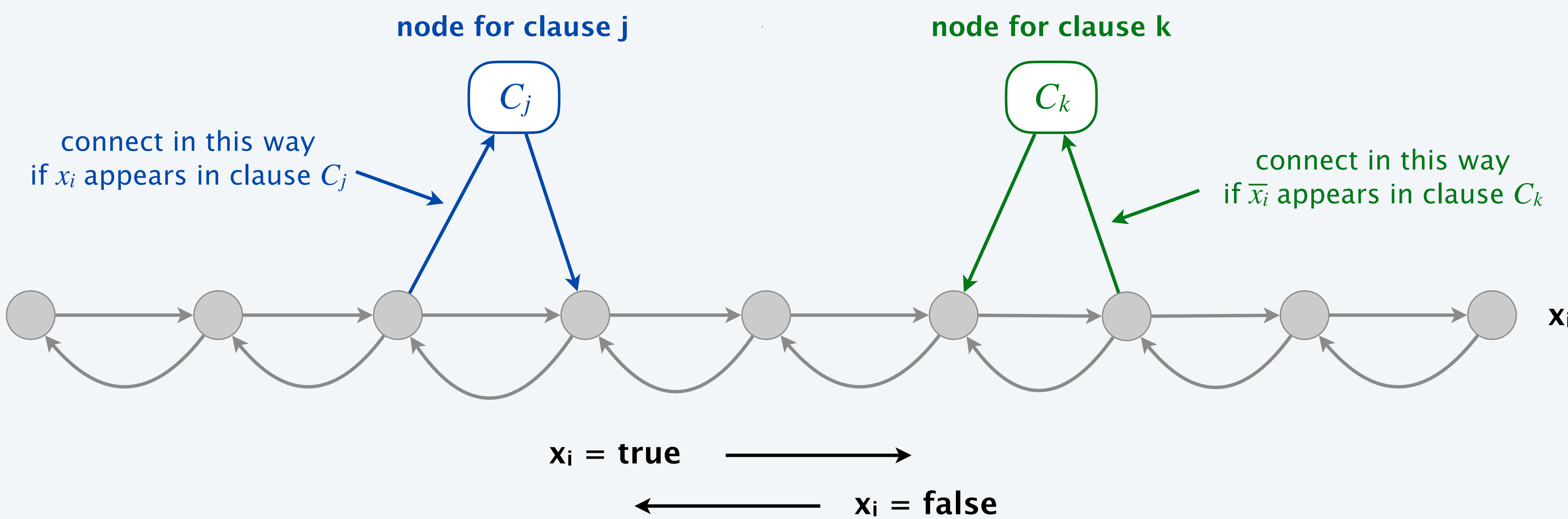
D. $x_1 = false, x_2 = false, x_3 = false$



3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

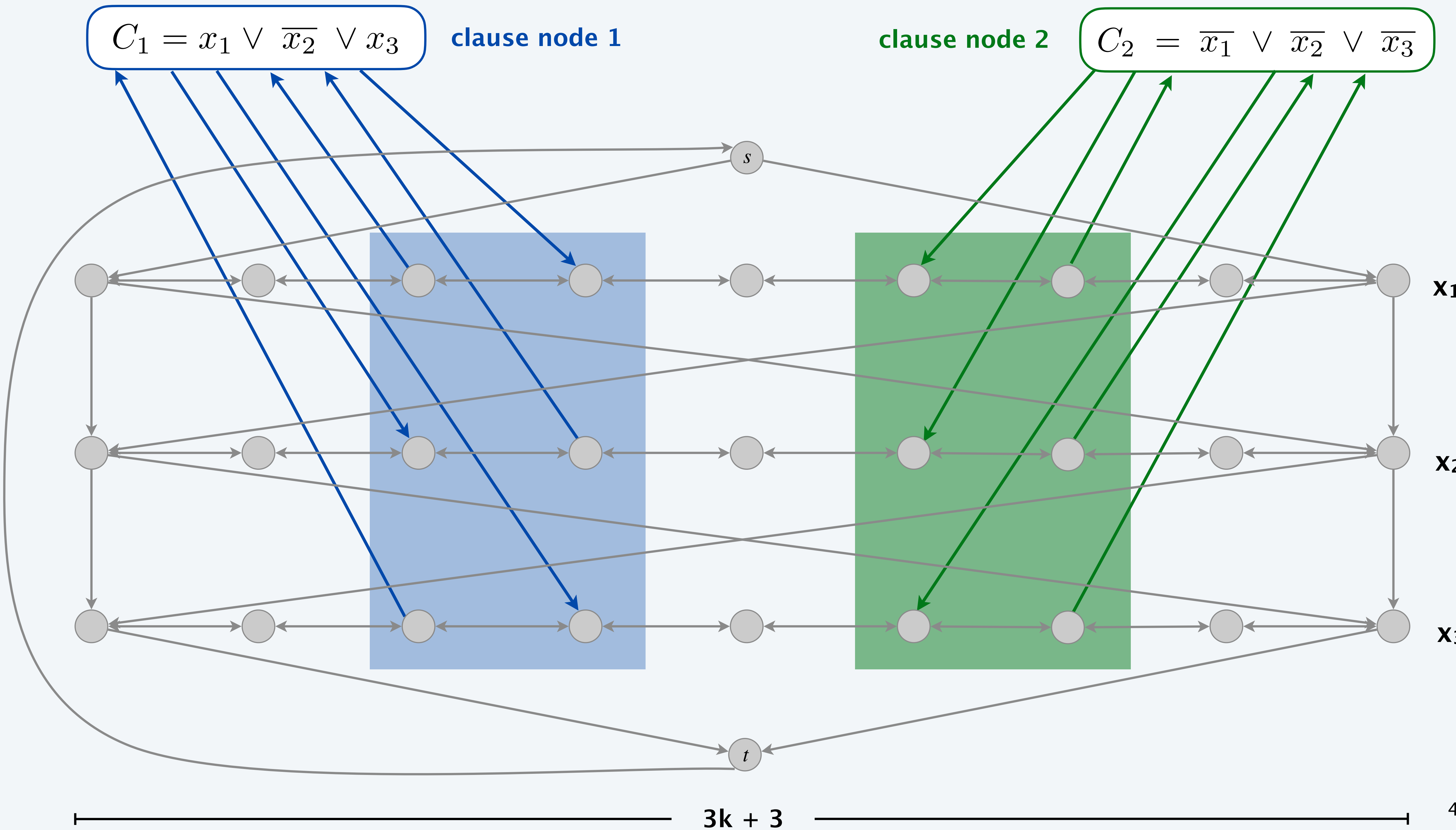
- For each clause: add a node and 2 edges per literal.



3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

- For each clause: add a node and 2 edges per literal.



3-satisfiability reduces to directed Hamilton cycle

Lemma. Φ is satisfiable iff G has a Hamilton cycle.

Pf. $\Rightarrow$

- Suppose 3-SAT instance Φ has satisfying assignment x^*.
- Then, define Hamilton cycle Γ in G as follows:
  - if $x_i^* = true$, traverse row i from left to right
  - if $x_i^* = false$, traverse row i from right to left
  - for each clause C_j , there will be at least one row i in which we are going in "correct" direction to splice clause node C_j into cycle (and we splice in C_j exactly once) ▪

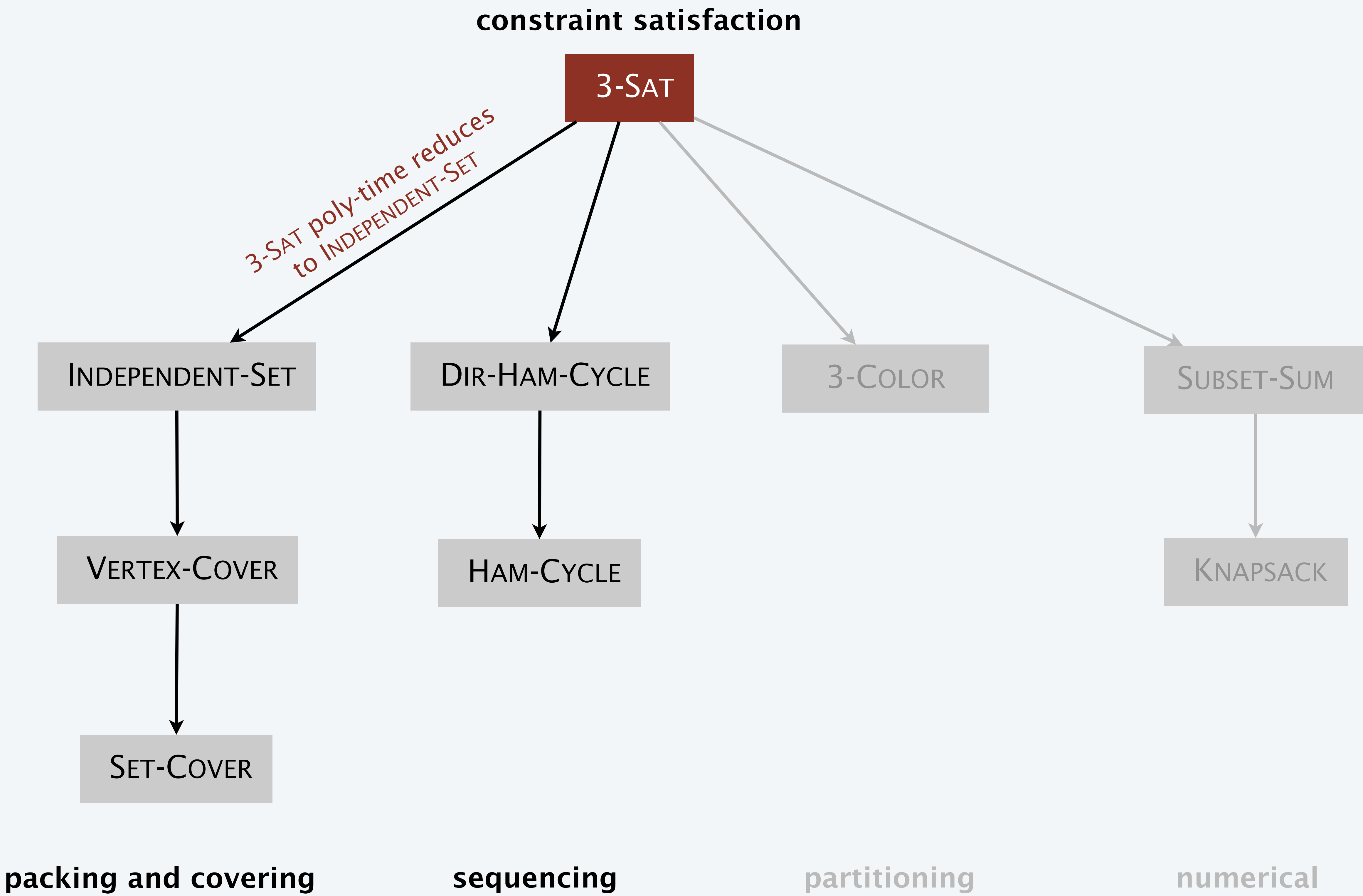
3-satisfiability reduces to directed Hamilton cycle

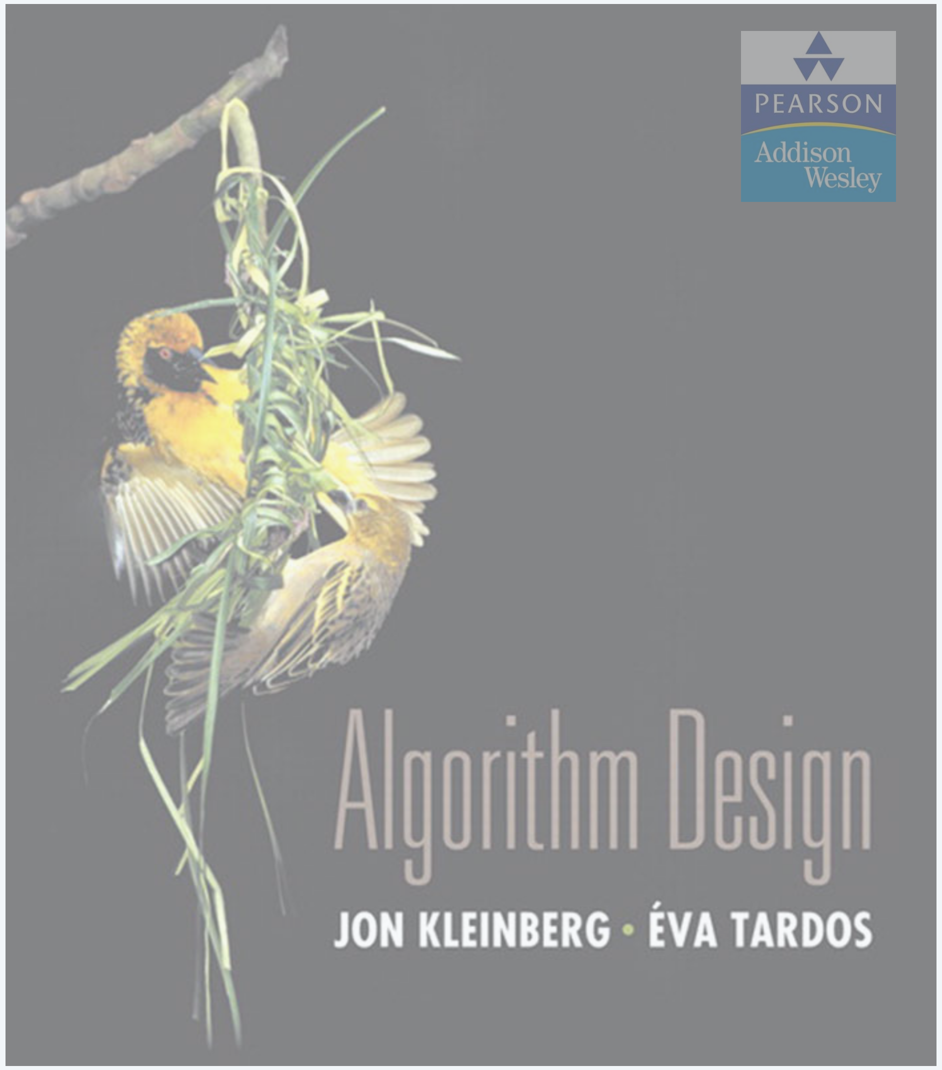
Lemma. Φ is satisfiable iff G has a Hamilton cycle.

Pf. $\Leftarrow$

- Suppose G has a Hamilton cycle Γ.
- If Γ enters clause node C_j, it must depart on mate edge.
  - nodes immediately before and after C_j are connected by an edge $e \in E$
  - removing C_j from cycle, and replacing it with edge e yields Hamilton cycle on $G - \{ C_j \}$
- Continuing in this way, we are left with a Hamilton cycle Γ' in $G - \{ C_1, C_2, \ldots, C_k \}$.
- Set $x_i^* = true$ if Γ' traverses row i left-to-right; otherwise, set $x_i^* = false$.
- traversed in "correct" direction, and each clause is satisfied. ▪

Poly-time reductions





SECTION 8.6

8. INTRACTABILITY I

- poly-time reductions
- packing and covering problems
- constraint satisfaction problems
- sequencing problems
- **partitioning problems**
- graph coloring
- numerical problems

3-dimensional matching

3D-MATCHING. Given n instructors, n courses, and n times, and a list of the possible courses and times each instructor is willing to teach, is it possible to make an assignment so that all courses are taught at different times?

instructor	course	time
Wayne	COS 226	TTh 11–12:20
Wayne	COS 423	MW 11–12:20
Wayne	COS 423	TTh 11–12:20
Tardos	COS 423	TTh 3–4:20
Tardos	COS 523	TTh 3–4:20
Kleinberg	COS 226	TTh 3–4:20
Kleinberg	COS 226	MW 11–12:20
Kleinberg	COS 423	MW 11–12:20

3-dimensional matching

3D-MATCHING. Given 3 disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

$$X = \{ x_1, x_2, x_3 \}, \quad Y = \{ y_1, y_2, y_3 \}, \quad Z = \{ z_1, z_2, z_3 \}$$

$$T_1 = \{ x_1, y_1, z_2 \}, \quad T_2 = \{ x_1, y_2, z_1 \}, \quad \boxed{T_3 = \{ x_1, y_2, z_2 \}}$$

$$T_4 = \{ x_2, y_2, z_3 \}, \quad \boxed{T_5 = \{ x_2, y_3, z_3 \},}$$

$$T_7 = \{ x_3, y_1, z_3 \}, \quad \boxed{T_8 = \{ x_3, y_1, z_1 \},} \quad T_9 = \{ x_3, y_2, z_1 \}$$

an instance of 3d-matching (with n = 3)

Remark. Generalization of bipartite matching.

3-dimensional matching

3D-MATCHING. Given 3 disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

Theorem. 3-SAT $\leq_{\text{P}}$ 3D-MATCHING.

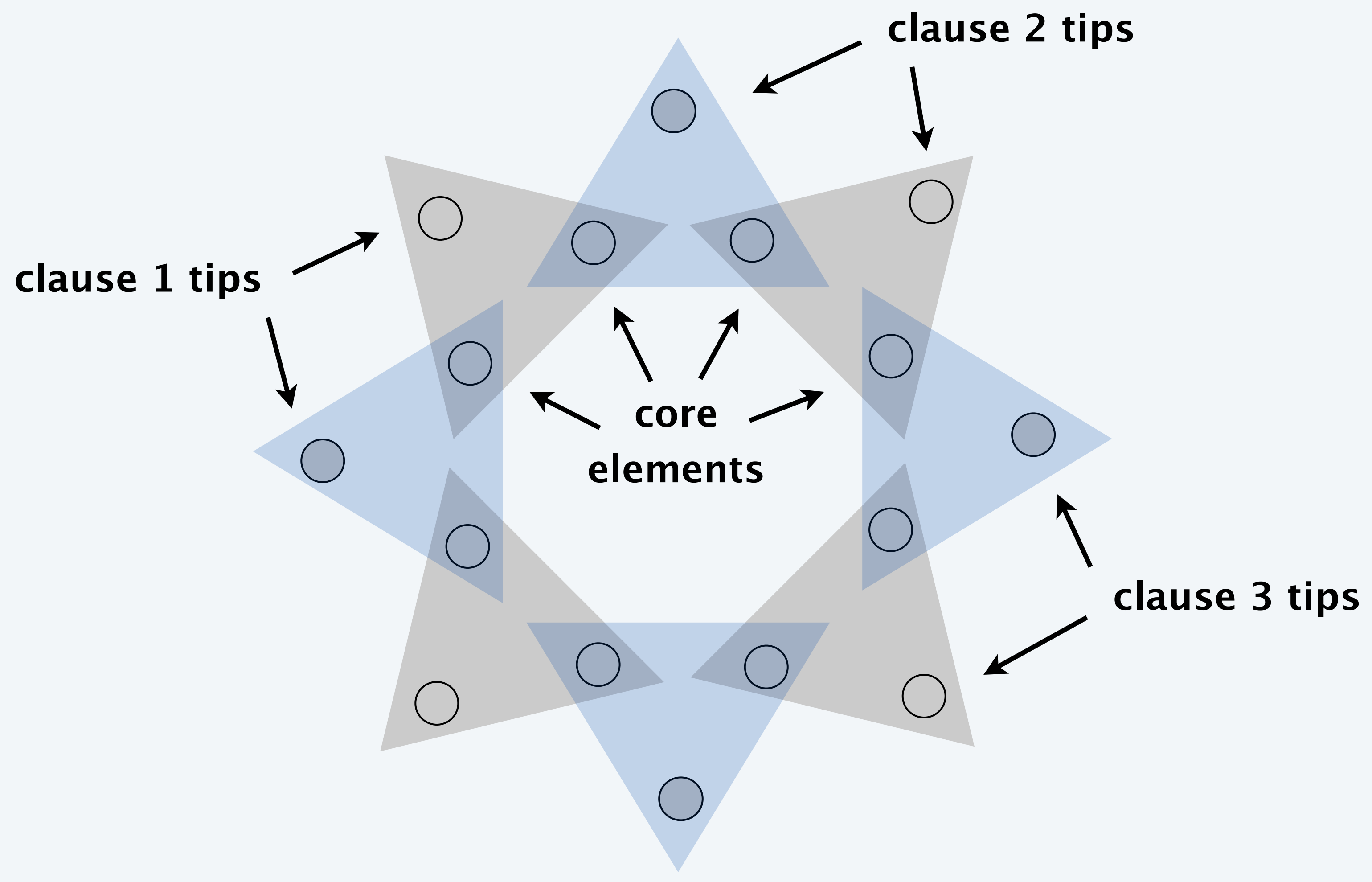
Pf. Given an instance Φ of 3-SAT, we construct an instance of 3D-MATCHING that has a perfect matching iff Φ is satisfiable.

3-satisfiability reduces to 3-dimensional matching

Construction. (part 1)

number of clauses

- Create gadget for each variable x_i with $2k$ core elements and $2k$ tip ones.



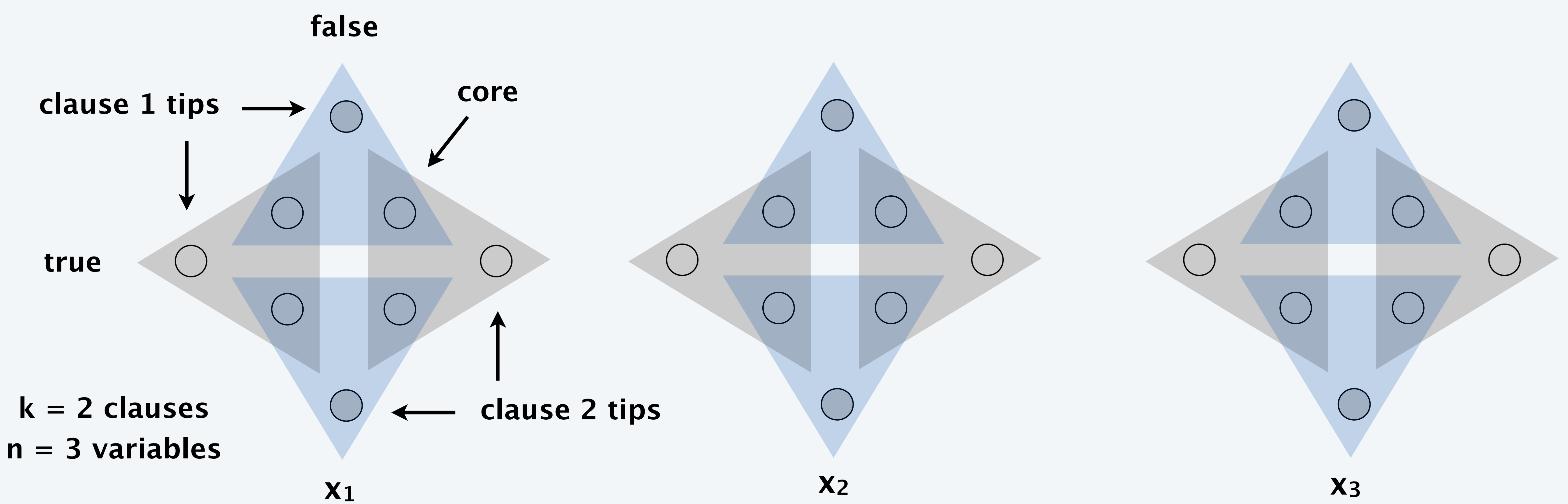
a gadget for variable x_i (k = 4)

3-satisfiability reduces to 3-dimensional matching

number of clauses

Construction. (part 1)

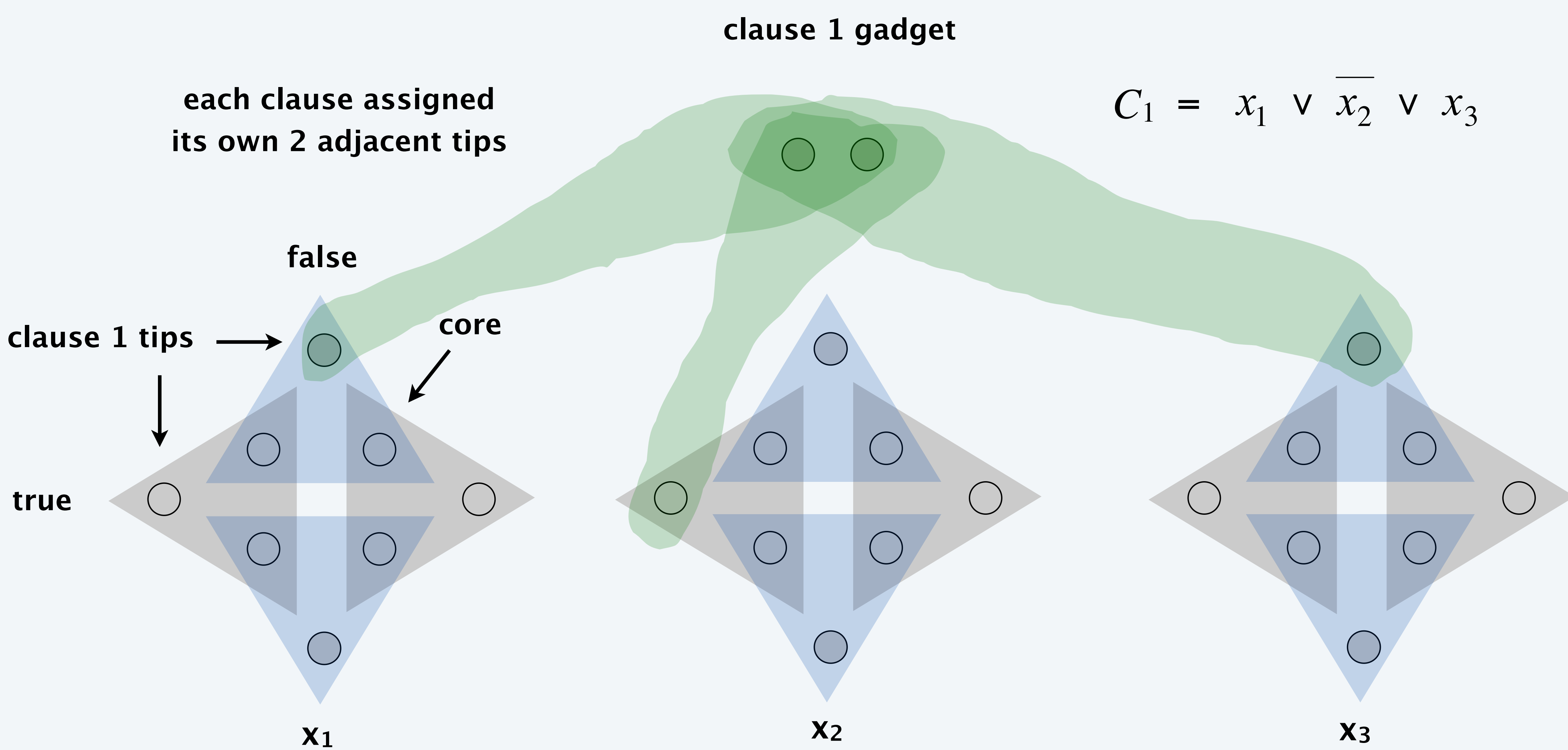
- Create gadget for each variable x_i with $2k$ core elements and $2k$ tip ones.
- No other triples will use core elements.
- In gadget for x_i, any perfect matching must use either all gray triples (corresponding to $x_i = true$) or all blue ones (corresponding to $x_i = false$).



3-satisfiability reduces to 3-dimensional matching

Construction. (part 2)

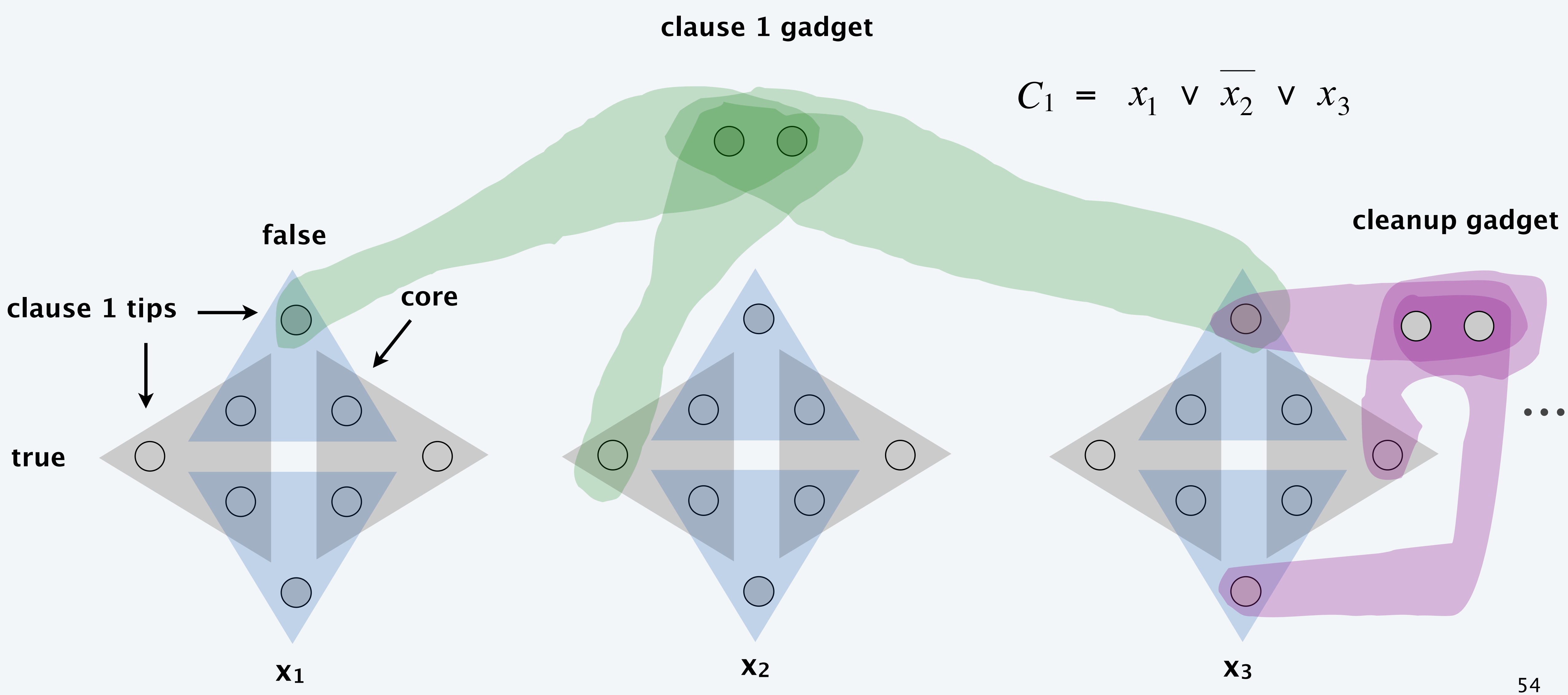
- Create gadget for each clause C_j with two elements and three triples.
- Exactly one of these triples will be used in any 3d-matching.
- Ensures any perfect matching uses either (i) grey core of x_1 or (ii) blue core of x_2 or (iii) grey core of x_3.



3-satisfiability reduces to 3-dimensional matching

Construction. (part 3)

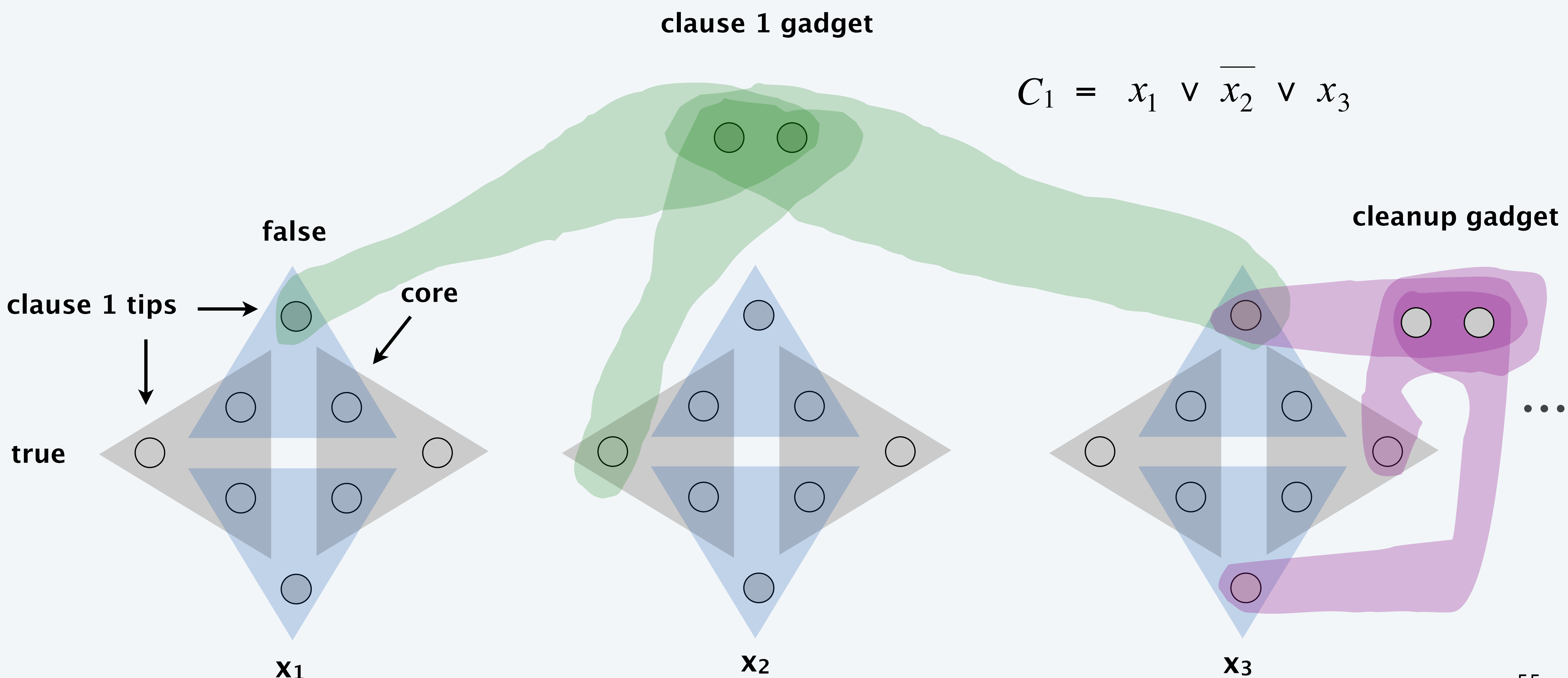
- There are $2nk$ tips: nk covered by blue/gray triples; k by clause triples.
- To cover remaining $(n-1)k$ tips, create $(n-1)k$ cleanup gadgets: same as clause gadget but with $2nk$ triples, connected to every tip.



3-satisfiability reduces to 3-dimensional matching

Lemma. Instance (X, Y, Z) has a perfect matching iff Φ is satisfiable.

Q. What are X, Y, and Z?

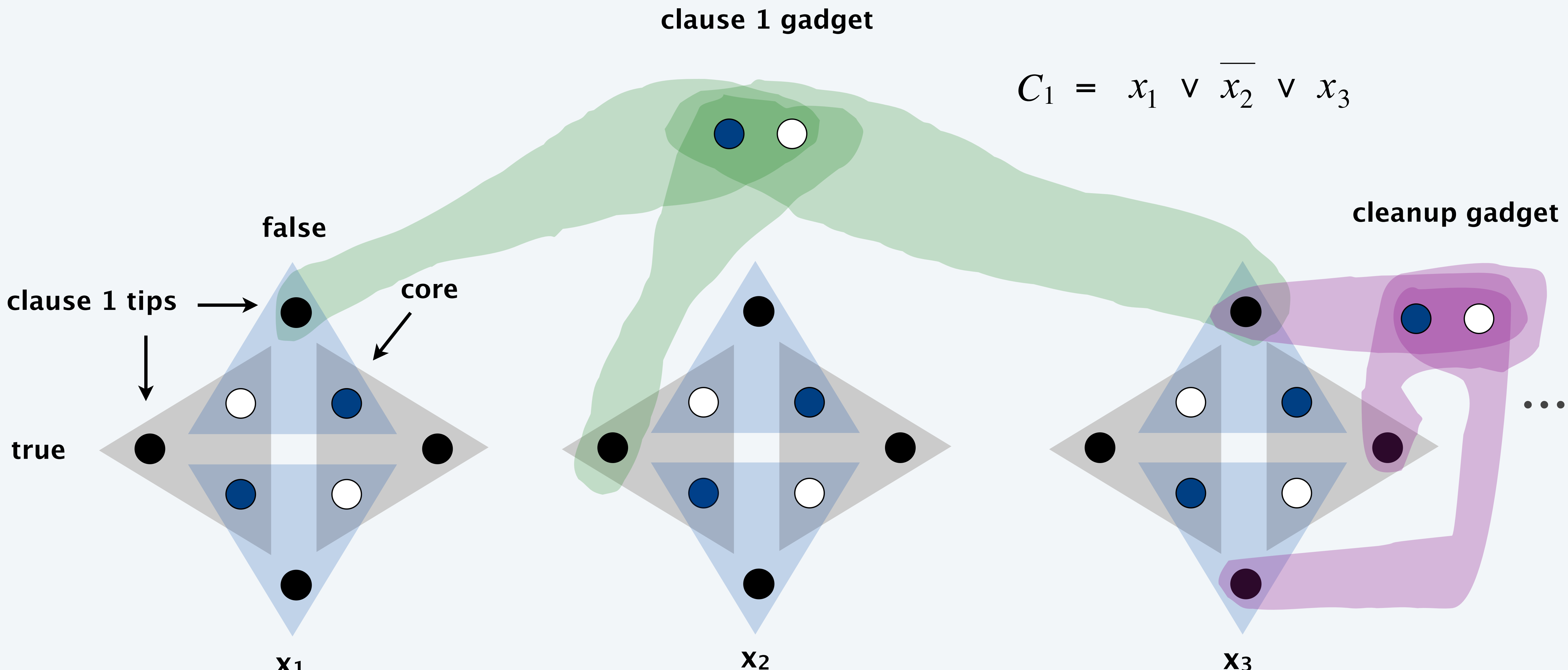


3-satisfiability reduces to 3-dimensional matching

Lemma. Instance (X, Y, Z) has a perfect matching iff Φ is satisfiable.

Q. What are X, Y, and Z?

A. $X = black$, $Y = white$, and $Z = blue$.

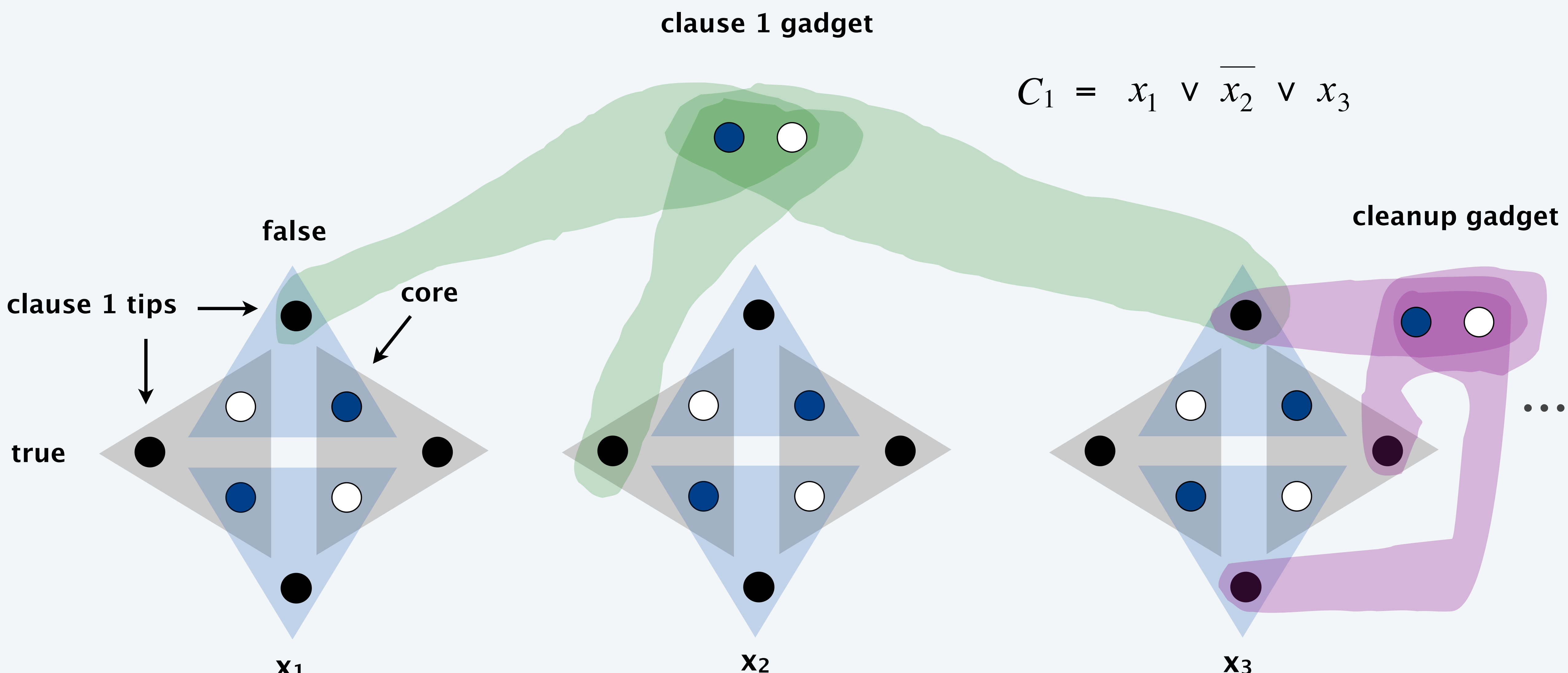


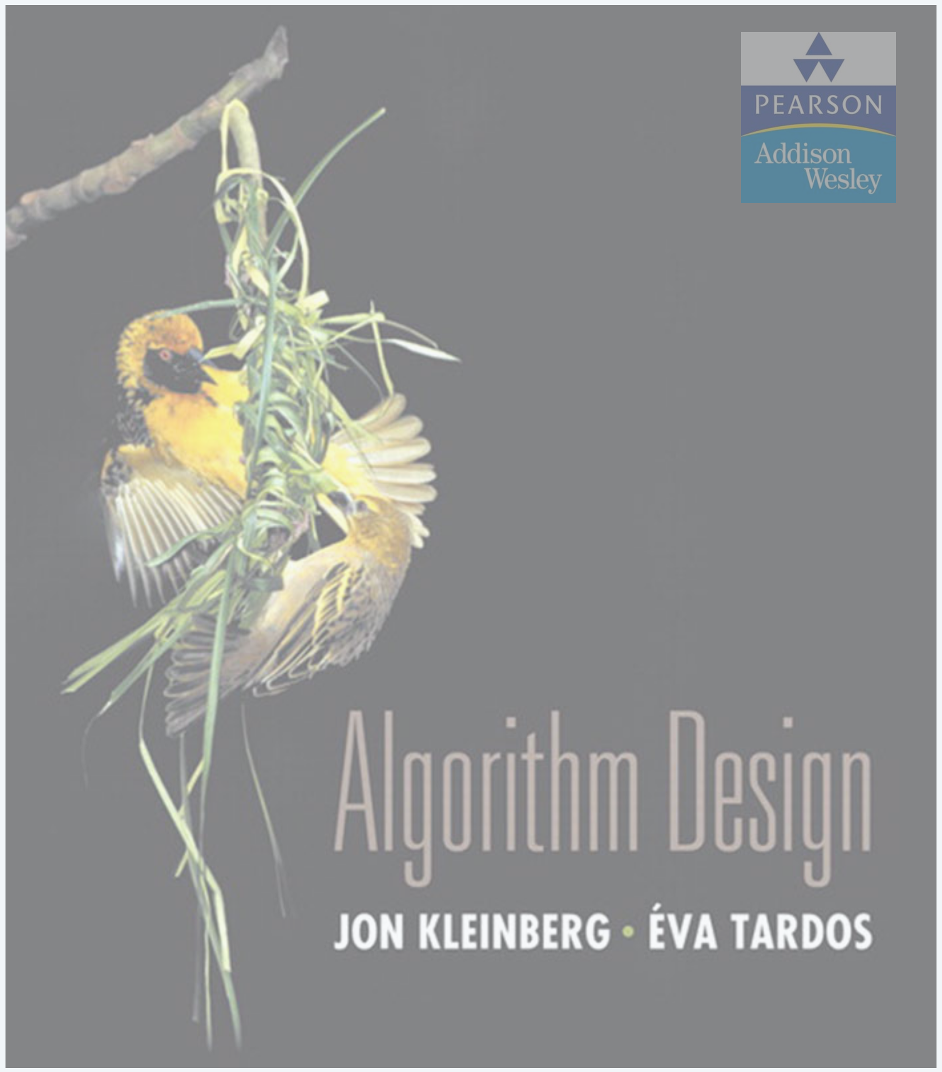
3-satisfiability reduces to 3-dimensional matching

Lemma. Instance (X, Y, Z) has a perfect matching iff Φ is satisfiable.

Pf. $\Rightarrow$ If 3d-matching, then assign x_i according to gadget x_i.

Pf. $\Leftarrow$ If Φ is satisfiable, use any true literal in C_j to select gadget C_j triple. ▪





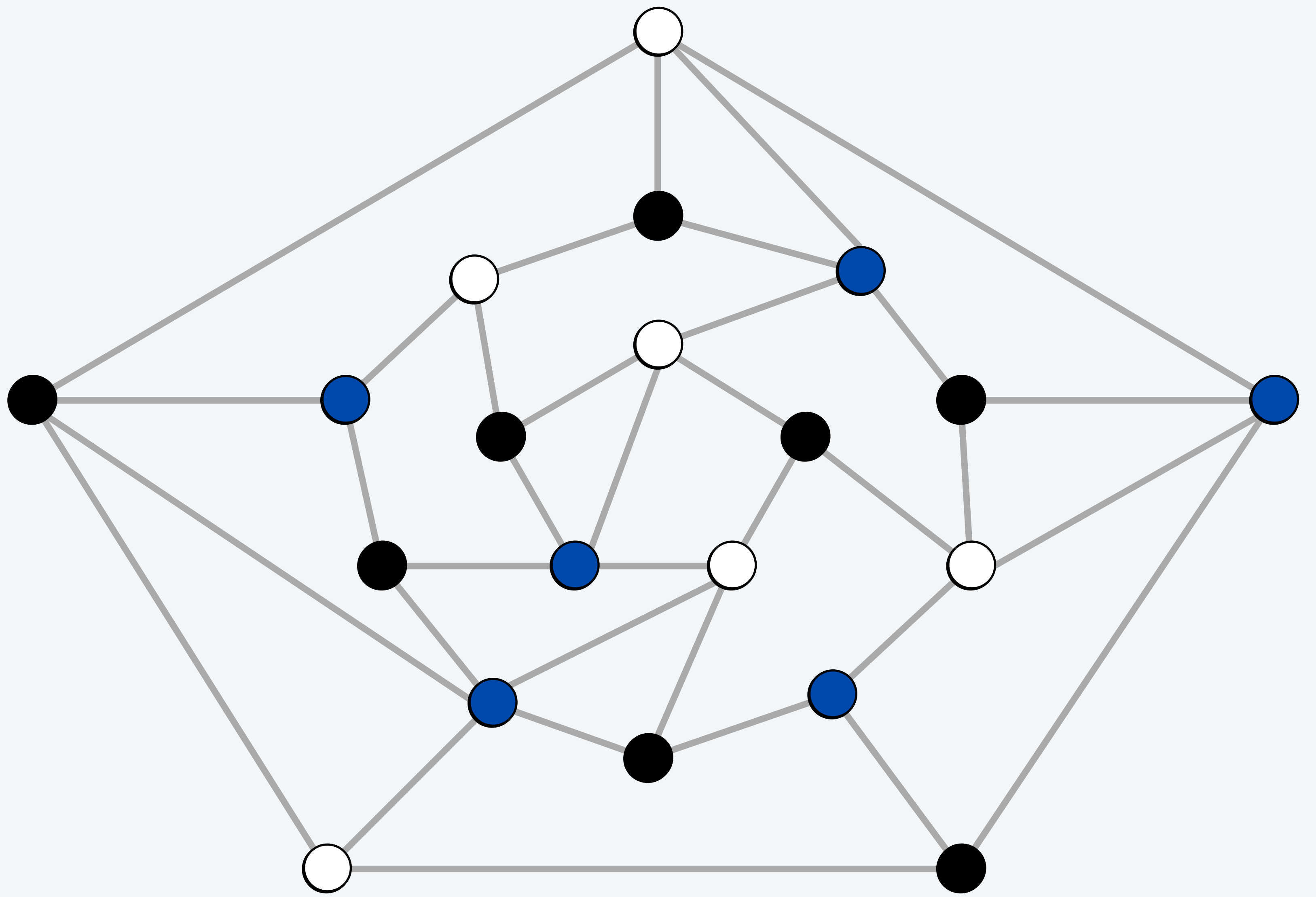
SECTION 8.7

8. INTRACTABILITY I

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *sequencing problems*
- *partitioning problems*
- *graph coloring*
- *numerical problems*

3-colorability

3-COLOR. Given an undirected graph G, can the nodes be colored black, white, and blue so that no adjacent nodes have the same color?



yes instance

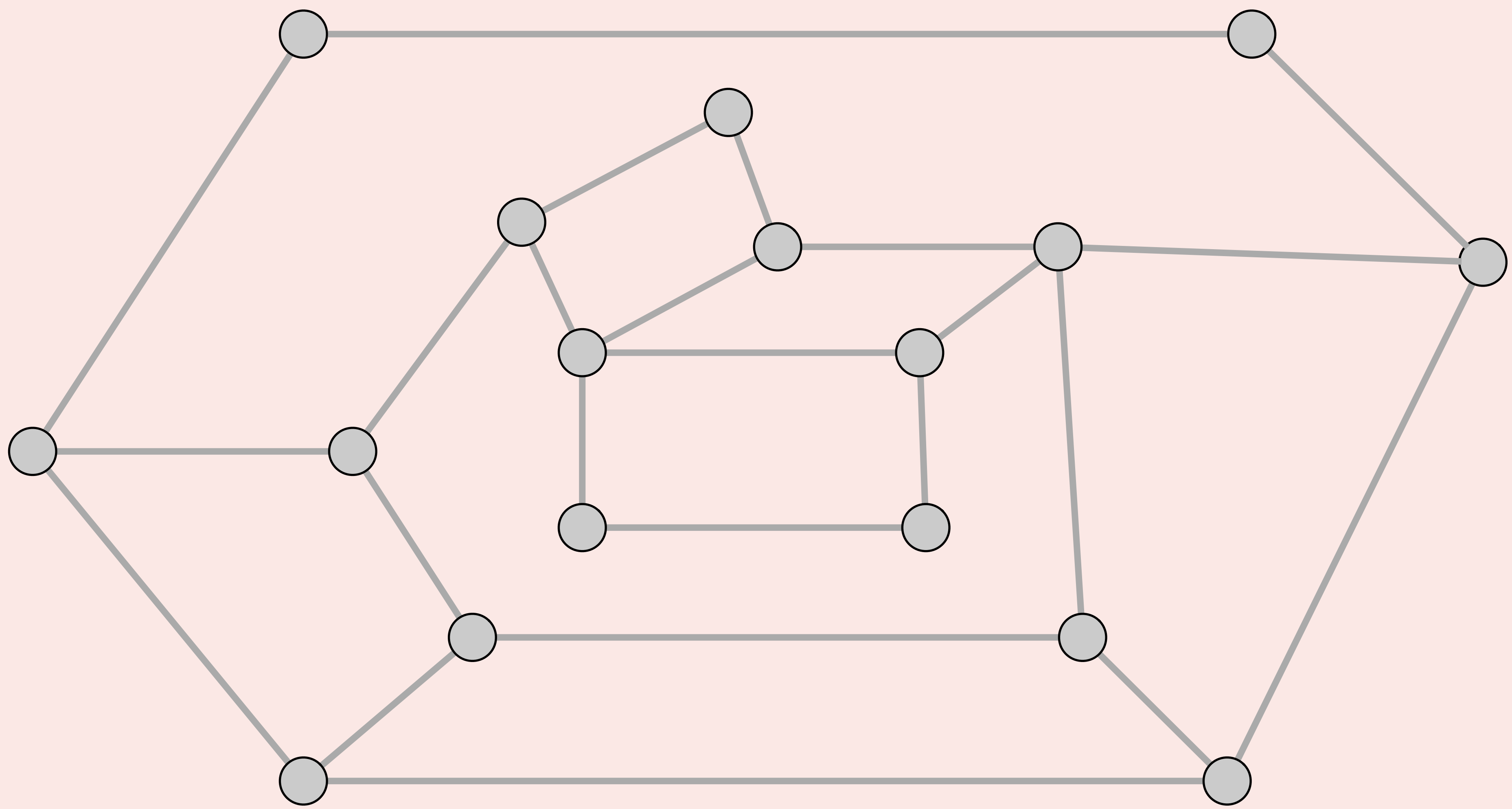
How difficult to solve 2-COLOR?

A. $O(m + n)$ using BFS or DFS.

B. $O(mn)$ using maximum flow.

C. $\Omega(2^n)$ using brute force.

D. Not even Tarjan knows.



Application: register allocation

Register allocation. Assign program variables to machine registers so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables; edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k-colorable.

Fact. 3-COLOR $\leq_P$ K-REGISTER-ALLOCATION for any constant $k \geq 3$.

REGISTER ALLOCATION & SPILLING VIA GRAPH COLORING

G. J. Chaitin
IBM Research
P.O.Box 218, Yorktown Heights, NY 10598

3-satisfiability reduces to 3-colorability

Theorem. 3-SAT $\le_P$ 3-COLOR.

Pf. Given 3-SAT instance Φ, we construct an instance of 3-COLOR that is 3-colorable iff Φ is satisfiable.

3-satisfiability reduces to 3-colorability

Construction.

(i) Create a graph G with a node for each literal.

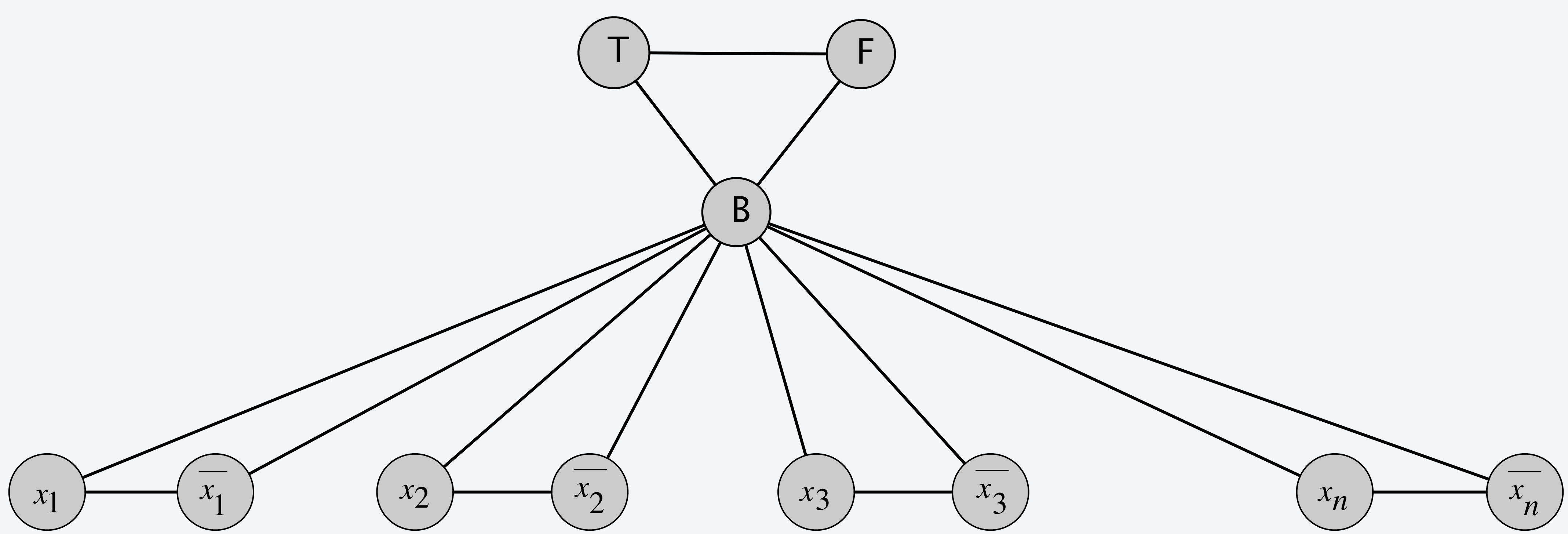
(ii) Connect each literal to its negation.

(iii) Create 3 new nodes T, F, and B; connect them in a triangle.

(iv) Connect each literal to B.

(v) For each clause C_j, add a gadget of 6 nodes and 13 edges.

↑ to be described later

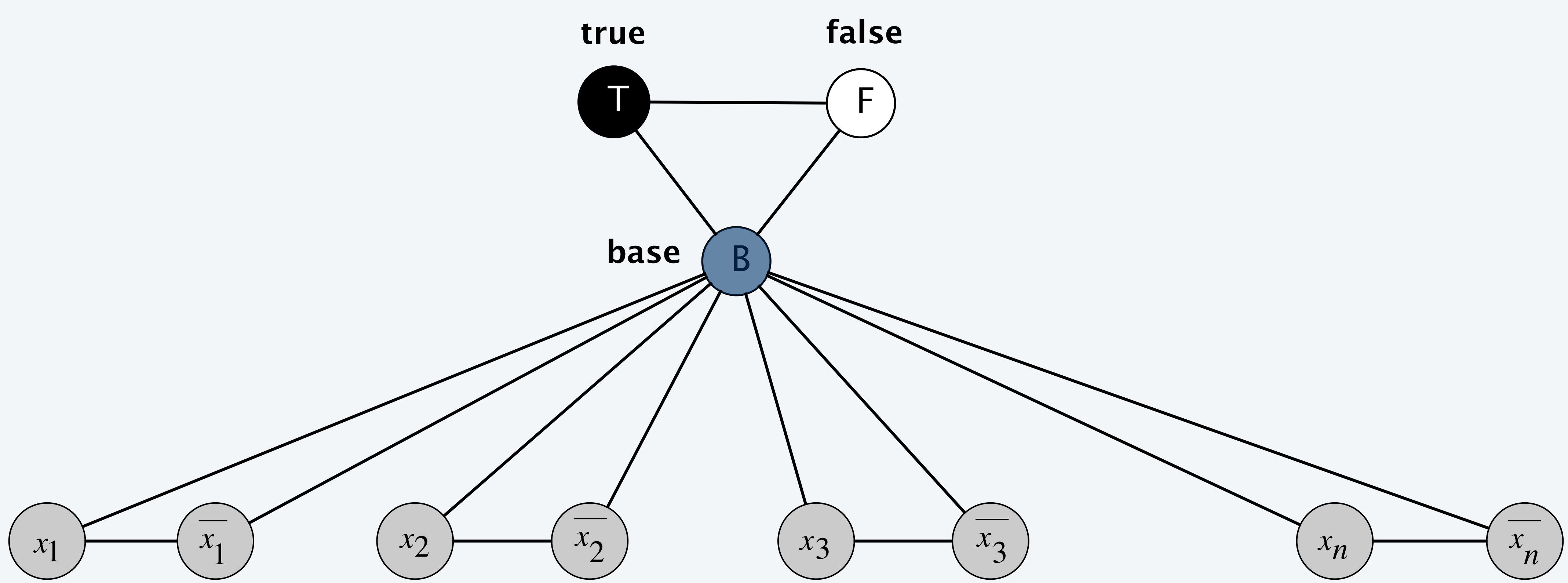


3-satisfiability reduces to 3-colorability

Lemma. Graph G is 3-colorable iff Φ is satisfiable.

Pf. $\Rightarrow$ Suppose graph G is 3-colorable.

- WLOG, assume that node T is colored *black*, F is *white*, and B is *blue*.
- Consider assignment that sets all *black* literals to *true* (and *white* to *false*).
- (iv) ensures each literal is colored either *black* or *white*.
- (ii) ensures that each literal is *white* if its negation is *black* (and vice versa).

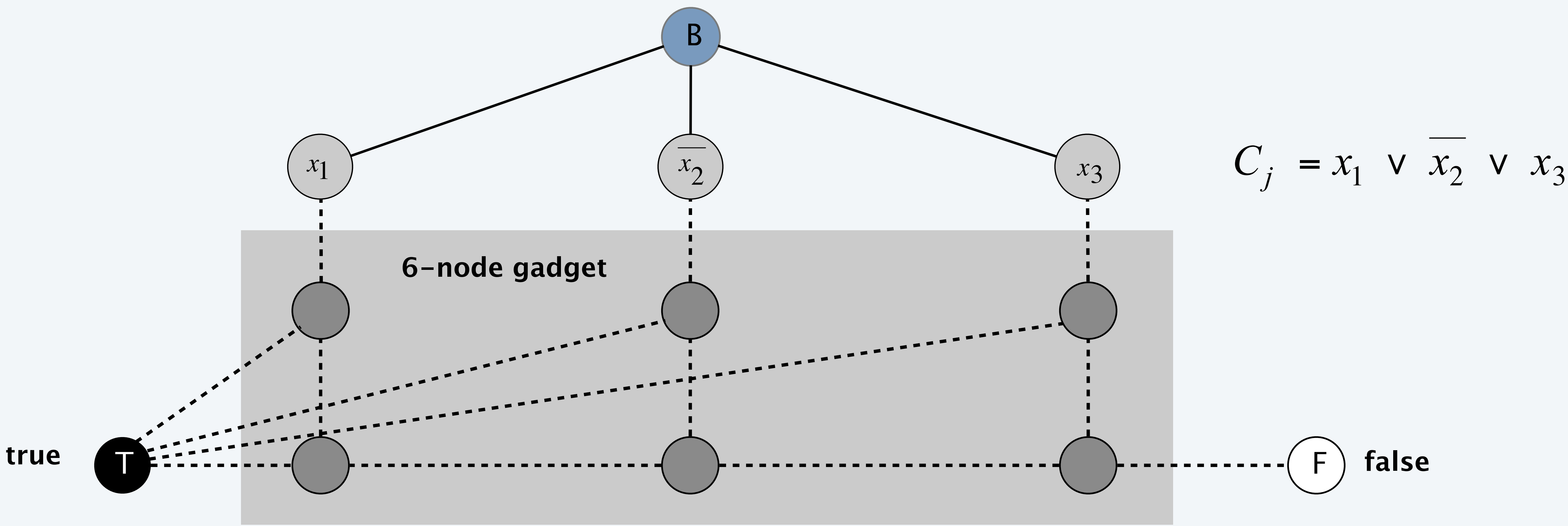


3-satisfiability reduces to 3-colorability

Lemma. Graph G is 3-colorable iff Φ is satisfiable.

Pf. $\Rightarrow$ Suppose graph G is 3-colorable.

- WLOG, assume that node T is colored *black*, F is *white*, and B is *blue*.
- Consider assignment that sets all *black* literals to *true* (and *white* to *false*).
- (iv) ensures each literal is colored either *black* or *white*.
- (ii) ensures that each literal is *white* if its negation is *black* (and vice versa).
- (v) ensures at least one literal in each clause is *black*.

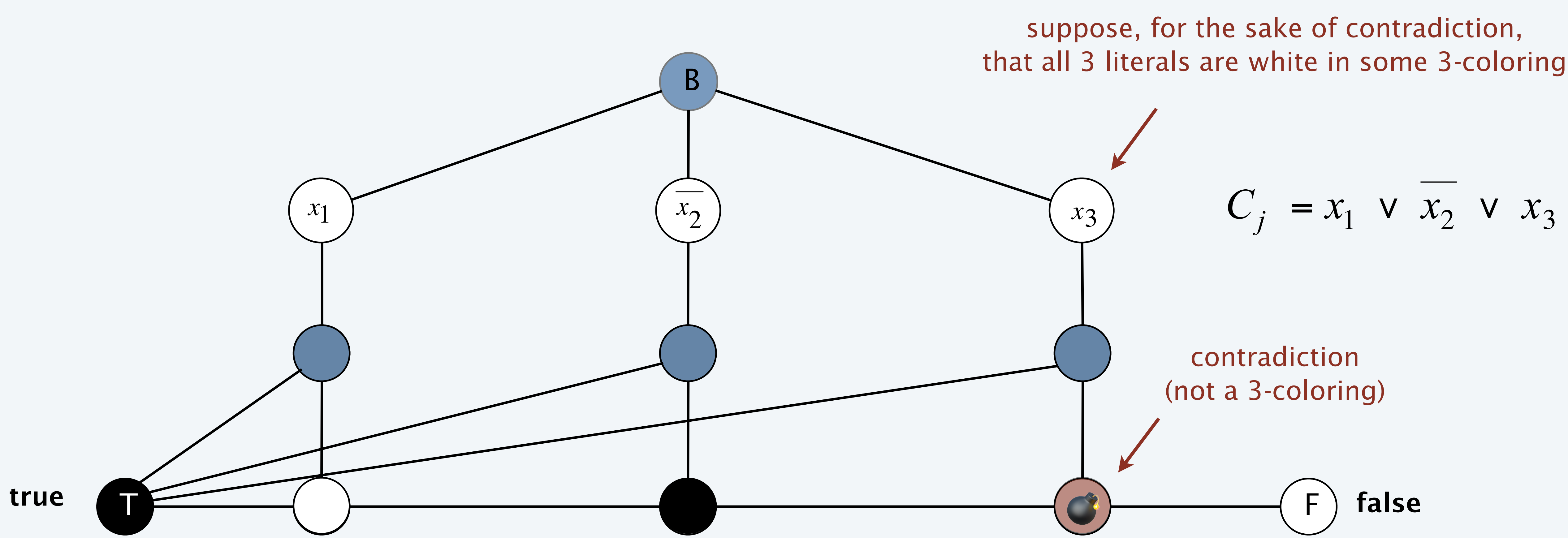


3-satisfiability reduces to 3-colorability

Lemma. Graph G is 3-colorable iff Φ is satisfiable.

Pf. $\Rightarrow$ Suppose graph G is 3-colorable.

- WLOG, assume that node T is colored *black*, F is *white*, and B is *blue*.
- Consider assignment that sets all *black* literals to *true* (and *white* to *false*).
- (iv) ensures each literal is colored either *black* or *white*.
- (ii) ensures that each literal is *white* if its negation is *black* (and vice versa).
- (v) ensures at least one literal in each clause is *black*. ▪

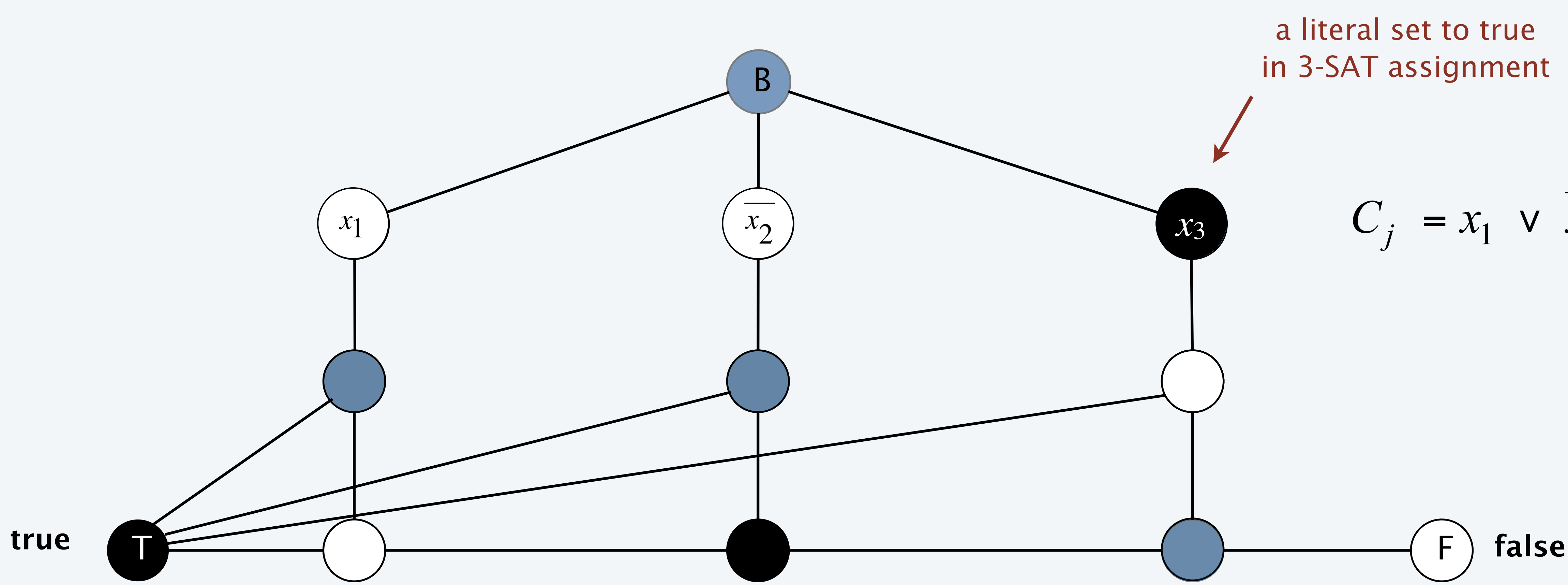


3-satisfiability reduces to 3-colorability

Lemma. Graph G is 3-colorable iff Φ is satisfiable.

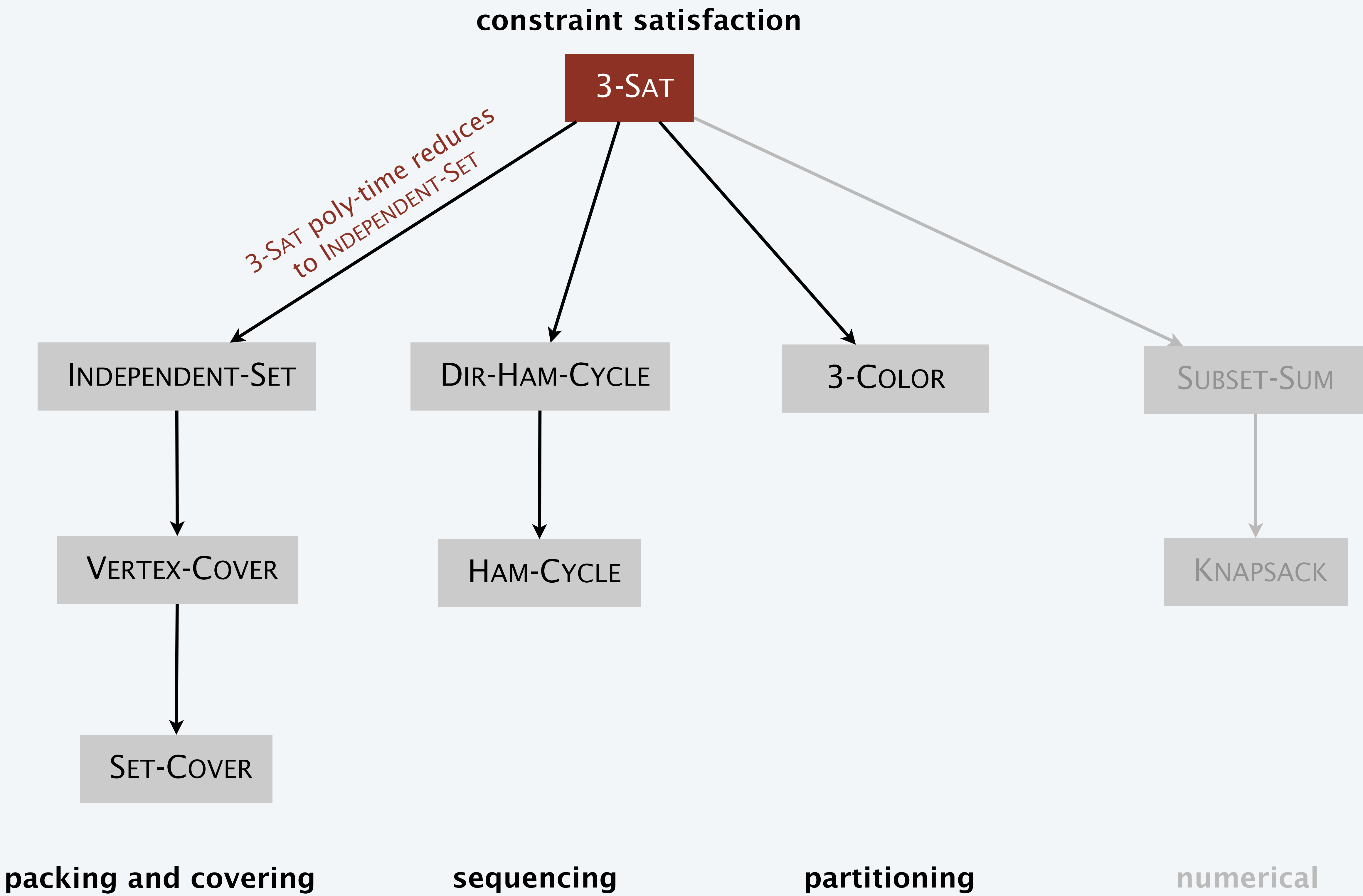
Pf. $\Leftarrow$ Suppose 3-SAT instance Φ is satisfiable.

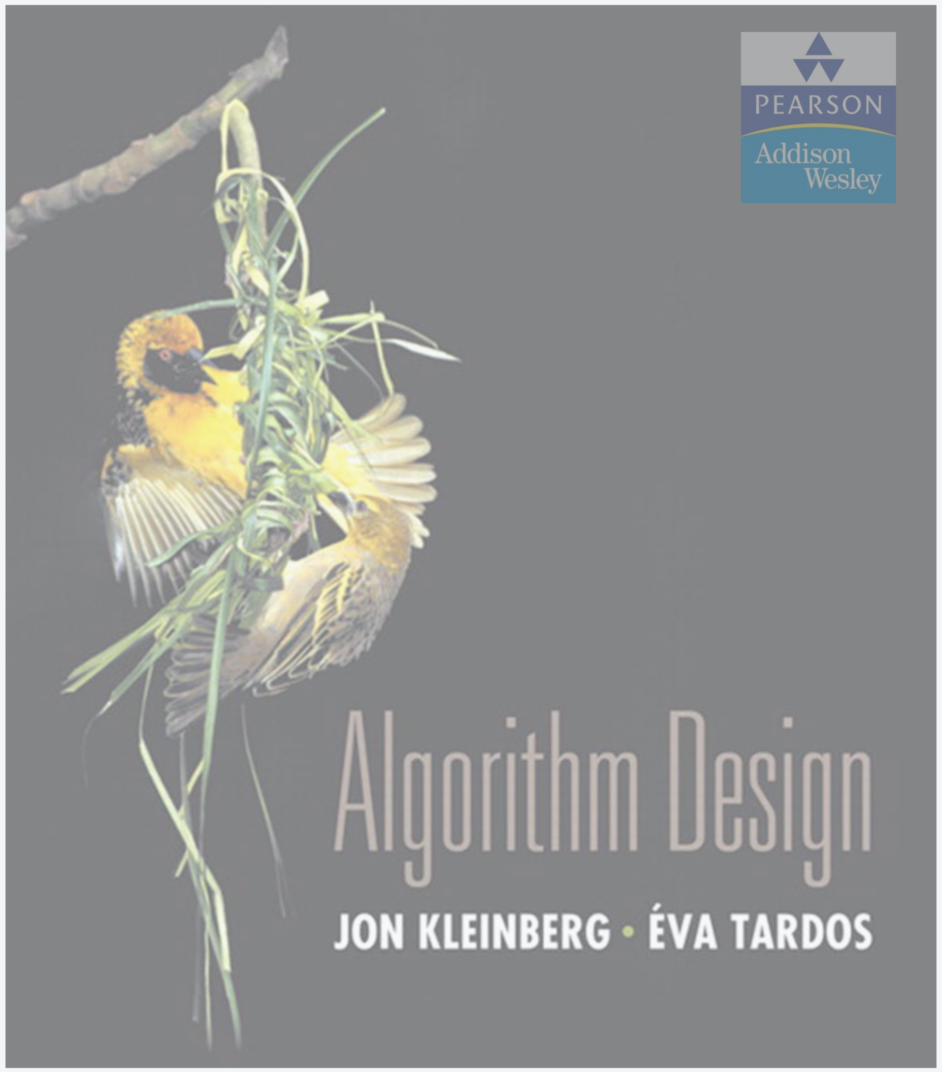
- Color all *true* literals *black* and all *false* literals *white*.
- Pick one *true* literal; color node below that node *white*, and node below that *blue*.
- Color remaining middle row nodes *blue*.
- Color remaining bottom nodes *black* or *white*, as forced. ▪



$C_j = x_1 \lor \overline{x_2} \lor x_3$

Poly-time reductions



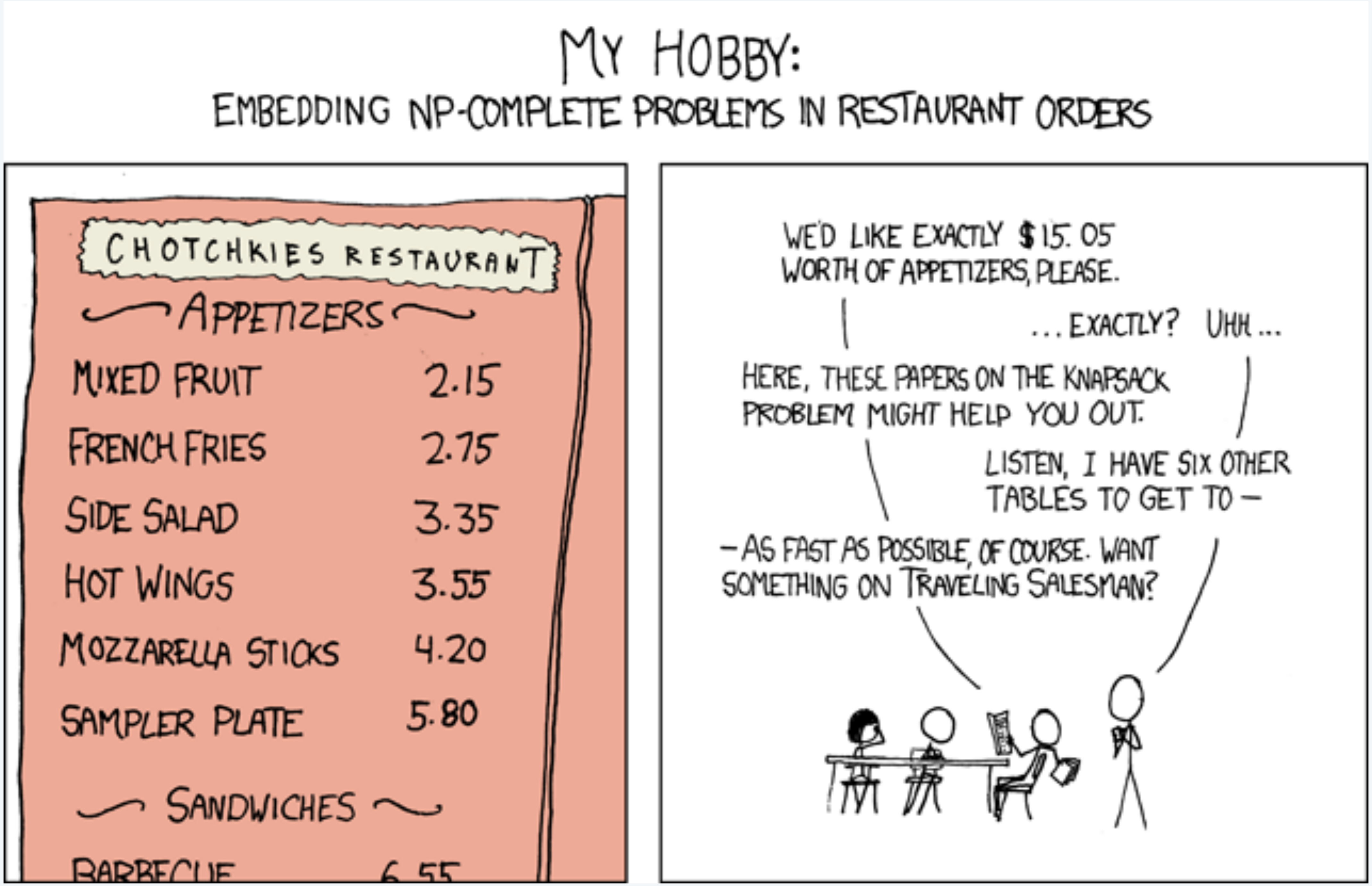


SECTION 8.8

8. INTRACTABILITY I

- poly-time reductions
- packing and covering problems
- constraint satisfaction problems
- sequencing problems
- partitioning problems
- graph coloring
- **numerical problems**

My hobby



NP-Complete by Randall Munro
http://xkcd.com/287

Subset sum

SUBSET-SUM. Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Ex. $\{ 215, 215, 275, 275, 355, 355, 420, 420, 580, 580, 655, 655 \}, \quad W = 1505.$

Yes. $215 + 355 + 355 + 580 = 1505.$

Remark. With arithmetic problems, input integers are encoded in binary. Poly-time reduction must be polynomial in binary encoding.

Subset sum

Theorem. 3-SAT $\le_P$ SUBSET-SUM.

Pf. Given an instance Φ of 3-SAT, we construct an instance of SUBSET-SUM that has a solution iff Φ is satisfiable.

3-satisfiability reduces to subset sum

Construction. Given 3-SAT instance Φ with n variables and k clauses, form $2n + 2k$ decimal integers, each having $n + k$ digits:

- Include one digit for each variable x_i and one digit for each clause C_j.
- Include two numbers for each variable x_i.
- Include two numbers for each clause C_j.
- Sum of each x_i digit is 1;
  sum of each C_j digit is 4.

Key property. No carries possible $\Rightarrow$ each digit yields one equation.

$$C_1 = \neg x_1 \vee x_2 \vee x_3$$
$$C_2 = x_1 \vee \neg x_2 \vee x_3$$
$$C_3 = \neg x_1 \vee \neg x_2 \vee \neg x_3$$

3-SAT instance

dummies to get clause columns to sum to 4

	x_1	x_2	x_3	C_1	C_2	C_3	
x_1	1	0	0	0	1	0	100,010
$\neg x_1$	1	0	0	1	0	1	100,101
x_2	0	1	0	1	0	0	10,100
$\neg x_2$	0	1	0	0	1	1	10,011
x_3	0	0	1	1	1	0	1,110
$\neg x_3$	0	0	1	0	0	1	1,001
	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

SUBSET-SUM instance

3-satisfiability reduces to subset sum

Lemma. Φ is satisfiable iff there exists a subset that sums to W.

Pf. $\Rightarrow$ Suppose 3-SAT instance Φ has satisfying assignment x^*.

- If $x_i^* = true$, select integer in row x_i; otherwise, select integer in row $\neg x_i$.
- Each x_i digit sums to 1.
- Since Φ is satisfiable, each C_j digit sums to at least 1 from x_i and $\neg x_i$ rows.
- Select dummy integers to make C_j digits sum to 4. ▪

$$
\begin{aligned}
C_1 &= \neg x_1 \vee x_2 \vee x_3 \\
C_2 &= x_1 \vee \neg x_2 \vee x_3 \\
C_3 &= \neg x_1 \vee \neg x_2 \vee \neg x_3
\end{aligned}
$$

3-SAT instance

	x_1	x_2	x_3	C_1	C_2	C_3	
x_1	1	0	0	0	1	0	100,010
$\neg x_1$	1	0	0	1	0	1	100,101
x_2	0	1	0	1	0	0	10,100
$\neg x_2$	0	1	0	0	1	1	10,011
x_3	0	0	1	1	1	0	1,110
$\neg x_3$	0	0	1	0	0	1	1,001
dummies to get clause columns to sum to 4	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

SUBSET-SUM instance

3-satisfiability reduces to subset sum

Lemma. Φ is satisfiable iff there exists a subset that sums to W.

Pf. $\Leftarrow$ Suppose there exists a subset S^* that sums to W.

- Digit x_i forces subset S^* to select either row x_i or row $\neg x_i$ (but not both).
- If row x_i selected, assign $x_i^* = \mathit{true}$; otherwise, assign $x_i^* = \mathit{false}$.

  Digit C_j forces subset S^* to select at least one literal in clause. ▪

$$\begin{array}{lllllllll}
C_1 & = & \neg x_1 & \vee & x_2 & \vee & x_3 \\
C_2 & = & x_1 & \vee & \neg x_2 & \vee & x_3 \\
C_3 & = & \neg x_1 & \vee & \neg x_2 & \vee & \neg x_3
\end{array}$$

3-SAT instance

	x_1	x_2	x_3	C_1	C_2	C_3	
x_1	1	0	0	0	1	0	100,010
$\neg x_1$	1	0	0	1	0	1	100,101
x_2	0	1	0	1	0	0	10,100
$\neg x_2$	0	1	0	0	1	1	10,011
x_3	0	0	1	1	1	0	1,110
$\neg x_3$	0	0	1	0	0	1	1,001
dummies to get clause columns to sum to 4	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

SUBSET-SUM instance

SUBSET SUM REDUCES TO KNAPSACK

SUBSET-SUM. Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

KNAPSACK. Given a set of items X, weights $u_i \geq 0$, values $v_i \geq 0$, a weight limit U, and a target value V, is there a subset $S \subseteq X$ such that:

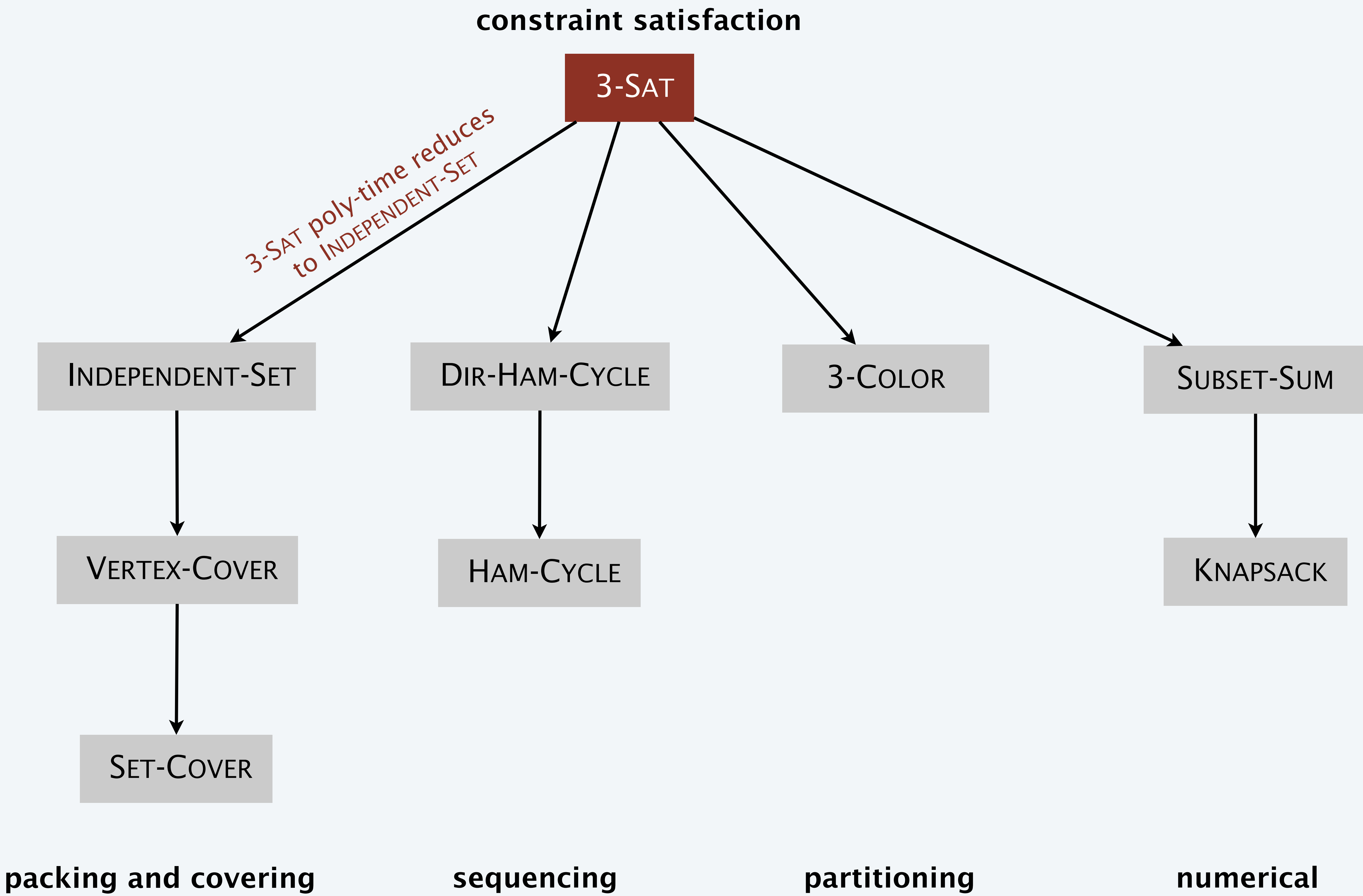
$$\sum_{i \in S} u_i \;\leq\; U, \quad \sum_{i \in S} v_i \;\geq\; V$$

Recall. $O(n\,U)$ dynamic programming algorithm for KNAPSACK.

Challenge. Prove SUBSET-SUM $\leq_P$ KNAPSACK.

Pf. Given instance $(w_1, \ldots, w_n, W)$ of SUBSET-SUM, create KNAPSACK instance:

Poly-time reductions



Karp's 20 poly-time reductions from satisfiability

96

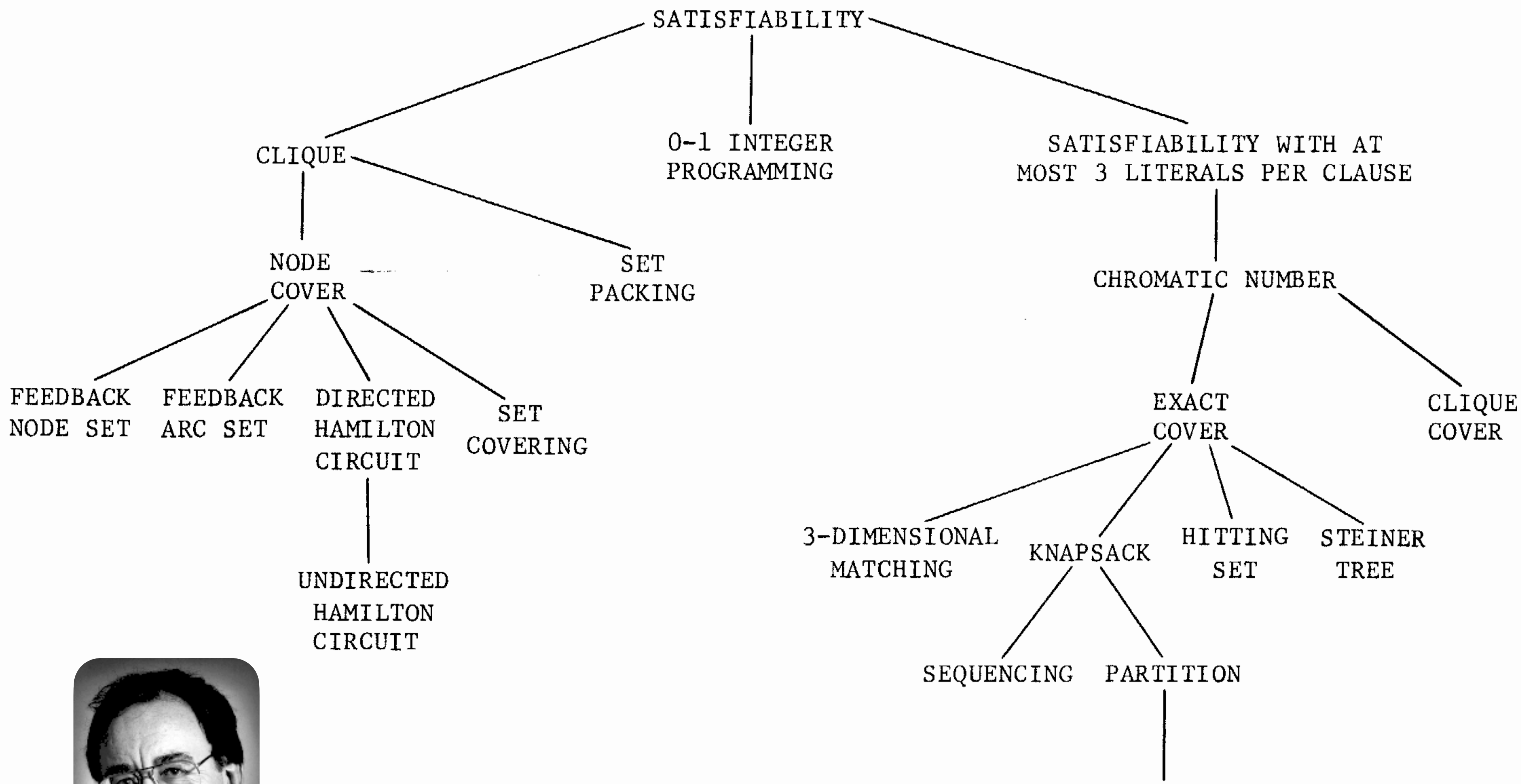
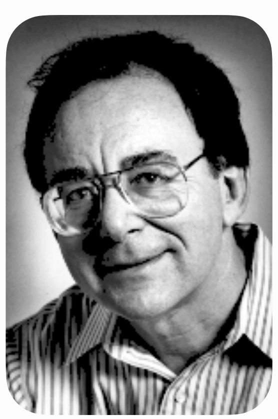


FIGURE 1 - Complete Problems

RICHARD M. KARP



Dick Karp (1972)
1985 Turing Award