

7. NETWORK FLOW I

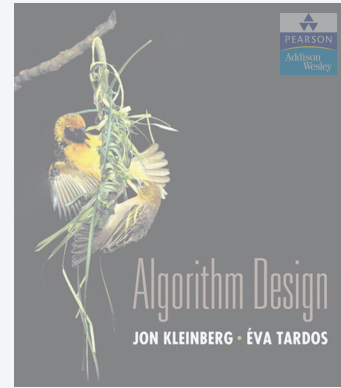
- ▶ Ford-Fulkerson demo
- ▶ exponential-time example
- ▶ pathological example

Lecture slides by Kevin Wayne

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<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>

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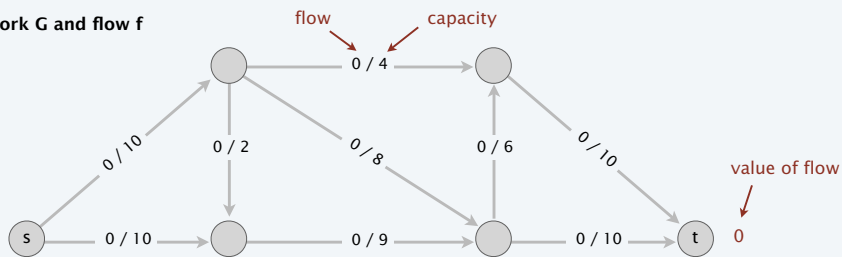
7. NETWORK FLOW I

- ▶ Ford-Fulkerson demo
- ▶ exponential-time example
- ▶ pathological example

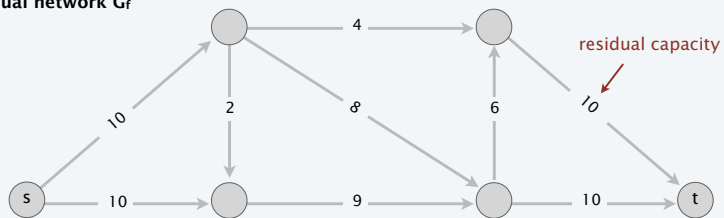
SECTION 7.1

Ford-Fulkerson algorithm demo

network G and flow f



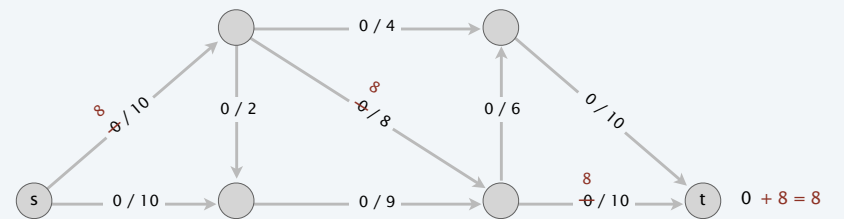
residual network G_f



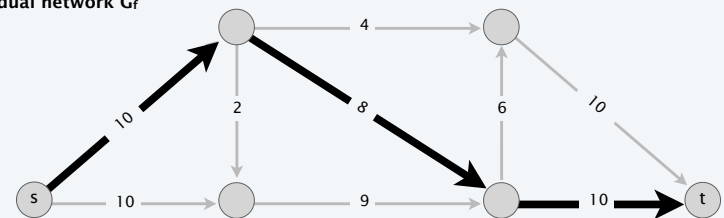
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Ford-Fulkerson algorithm demo

network G and flow f



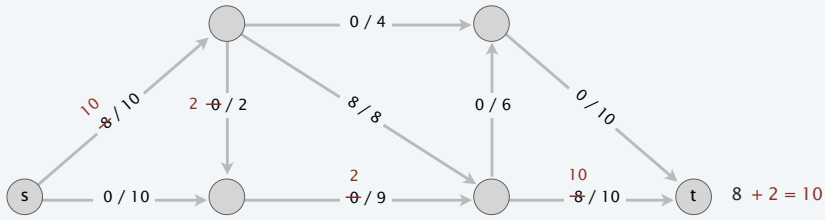
residual network G_f



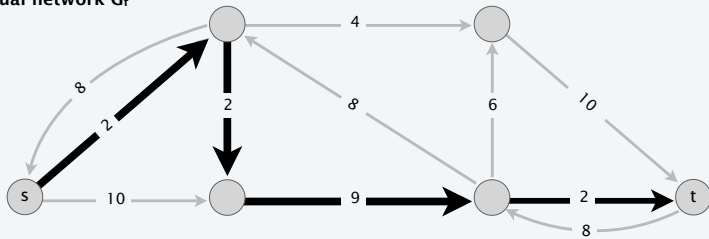
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Ford-Fulkerson algorithm demo

network G and flow f



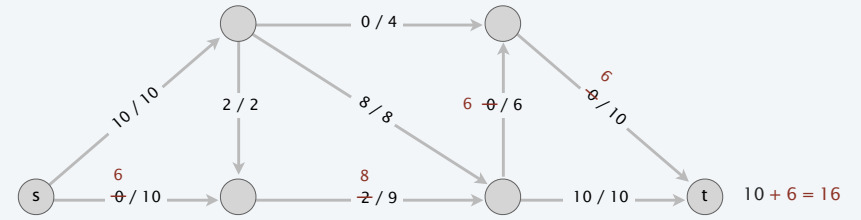
residual network G_r



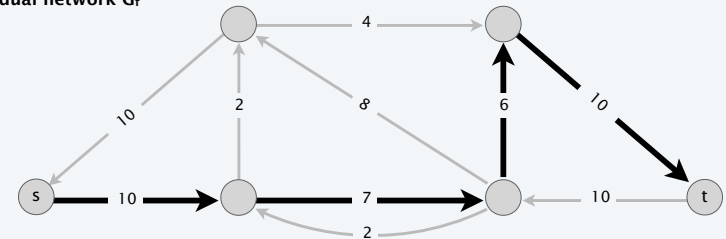
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Ford-Fulkerson algorithm demo

network G and flow f



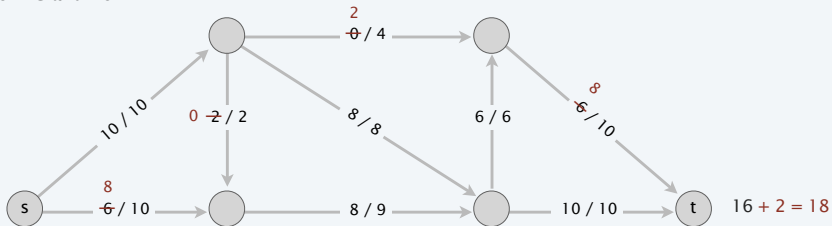
residual network G_r



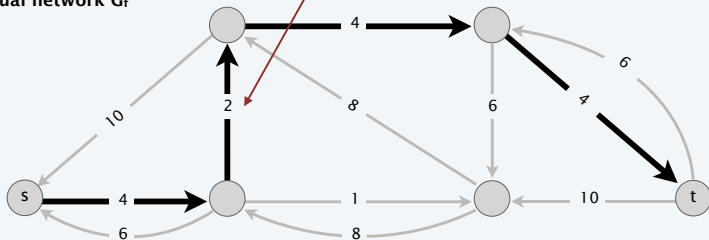
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Ford-Fulkerson algorithm demo

network G and flow f



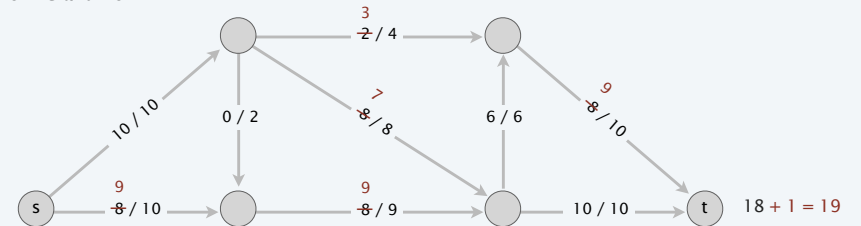
residual network G_r



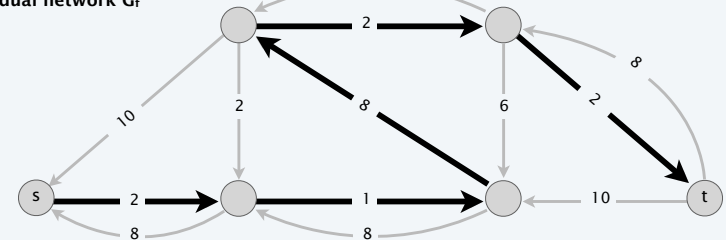
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Ford-Fulkerson algorithm demo

network G and flow f



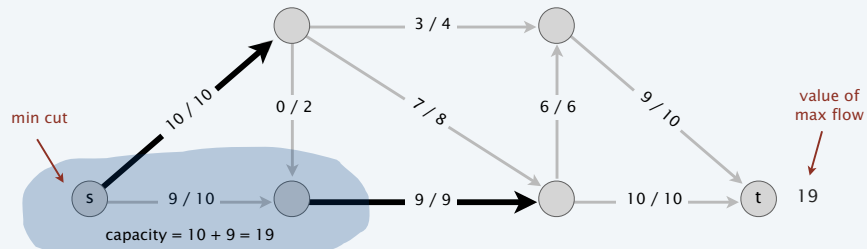
residual network G_r



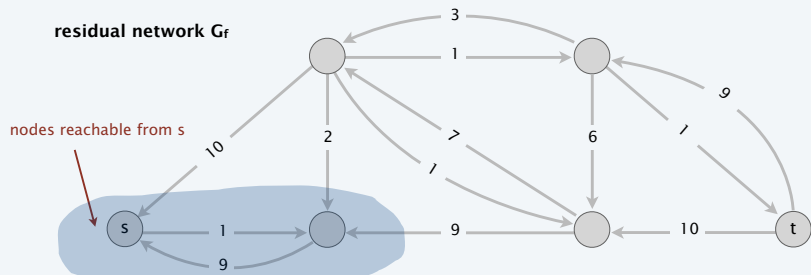
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Ford-Fulkerson algorithm demo

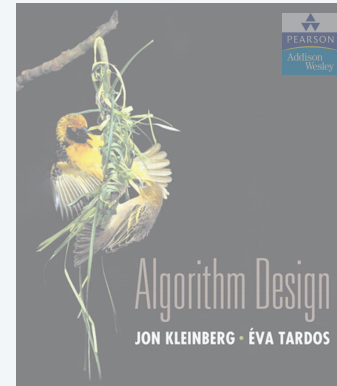
network G and flow f



residual network G_f



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SECTION 7.1

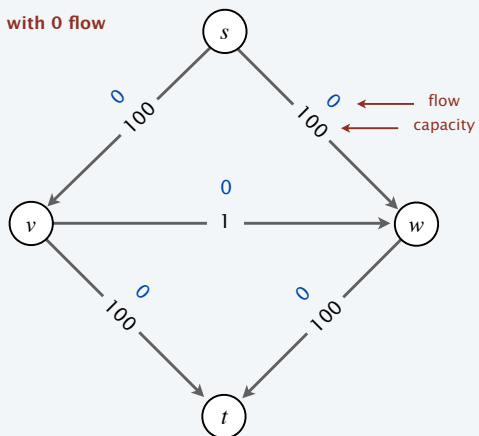
7. NETWORK FLOW I

- ▶ Ford-Fulkerson demo
- ▶ exponential-time example
- ▶ pathological example

Ford-Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.

initialize with 0 flow

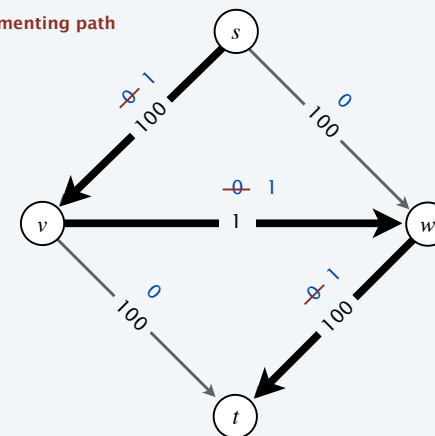


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Ford-Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.

1st augmenting path

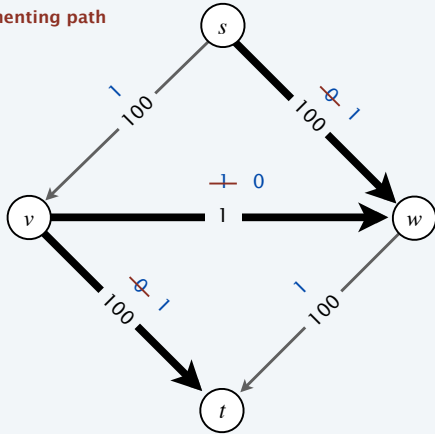


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Ford-Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.

2nd augmenting path

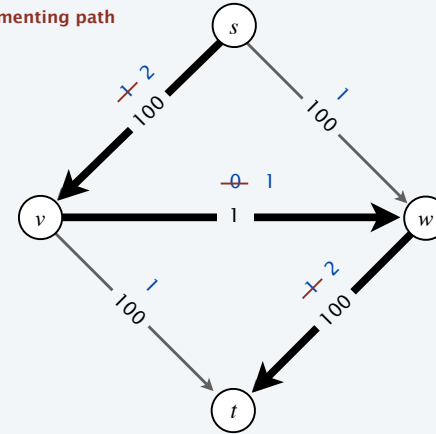


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Ford-Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.

3rd augmenting path

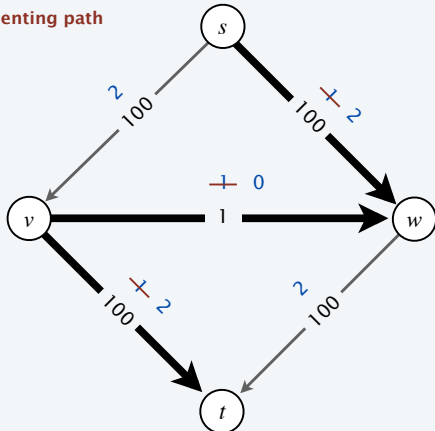


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Ford-Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.

4th augmenting path



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Ford-Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.

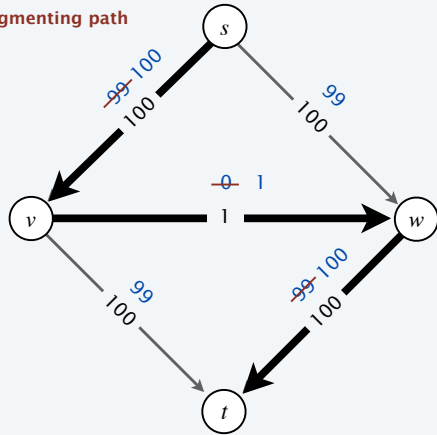
...

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Ford-Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.

199th augmenting path

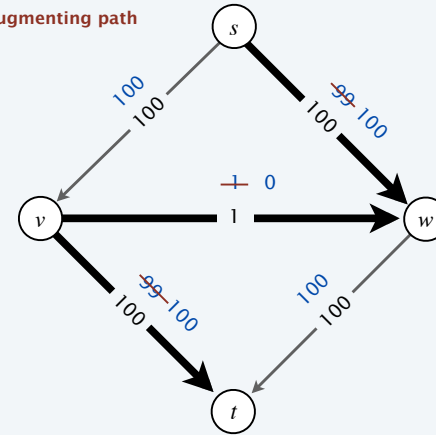


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Ford-Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.

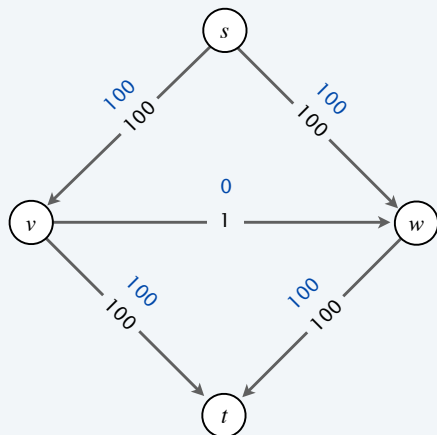
200th augmenting path



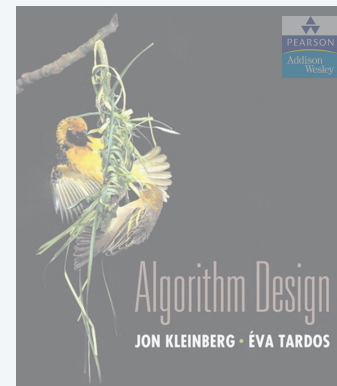
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Ford-Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.



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SECTION 7.1

7. NETWORK FLOW I

- ▶ *Ford-Fulkerson demo*
- ▶ *exponential-time example*
- ▶ *pathological example*

Ford-Fulkerson algorithm: pathological example

Intuition. Let $r > 0$ satisfy $r^2 = 1 - r$.

- Initially, some residual capacities are 1 and r .
- After two augmenting paths, some residual capacities are r and r^2 .
- After two more augmenting paths, some residual capacities are r^2 and r^3 .
- After two more, some residual capacities are r^3 and r^4 .
- By carefully choreographing the augmenting paths, infinitely many residual capacities arise!

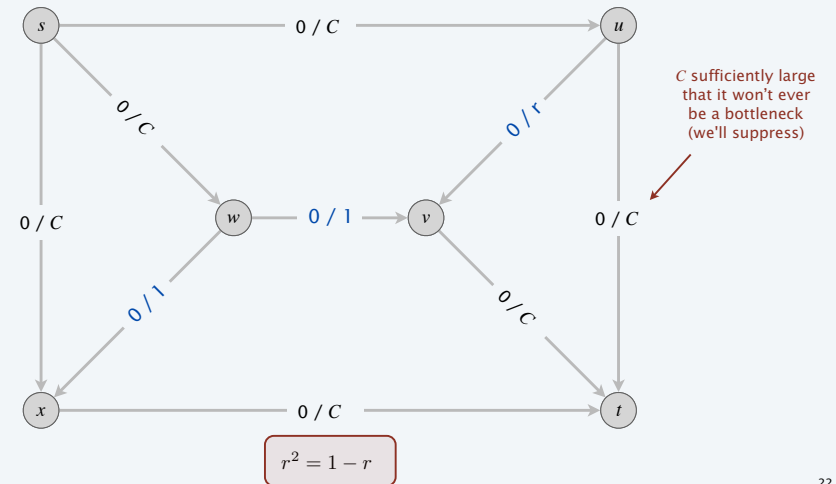
$$r = \frac{\sqrt{5}-1}{2} \implies r^2 = 1 - r$$

$$r \approx 0.618 \implies r^4 < r^3 < r^2 < r < 1$$

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Ford-Fulkerson algorithm: pathological example

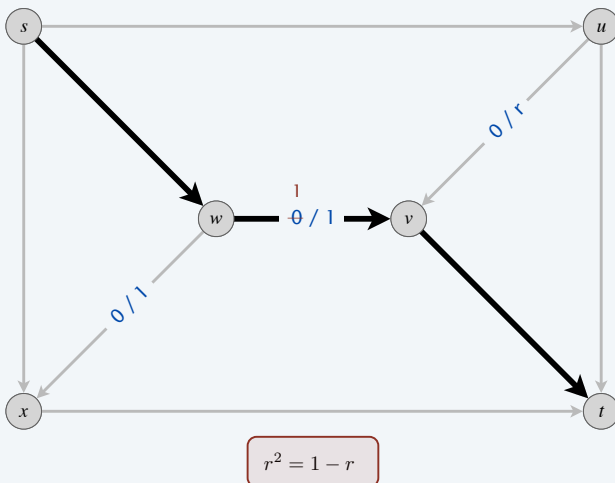
flow network G



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Ford-Fulkerson algorithm: pathological example

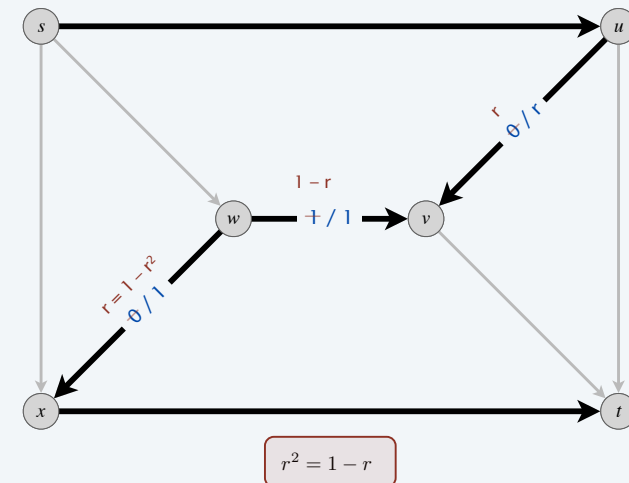
augmenting path 1: $s \rightarrow w \rightarrow v \rightarrow t$ (bottleneck capacity = 1)



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Ford-Fulkerson algorithm: pathological example

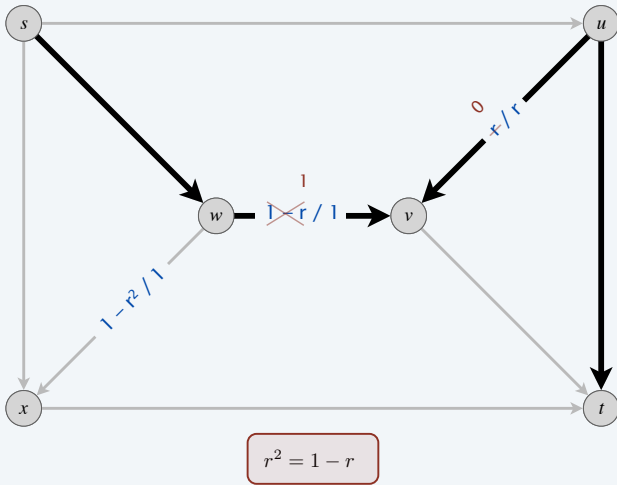
augmenting path 2: $s \rightarrow u \rightarrow v \rightarrow w \rightarrow x \rightarrow t$ (bottleneck capacity = r)



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Ford-Fulkerson algorithm: pathological example

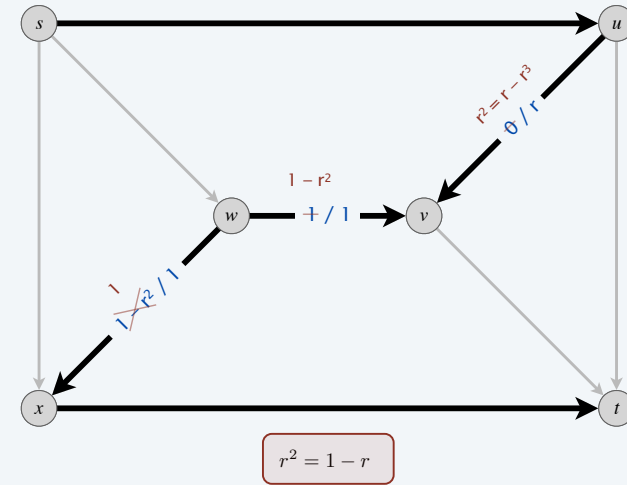
augmenting path 3: $s \rightarrow w \rightarrow v \rightarrow u \rightarrow t$ (bottleneck capacity = r)



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Ford-Fulkerson algorithm: pathological example

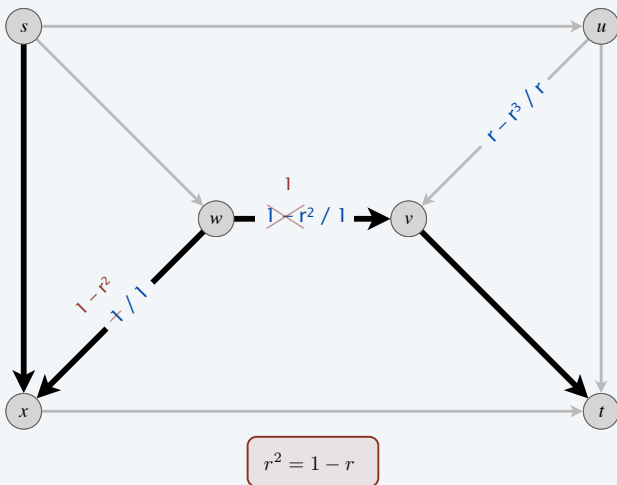
augmenting path 4: $s \rightarrow u \rightarrow v \rightarrow w \rightarrow x \rightarrow t$ (bottleneck capacity = r^2)



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Ford-Fulkerson algorithm: pathological example

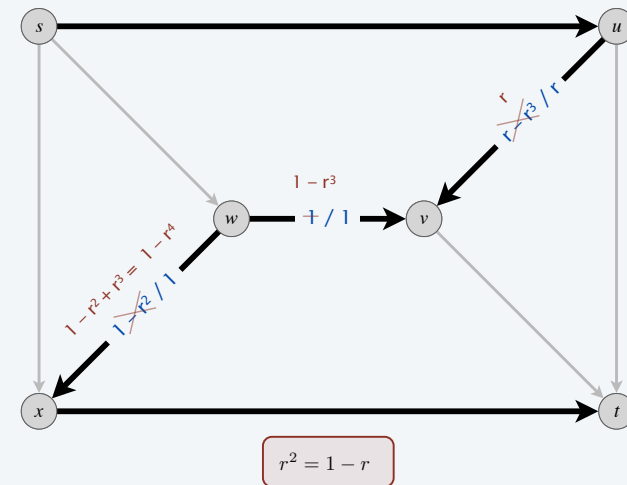
augmenting path 5: $s \rightarrow x \rightarrow w \rightarrow v \rightarrow t$ (bottleneck capacity = r^2)



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Ford-Fulkerson algorithm: pathological example

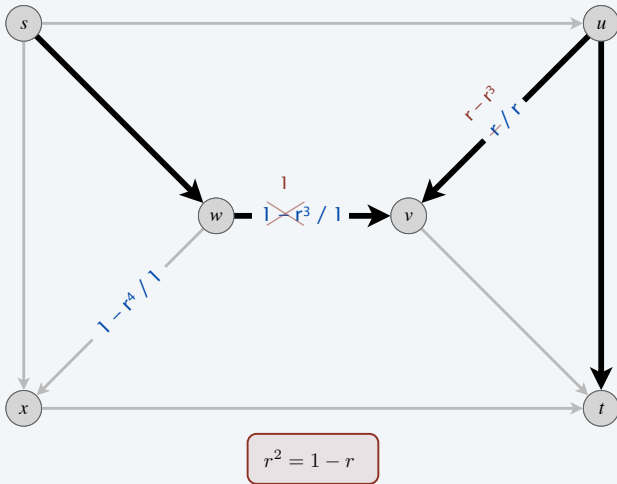
augmenting path 6: $s \rightarrow u \rightarrow v \rightarrow w \rightarrow x \rightarrow t$ (bottleneck capacity = r^3)



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Ford-Fulkerson algorithm: pathological example

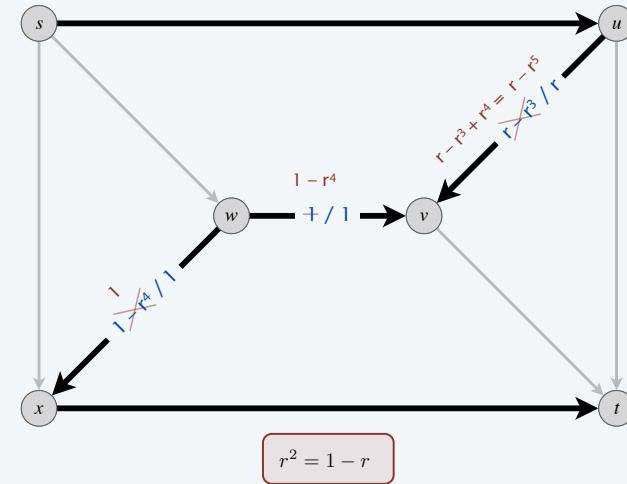
augmenting path 7: $s \rightarrow w \rightarrow v \rightarrow u \rightarrow t$ (bottleneck capacity = r^3)



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Ford-Fulkerson algorithm: pathological example

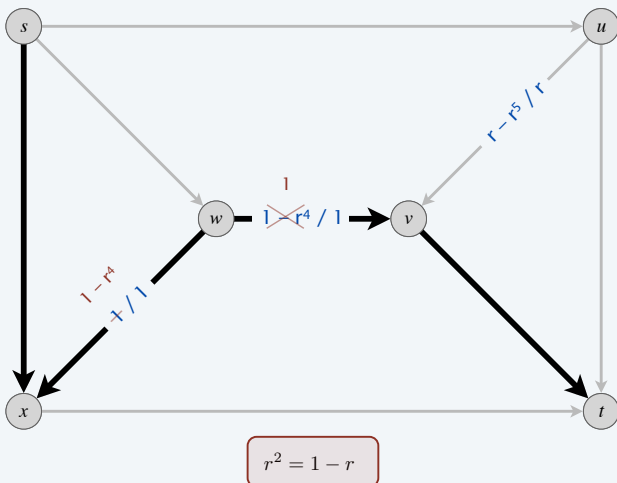
augmenting path 8: $s \rightarrow u \rightarrow v \rightarrow w \rightarrow x \rightarrow t$ (bottleneck capacity = r^4)



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Ford-Fulkerson algorithm: pathological example

augmenting path 9: $s \rightarrow x \rightarrow w \rightarrow v \rightarrow t$ (bottleneck capacity = r^4)



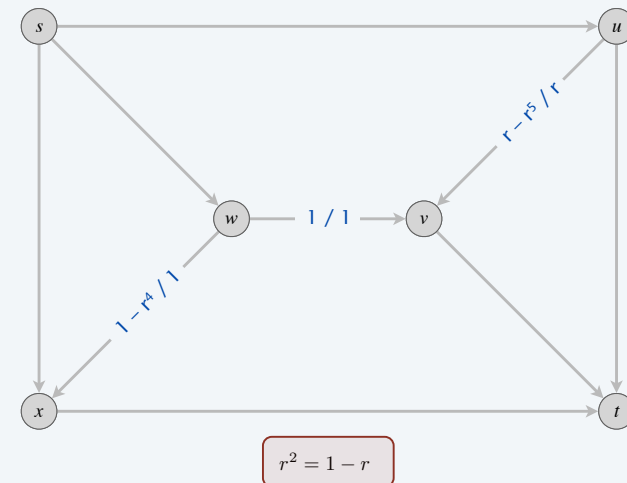
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Ford-Fulkerson algorithm: pathological example

flow after augmenting path 1: $\{ r - r^1, 1, 1 - r^0 \}$ (value of flow = 1)

flow after augmenting path 5: $\{ r - r^3, 1, 1 - r^2 \}$ (value of flow = $1 + 2r + 2r^2$)

flow after augmenting path 9: $\{ r - r^5, 1, 1 - r^4 \}$ (value of flow = $1 + 2r + 2r^2 + 2r^3 + 2r^4$)



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Ford–Fulkerson algorithm: pathological example

Theorem. The Ford–Fulkerson algorithm may not terminate; moreover, it may converge to a value not equal to the value of the maximum flow.

Pf.

- After $(1 + 4k)$ augmenting paths of the form just described, the value of the flow

$$= 1 + 2 \sum_{i=1}^{2k} r^i$$

$$\leq 1 + 2 \sum_{i=1}^{\infty} r^i$$

$$= 1 + \frac{2r}{1-r}$$

$$< 5$$

$$r = \frac{\sqrt{5}-1}{2}$$

- Value of maximum flow = $2C + 1$. ▀

Reference

