Dynamic programming

Bellman. Pioneered the systematic study of dynamic programming in 1950s.

Etymology.
- Dynamic programming = planning over time.
- Secretary of Defense had pathological fear of mathematical research.
- Bellman sought a “dynamic” adjective to avoid conflict.

Dynamic programming applications

Application areas.
- Computer science: AI, compilers, systems, graphics, theory, ....
- Operations research.
- Information theory.
- Control theory.
- Bioinformatics.

Some famous dynamic programming algorithms.
- Avidan–Shamir for seam carving.
- Unix diff for comparing two files.
- Viterbi for hidden Markov models.
- De Boor for evaluating spline curves.
- Bellman–Ford–Moore for shortest path.
- Knuth–Plass for word wrapping text in TeX.
- Cocke–Kasami–Younger for parsing context-free grammars.
- Needleman–Wunsch/Smith–Waterman for sequence alignment.
6. Dynamic Programming I

- weighted interval scheduling
- segmented least squares
- knapsack problem
- RNA secondary structure

Sections 6.1–6.2

Weighted interval scheduling

- Job \( j \) starts at \( s_j \), finishes at \( f_j \), and has weight \( w_j > 0 \).
- Two jobs are compatible if they don’t overlap.
- Goal: find max-weight subset of mutually compatible jobs.

Earliest-finish-time first algorithm

Earliest finish-time first.
- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Recall. Greedy algorithm is correct if all weights are 1.

Observation. Greedy algorithm fails spectacularly for weighted version.
Weighted interval scheduling

**Convention.** Jobs are in ascending order of finish time: \( f_1 \leq f_2 \leq \ldots \leq f_n \).

**Def.** \( p(j) \) = largest index \( i < j \) such that job \( i \) is compatible with \( j \).

**Ex.** \( p(8) = 1, p(7) = 3, p(2) = 0 \).

\[ \text{time} \]

\[ \begin{array}{cccccccccc}
    & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
0 &   &   &   &   &   &   &   &   &   &   &   &   \\
1 &   &   &   &   &   &   &   &   &   &   &   &   \\
2 &   &   &   &   &   &   &   &   &   &   &   &   \\
3 &   &   &   &   &   &   &   &   &   &   &   &   \\
4 &   &   &   &   &   &   &   &   &   &   &   &   \\
5 &   &   &   &   &   &   &   &   &   &   &   &   \\
6 &   &   &   &   &   &   &   &   &   &   &   &   \\
7 &   &   &   &   &   &   &   &   &   &   &   &   \\
8 &   &   &   &   &   &   &   &   &   &   &   &   \\
\end{array} \]

\( j \) is leftmost interval that ends before \( j \) begins

Dynamic programming: binary choice

**Def.** \( OPT(j) = \max \text{ weight of any subset of mutually compatible jobs for subproblem consisting only of jobs } 1, 2, \ldots, j \).

**Goal.** \( OPT(n) = \max \text{ weight of any subset of mutually compatible jobs.} \)

**Case 1.** \( OPT(j) \) does not select job \( j \).
- Must be an optimal solution to problem consisting of remaining jobs \( 1, 2, \ldots, j - 1 \).

**Case 2.** \( OPT(j) \) selects job \( j \).
- Collect profit \( w_j \).
- Can’t use incompatible jobs \( \{ p(j) + 1, p(j) + 2, \ldots, j - 1 \} \).
- Must include optimal solution to problem consisting of remaining compatible jobs \( 1, 2, \ldots, p(j) \).

**Bellman equation.**
\[
OPT(j) = \begin{cases} 
0 & \text{if } j = 0 \\
\max \{ OPT(j - 1), w_j + OPT(p(j)) \} & \text{if } j > 0 
\end{cases}
\]

Dynamic programming: quiz 1

What is running time of \( \text{COMPUTE–OPT}(n) \) in the worst case?

A. \( \Theta(n \log n) \)

B. \( \Theta(n^2) \)

C. \( \Theta(1.618^n) \)

D. \( \Theta(2^n) \)

\[
\text{COMPUTE–OPT}(j) \]

\[
\begin{cases} 
0 & \text{if } j = 0 \\
\max \{ \text{COMPUTE–OPT}(j - 1), w_j + \text{COMPUTE–OPT}(p(j)) \} & \text{if } j > 0 
\end{cases}
\]

**Brute-force** \((n, s_1, \ldots, s_n, f_1, \ldots, f_n, w_1, \ldots, w_n)\)

Sort jobs by finish time and renumber so that \( f_1 \leq f_2 \leq \ldots \leq f_n \).
Compute \( p[1], p[2], \ldots, p[n] \) via binary search.

**Return:** \( \text{COMPUTE–OPT}(n) \).

**Compute–OPT(j)**

\[
\begin{cases} 
0 & \text{if } j = 0 \\
\max \{ \text{COMPUTE–OPT}(j - 1), w_j + \text{COMPUTE–OPT}(p[j]) \} & \text{if } j > 0 
\end{cases}
\]
Weighted interval scheduling: brute force

Observation. Recursive algorithm is spectacularly slow because of overlapping subproblems ⇒ exponential-time algorithm.

Ex. Number of recursive calls for family of “layered” instances grows like Fibonacci sequence.

---

Weighted interval scheduling: memoization

Top-down dynamic programming (memoization).

- Cache result of subproblem $j$ in $M[j]$.
- Use $M[j]$ to avoid solving subproblem $j$ more than once.

\[
\text{TOP-DOWN}(n, s_1, \ldots, s_n, f_1, \ldots, f_n, w_1, \ldots, w_n)
\]

Sort jobs by finish time and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$. Compute $p[1], p[2], \ldots, p[n]$ via binary search.

$M[0] \leftarrow 0$. --- global array

\[
\text{RETURN } \text{M-COMPUTE-OPT}(n).
\]

\[
\begin{align*}
\text{M-COMPUTE-OPT}(j) \\
\text{IF } (M[j] \text{ is uninitialized}) \\
M[j] & \leftarrow \max \{ \text{M-COMPUTE-OPT}(j-1), w_j + \text{M-COMPUTE-OPT}(p[j]) \}.
\end{align*}
\]

\[
\text{RETURN } M[j].
\]

---

Weighted interval scheduling: running time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.

Pf.

- Sort by finish time: $O(n \log n)$ via mergesort.
- Compute $p[j]$ for each $j$: $O(n \log n)$ via binary search.

- $\text{M-COMPUTE-OPT}(j)$: each invocation takes $O(1)$ time and either
  - (1) returns an initialized value $M[j]$
  - (2) initializes $M[j]$ and makes two recursive calls

- Progress measure $\Phi = \# \text{initialized entries among } M[1..n]$.
  - initially $\Phi = 0$; throughout $\Phi \leq n$.
  - (2) increases $\Phi$ by 1 ⇒ $\leq 2n$ recursive calls.

- Overall running time of $\text{M-COMPUTE-OPT}(n)$ is $O(n)$.  •
Weighted interval scheduling: finding a solution

Q. DP algorithm computes optimal value. How to find optimal solution?
A. Make a second pass by calling FIND-SOLUTION(n).

**FIND-SOLUTION(j)**

IF (j = 0)  
RETURN ∅.
ELSE IF (w_j + M[p(j)] > M[j-1])
RETURN (j) ∪ FIND-SOLUTION(p(j)).
ELSE
RETURN FIND-SOLUTION(j - 1).

M[j] = max { M[j - 1], w_j + M[p(j)] }.

**Analysis.** # of recursive calls ≤ n ⇒ O(n).

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Weighted interval scheduling: bottom-up dynamic programming

**Bottom-up dynamic programming.** Unwind recursion.

**BOTTOM-UP(n, s1, ..., sn, f1, ..., fn, w1, ..., wn)**

Sort jobs by finish time and renumber so that f_1 ≤ f_2 ≤ ... ≤ f_n.
Compute p[1], p[2], ..., p[n].

M[0] ← 0.

FOR j = 1 TO n
M[j] ← max { M[j - 1], w_j + M[p(j)] }.

**Running time.** The bottom-up version takes O(n log n) time.

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**MAXIMUM SUBARRAY PROBLEM**

**Goal.** Given an array x of n integer (positive or negative), find a contiguous subarray whose sum is maximum.

<table>
<thead>
<tr>
<th>12</th>
<th>5</th>
<th>-1</th>
<th>31</th>
<th>-61</th>
<th>59</th>
<th>26</th>
<th>-53</th>
<th>58</th>
<th>97</th>
<th>-93</th>
<th>-23</th>
<th>84</th>
<th>-15</th>
<th>6</th>
</tr>
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</tr>
</tbody>
</table>

**Applications.** Computer vision, data mining, genomic sequence analysis, technical job interviews, ...

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**MAXIMUM RECTANGLE PROBLEM**

**Goal.** Given an n-by-n matrix A, find a rectangle whose sum is maximum.

A = $\begin{bmatrix}
-2 & 5 & 0 & -5 & -2 & 2 & -3 \\
4 & -3 & 1 & 3 & 2 & 1 & -1 \\
-5 & 6 & 3 & -5 & -1 & -4 & -2 \\
-1 & -1 & 3 & -1 & 4 & 1 & 1 \\
3 & -3 & 2 & 0 & 3 & -3 & -2 \\
-2 & 1 & -2 & 1 & 1 & 3 & -1 \\
2 & -4 & 0 & 1 & 0 & -3 & -1
\end{bmatrix}$

*13*

**Applications.** Databases, image processing, maximum likelihood estimation, technical job interviews, ...
Section 6.3

Least squares

**Least squares.** Foundational problem in statistics.
- Given \( n \) points in the plane: \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\).
- Find a line \( y = ax + b \) that minimizes the sum of the squared error:

\[
SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2
\]

**Solution.** Calculus \( \Rightarrow \) min error is achieved when

\[
a = \frac{\sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{\sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}
\]

Segmented least squares

**Segmented least squares.**
- Points lie roughly on a sequence of several line segments.
- Given \( n \) points in the plane: \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) with \( x_1 < x_2 < \ldots < x_n \), find a sequence of lines that minimizes \( f(x) \).

Q. What is a reasonable choice for \( f(x) \) to balance accuracy and parsimony?

**Goal.** Minimize \( f(x) = E + cL \) for some constant \( c > 0 \), where
- \( E \) = sum of the sums of the squared errors in each segment.
- \( L \) = number of lines.
Dynamic programming: multiway choice

Notation.
- \( OPT(j) \) = minimum cost for points \( p_1, p_2, \ldots, p_j \).
- \( e_{ij} \) = SSE for for points \( p_i, p_{i+1}, \ldots, p_j \).

To compute \( OPT(j) \):
- Last segment uses points \( p_i, p_{i+1}, \ldots, p_j \) for some \( i \neq j \).
- Cost = \( e_{ij} + c + OPT(i-1) \).  

Bellman equation.

\[
OPT(j) = \begin{cases} 
0 & \text{if } j = 0 \\
\min_{1 \leq i \leq j} \{ e_{ij} + c + OPT(i-1) \} & \text{if } j > 0 
\end{cases}
\]

Segmented least squares algorithm

SEGMENTED-LEAST-SQUARES\( (n, p_1, \ldots, p_n, c) \)

FOR \( j = 1 \) TO \( n \)

FOR \( i = 1 \) TO \( j \)

Compute the SSE \( e_{ij} \) for the points \( p_i, p_{i+1}, \ldots, p_j \).

\( M[0] \leftarrow 0 \).

FOR \( j = 1 \) TO \( n \)

\( M[j] \leftarrow \min_{1 \leq i \leq j} \{ e_{ij} + c + M[i-1] \} \).

RETURN \( M[n] \).

Segmented least squares analysis

Theorem. [Bellman 1961] DP algorithm solves the segmented least squares problem in \( O(n^3) \) time and \( O(n^2) \) space.

Pf.
- Bottleneck = computing SSE \( e_{ij} \) for each \( i \) and \( j \).

\[
a_{ij} = \frac{n \sum_k x_k y_k - (\sum_k x_k)(\sum_k y_k)}{n \sum_k x_k^2 - (\sum_k x_k)^2}, \quad b_{ij} = \frac{\sum_k y_k - a_{ij} \sum_k x_k}{n}
\]

- \( O(n) \) to compute \( e_{ij} \).

Remark. Can be improved to \( O(n^2) \) time.
- For each \( i \): precompute cumulative sums \( \sum_{k=1}^i x_k, \sum_{k=1}^i y_k, \sum_{k=1}^i x_k^2, \sum_{k=1}^i x_k y_k \).
- Using cumulative sums, can compute \( e_{ij} \) in \( O(1) \) time.

6. Dynamic Programming I

- weighted interval scheduling
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- RNA secondary structure
Knapsack problem

Goal. Pack knapsack so as to maximize total value of items taken.
- There are \( n \) items: item \( i \) provides value \( v_i > 0 \) and weighs \( w_i > 0 \).
- Value of a subset of items = sum of values of individual items.
- Knapsack has weight limit of \( W \).

Ex. The subset \( \{1, 2, 5\} \) has value $35 (and weight 10).
Ex. The subset \( \{3, 4\} \) has value $40 (and weight 11).

Assumption. All values and weights are integral.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( v_i )</th>
<th>( w_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1</td>
<td>1 kg</td>
</tr>
<tr>
<td>2</td>
<td>$6</td>
<td>2 kg</td>
</tr>
<tr>
<td>3</td>
<td>$18</td>
<td>5 kg</td>
</tr>
<tr>
<td>4</td>
<td>$22</td>
<td>6 kg</td>
</tr>
<tr>
<td>5</td>
<td>$28</td>
<td>7 kg</td>
</tr>
</tbody>
</table>

Knapsack instance (weight limit \( W = 11 \))

Dynamic programming: quiz 3

Which subproblems?

A. \( OPT(w) \) = optimal value of knapsack problem with weight limit \( w \).
B. \( OPT(i) \) = optimal value of knapsack problem with items \( 1, \ldots, i \) subject to weight limit \( w \).
C. \( OPT(i, w) \) = optimal value of knapsack problem with items \( 1, \ldots, i \) subject to weight limit \( w \).
D. Any of the above.

Dynamic programming: two variables

Def. \( OPT(i, w) = \) optimal value of knapsack problem with items \( 1, \ldots, i \), subject to weight limit \( w \).

Goal. \( OPT(n, W) \).

Case 1. \( OPT(i, w) \) does not select item \( i \).
- \( OPT(i, w) \) selects best of \( \{1, 2, \ldots, i - 1\} \) subject to weight limit \( w \).

Case 2. \( OPT(i, w) \) selects item \( i \).
- Collect value \( v_i \).
- New weight limit = \( w - w_i \).
- \( OPT(i, w) \) selects best of \( \{1, 2, \ldots, i - 1\} \) subject to new weight limit.

Bellman equation.

\[
OPT(i, w) = \begin{cases} 
0 & \text{if } i = 0 \\
OPT(i - 1, w) & \text{if } w_i > w \\
\max \{ OPT(i - 1, w), v_i + OPT(i - 1, w - w_i) \} & \text{otherwise}
\end{cases}
\]
Knapsack problem: bottom-up dynamic programming

**Knapsack** \((n, W, w_1, \ldots, w_n, v_1, \ldots, v_n)\)

**FOR** \(w = 0\) **TO** \(W\)

\[ M[0, w] \leftarrow 0. \]

**FOR** \(i = 1\) **TO** \(n\)

**FOR** \(w = 0\) **TO** \(W\)

**IF** \((w_i > w)\)

\[ M[i, w] \leftarrow M[i-1, w]. \]

**ELSE**

\[ M[i, w] \leftarrow \max \{ M[i-1, w], v_i + M[i-1, w-w_i] \}. \]

**RETURN** \(M[n, W]\).

\[ \text{OPT}(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ \text{OPT}(i-1, w) & \text{if } w_i > w \\ \max \{ \text{OPT}(i-1, w), v_i + \text{OPT}(i-1, w-w_i) \} & \text{otherwise} \end{cases} \]

Knapsack problem: running time

**Theorem.** The DP algorithm solves the knapsack problem with \(n\) items and maximum weight \(W\) in \(\Theta(n W)\) time and \(\Theta(n W)\) space.

**Pf.**

- Takes \(O(1)\) time per table entry.
- There are \(\Theta(n W)\) table entries.
- After computing optimal values, can trace back to find solution: \(\text{OPT}(i, w)\) takes item \(i\) iff \(M[i, w] > M[i-1, w]\).

**Remarks.**

- Algorithm depends critically on assumption that weights are integral.
- Assumption that values are integral was not used.

Dynamic programming: quiz 4

**Does there exist a poly-time algorithm for the knapsack problem?**

A. Yes, because the DP algorithm takes \(\Theta(n W)\) time.

B. No, because \(\Theta(n W)\) is not a polynomial function of the input size.

C. No, because the problem is \(\text{NP}-\text{hard}\).

D. Unknown.
**COIN CHANGING**

**Problem.** Given $n$ coin denominations $\{c_1, c_2, \ldots, c_n\}$ and a target value $V$, find the fewest coins needed to make change for $V$ (or report impossible).

**Recall.** Greedy cashier’s algorithm is optimal for U.S. coin denominations, but not for arbitrary coin denominations.

**Ex.** $\{1, 10, 21, 34, 70, 100, 350, 1295, 1500\}$.

**Optimal.** $140\text{€} = 70 + 70$.

**RNA secondary structure**

**RNA.** String $B = b_1b_2\ldots b_n$ over alphabet $\{A, C, G, U\}$.

**Secondary structure.** RNA is single-stranded so it tends to loop back and form base pairs with itself. This structure is essential for understanding behavior of molecule.

**6. Dynamic Programming I**

- weighted interval scheduling
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- RNA secondary structure

**RNA secondary structure**

**Secondary structure.** A set of pairs $S = \{(b_i, b_j)\}$ that satisfy:

- [Watson–Crick] $S$ is a matching and each pair in $S$ is a Watson–Crick complement: A–U, U–A, C–G, or G–C.

**RNA secondary structure for** GUCGAUUGAGCGAAUGUAACAACGUGGCUACGGCGAGA

**RNA secondary structure for** ACGUGGCCCCAU

$B = ACGUGGCCCCAU$

$S = \{(b_1, b_{10}), (b_2, b_9), (b_3, b_8)\}$
RNA secondary structure

Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

- [Watson–Crick] $S$ is a matching and each pair in $S$ is a Watson–Crick complement: A–U, U–A, C–G, or G–C.
- [No sharp turns] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.

Free-energy hypothesis. RNA molecule will form the secondary structure with the minimum total free energy.

**Goal.** Given an RNA molecule $B = b_1b_2\ldots b_n$, find a secondary structure $S$ that maximizes the number of base pairs.
Is the following a secondary structure?

A. Yes.

B. No, violates Watson–Crick condition.

C. No, violates no-sharp-turns condition.

D. No, violates no-crossing condition.

RNA secondary structure: subproblems

First attempt. \( OPT(j) = \) maximum number of base pairs in a secondary structure of the substring \( b_1b_2 \ldots b_j \).

Goal. \( OPT(n) \).

Choice. Match bases \( b_i \) and \( b_j \).

Difficulty. Results in two subproblems (but one of wrong form).
- Find secondary structure in \( b_1b_2 \ldots b_{i-1} \).
- Find secondary structure in \( b_{i+1}b_{i+2} \ldots b_{j-1} \).

Dynamic programming: quiz 5

Dynamic programming: quiz 6

Which subproblems?

A. \( OPT(j) = \) max number of base pairs in secondary structure of the substring \( b_1b_2 \ldots b_j \).

B. \( OPT(j) = \) max number of base pairs in secondary structure of the substring \( b_jb_{j+1} \ldots b_n \).

C. Either A or B.

D. Neither A nor B.

Dynamic programming over intervals

Def. \( OPT(i, j) = \) maximum number of base pairs in a secondary structure of the substring \( b_ib_{i+1} \ldots b_j \).

Case 1. If \( i \geq j - 4 \).
- \( OPT(i, j) = 0 \) by no-sharp-turns condition.

Case 2. Base \( b_j \) is not involved in a pair.
- \( OPT(i, j) = OPT(i, j - 1) \).

Case 3. Base \( b_j \) pairs with \( b_t \) for some \( i \leq t < j - 4 \).
- Non-crossing condition decouples resulting two subproblems.
- \( OPT(i, j) = 1 + \max \{ OPT(i, t - 1) + OPT(t + 1, j - 1) \} \).

Dynamic programming over intervals
Dynamic programming: quiz 7

In which order to compute $OPT(i, j)$?

A. Increasing $i$, then $j$.
B. Increasing $j$, then $i$.
C. Either A or B.
D. Neither A nor B.

Bottom-up dynamic programming over intervals

Q. In which order to solve the subproblems?
A. Do shortest intervals first—increasing order of $|j - i|$.

\[ \text{RNA-SECONDARY-STRUCTURE}(n, b_1, \ldots, b_n) \]
\[ \text{FOR } k = 5 \text{ TO } n - 1 \]
\[ \text{FOR } i = 1 \text{ TO } n - k \]
\[ j \leftarrow i + k. \]
\[ \text{Compute } M[i, j] \text{ using formula.} \]
\[ \text{RETURN } M[1, n]. \]

Theorem. The DP algorithm solves the RNA secondary structure problem in $O(n^3)$ time and $O(n^2)$ space.

Dynamic programming summary

Outline.
- Define a collection of subproblems.
- Solution to original problem can be computed from subproblems.
- Natural ordering of subproblems from “smallest” to “largest” that enables determining a solution to a subproblem from solutions to smaller subproblems.

Techniques.
- Binary choice: weighted interval scheduling.
- Multiway choice: segmented least squares.
- Adding a new variable: knapsack problem.
- Intervals: RNA secondary structure.

Top-down vs. bottom-up dynamic programming. Opinions differ.