DATA STRUCTURES I, II, III, AND IV

I. Amortized Analysis
II. Binary and Binomial Heaps
III. Fibonacci Heaps
IV. Union–Find

Lecture slides by Kevin Wayne
http://www.cs.princeton.edu/~wayne/kleinberg-tardos
Data structures

**Static problems.** Given an input, produce an output.
*Ex.* Sorting, FFT, edit distance, shortest paths, MST, max-flow, ...

**Dynamic problems.** Given a sequence of operations (given one at a time), produce a sequence of outputs.
*Ex.* Stack, queue, priority queue, symbol table, union–find, ....

**Algorithm.** Step-by-step procedure to solve a problem.

**Data structure.** Way to store and organize data.
*Ex.* Array, linked list, binary heap, binary search tree, hash table, ...

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1 2 3 4 5 6 7 8
33 22 55 23 16 63 86 9
```

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1 • → 4 • → 1 • → 3 •
```

```
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Appetizer

**Goal.** Design a data structure to support all operations in $O(1)$ time.

- **INIT($n$):** create and return an *initialized* array (all zero) of length $n$.
- **READ($A, i$):** return element $i$ in array.
- **WRITE($A, i, value$):** set element $i$ in array to $value$.

**Assumptions.**

- Can **malloc** an uninitialized array of length $n$ in $O(1)$ time.
- Given an array, can read or write element $i$ in $O(1)$ time.

**Remark.** An array does **INIT** in $\Theta(n)$ time and **READ** and **WRITE** in $\Theta(1)$ time.
Appetizer

- $A[i]$ stores the current value for READ (if initialized).
- $k = \text{number of initialized entries}$.
- $C[j] = \text{index of } j^{\text{th}} \text{ initialized element for } j = 1, \ldots, k$.
- If $C[j] = i$, then $B[i] = j$ for $j = 1, \ldots, k$.

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. Ahead.

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$k = 4$

## Appetizer

**INIT** \((A, n)\)

\[
k \leftarrow 0.
\]

\[
A \leftarrow \text{ALLOC}(n).
\]

\[
B \leftarrow \text{ALLOC}(n).
\]

\[
C \leftarrow \text{ALLOC}(n).
\]

**READ** \((A, i)\)

\[
\begin{array}{l}
\text{IF (IS-INITIALIZED (A[i]))} \\
\quad \text{RETURN } A[i]. \\
\text{ELSE} \\
\quad \text{RETURN } 0.
\end{array}
\]

**WRITE** \((A, i, value)\)

\[
\begin{array}{l}
\text{IF (IS-INITIALIZED (A[i]))} \\
\quad A[i] \leftarrow value. \\
\text{ELSE} \\
\quad k \leftarrow k + 1. \\
\quad A[i] \leftarrow value. \\
\quad B[i] \leftarrow k. \\
\quad C[k] \leftarrow i.
\end{array}
\]

**IS-INITIALIZED** \((A, i)\)

\[
\begin{array}{l}
\text{IF (} 1 \leq B[i] \leq k \text{ and } (C[B[i]] = i) \text{)} \\
\quad \text{RETURN } true. \\
\text{ELSE} \\
\quad \text{RETURN } false.
\end{array}
\]
Theorem. \( A[i] \) is initialized iff both \( 1 \leq B[i] \leq k \) and \( C[B[i]] = i \).

Pf. \( \Rightarrow \)

- Suppose \( A[i] \) is the \( j^{th} \) entry to be initialized.
- Then \( C[j] = i \) and \( B[i] = j \).
- Thus, \( C[B[i]] = i \).

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\end{array}
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\( k = 4 \)

Theorem. $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

Pf. $\Leftarrow$

• Suppose $A[i]$ is uninitialized.

• If $B[i] < 1$ or $B[i] > k$, then $A[i]$ clearly uninitialized.

• If $1 \leq B[i] \leq k$ by coincidence, then we still can’t have $C[B[i]] = i$ because none of the entries $C[1..k]$ can equal $i$. ■

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\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
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k = 4
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Amortized Analysis

- binary counter
- multi-pop stack
- dynamic table
Amortized analysis

Worst-case analysis. Determine worst-case running time of a data structure operation as function of the input size $n$. 

Amortized analysis. Determine worst-case running time of a sequence of $n$ data structure operations.

Ex. Starting from an empty stack implemented with a dynamic table, any sequence of $n$ push and pop operations takes $O(n)$ time in the worst case.
Amortized analysis: applications

- Splay trees.
- Dynamic table.
- Fibonacci heaps.
- Garbage collection.
- Move-to-front list updating.
- Push–relabel algorithm for max flow.
- Path compression for disjoint-set union.
- Structural modifications to red–black trees.
- Security, databases, distributed computing, ...

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AMORTIZED COMPUTATIONAL COMPLEXITY*

ROBERT ENDRE TARJAN†

Abstract. A powerful technique in the complexity analysis of data structures is amortization, or averaging over time. Amortized running time is a realistic but robust complexity measure for which we can obtain surprisingly tight upper and lower bounds on a variety of algorithms. By following the principle of designing algorithms whose amortized complexity is low, we obtain “self-adjusting” data structures that are simple, flexible and efficient. This paper surveys recent work by several researchers on amortized complexity.

ASM(MOS) subject classifications. 68C25, 68E05
CHAPTER 17

AMORTIZED ANALYSIS

‣ binary counter
‣ multi-pop stack
‣ dynamic table
Binary counter

**Goal.** Increment a $k$-bit binary counter (mod $2^k$).

**Representation.** $A[j] = j^{th}$ least significant bit of counter.

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**Cost model.** Number of bits flipped.
Binary counter

**Goal.** Increment a $k$-bit binary counter (mod $2^k$).

**Representation.** $A[j] = j^{th}$ least significant bit of counter.

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**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(nk)$ bits. ❯ overly pessimistic upper bound

**Pf.** At most $k$ bits flipped per increment. ■
### Aggregate method (brute force)

**Aggregate method.** Analyze cost of a sequence of operations.

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Figure 17.2 An 8-bit binary counter as its value goes from 0 to 16 by a sequence of 16 $INCREMENT$ operations. Bits that flip to achieve the next value are shaded. The running cost for flipping bits is shown at the right. Notice that the total cost is always less than twice the total number of $INCREMENT$ operations.

Operations on an initially zero counter causes $A[0]$ to flip $b_{n=2}$ times. Similarly, $A[2]$ flips only every fourth time, or $b_{n=4}$ times in a sequence of $n$ $INCREMENT$ operations. In general, for $i = 0, 1, \ldots, k$, $A[i]$ flips $b_{n=2^i}$ times in a sequence of $n$ $INCREMENT$ operations on an initially zero counter. For $i \geq k$, $A[i]$ does not exist, and so it cannot flip. The total number of flips in the sequence is thus $\sum_{i=0}^{k-1} b_{n=2^i}$ by equation (A.6). The worst-case time for a sequence of $n$ $INCREMENT$ operations on an initially zero counter is therefore $O(n)$. The average cost to feed each operation, and therefore the amortized cost per operation, is $O(n/n) = O(1)$. 


Binary counter: aggregate method

Starting from the zero counter, in a sequence of $n$ INCREMENT operations:

- Bit 0 flips $n$ times.
- Bit 1 flips $\lfloor n/2 \rfloor$ times.
- Bit 2 flips $\lfloor n/4 \rfloor$ times.
- ...

**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(n)$ bits.

**Pf.**

- Bit $j$ flips $\lfloor n/2^j \rfloor$ times.
- The total number of bits flipped is $\sum_{j=0}^{k-1} \left\lfloor \frac{n}{2^j} \right\rfloor < n \sum_{j=0}^{\infty} \frac{1}{2^j} = 2n$.

**Remark.** Theorem may be false if initial counter is not zero.
Accounting method (banker’s method)

Assign (potentially) different charges to each operation.

- $D_i$ = data structure after $i^{th}$ operation.
- $c_i$ = actual cost of $i^{th}$ operation.
- $\hat{c}_i$ = amortized cost of $i^{th}$ operation $\neq$ amount we charge operation $i$.
- When $\hat{c}_i > c_i$, we store credits in data structure $D_i$ to pay for future ops; when $\hat{c}_i < c_i$, we consume credits in data structure $D_i$.
- Initial data structure $D_0$ starts with 0 credits.

Credit invariant. The total number of credits in the data structure $\geq 0$.

$$\sum_{i=1}^{\hat{c}_i} - \sum_{i=1}^{c_i} \geq 0$$

our job is to choose suitable amortized costs so that this invariant holds
Accounting method (banker’s method)

Assign (potentially) different charges to each operation.

- \( D_i \) = data structure after \( i^{th} \) operation.
- \( c_i \) = actual cost of \( i^{th} \) operation.
- \( \hat{c}_i \) = amortized cost of \( i^{th} \) operation = amount we charge operation \( i \).
- When \( \hat{c}_i > c_i \), we store credits in data structure \( D_i \) to pay for future ops; when \( \hat{c}_i < c_i \), we consume credits in data structure \( D_i \).
- Initial data structure \( D_0 \) starts with 0 credits.

Credit invariant. The total number of credits in the data structure \( \geq 0 \).

\[
\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0
\]

Theorem. Starting from the initial data structure \( D_0 \), the total actual cost of any sequence of \( n \) operations is at most the sum of the amortized costs.

\[ \text{Pf.} \quad \text{The amortized cost of the sequence of } n \text{ operations is: } \sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i. \]

Intuition. Measure running time in terms of credits (time = money).
Binary counter: accounting method

**Credits.** One credit pays for a bit flip.

**Invariant.** Each 1 bit has one credit; each 0 bit has zero credits.

**Accounting.**

- Flip bit $j$ from 0 to 1: charge 2 credits (use one and save one in bit $j$).

---

**increment**

<table>
<thead>
<tr>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>
Binary counter: accounting method

Credits. One credit pays for a bit flip.

Invariant. Each 1 bit has one credit; each 0 bit has zero credits.

Accounting.
• Flip bit $j$ from 0 to 1: charge 2 credits (use one and save one in bit $j$).
• Flip bit $j$ from 1 to 0: pay for it with the 1 credit saved in bit $j$.

increment

<table>
<thead>
<tr>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
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Binary counter: accounting method

Credits. One credit pays for a bit flip.

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Accounting.

- Flip bit $j$ from 0 to 1: charge 2 credits (use one and save one in bit $j$).
- Flip bit $j$ from 1 to 0: pay for it with the 1 credit saved in bit $j$. 

```
  7  6  5  4  3  2  1  0
 0  1  0  1  0  0  0  0
```

![Image of binary counter]
Binary counter: accounting method

Credits. One credit pays for a bit flip.

Invariant. Each 1 bit has one credit; each 0 bit has zero credits.

Accounting.

- Flip bit \( j \) from 0 to 1: charge 2 credits (use one and save one in bit \( j \)).
- Flip bit \( j \) from 1 to 0: pay for it with the 1 credit saved in bit \( j \).

Theorem. Starting from the zero counter, a sequence of \( n \) INCREMENT operations flips \( O(n) \) bits.

Pf.

- Each INCREMENT operation flips at most one 0 bit to a 1 bit, so the amortized cost per INCREMENT \( \leq 2 \).
- Invariant \( \Rightarrow \) number of credits in data structure \( \geq 0 \).
- Total actual cost of \( n \) operations \( \leq \) sum of amortized costs \( \leq 2n \).
Potential method (physicist’s method)

Potential function.  \( \Phi(D_i) \) maps each data structure \( D_i \) to a real number s.t.:

- \( \Phi(D_0) = 0 \).
- \( \Phi(D_i) \geq 0 \) for each data structure \( D_i \).

Actual and amortized costs.

- \( c_i = \text{actual cost of } i^{th} \text{ operation.} \)
- \( \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = \text{amortized cost of } i^{th} \text{ operation.} \)

our job is to choose a potential function so that the amortized cost of each operation is low
Potential method (physicist’s method)

Potential function. \( \Phi(D_i) \) maps each data structure \( D_i \) to a real number s.t.:

- \( \Phi(D_0) = 0 \).
- \( \Phi(D_i) \geq 0 \) for each data structure \( D_i \).

Actual and amortized costs.

- \( c_i \) = actual cost of \( i^{th} \) operation.
- \( \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \) = amortized cost of \( i^{th} \) operation.

Theorem. Starting from the initial data structure \( D_0 \), the total actual cost of any sequence of \( n \) operations is at most the sum of the amortized costs.

Pf. The amortized cost of the sequence of operations is:

\[
\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1})) \\
= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0) \\
\geq \sum_{i=1}^{n} c_i \quad \blacksquare
\]
Binary counter: potential method

Potential function. Let $\Phi(D) =$ number of 1 bits in the binary counter $D$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

increment

<table>
<thead>
<tr>
<th>7</th>
<th>6</th>
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<td>1</td>
</tr>
</tbody>
</table>
Binary counter: potential method

**Potential function.** Let $\Phi(D) = \text{number of 1 bits in the binary counter } D$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**increment**

<table>
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<th>4</th>
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Binary counter: potential method

Potential function. Let $\Phi(D) = \text{number of 1 bits in the binary counter } D$.

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Binary counter: potential method

**Potential function.** Let $\Phi(D) =$ number of 1 bits in the binary counter $D$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(n)$ bits.

**Pf.**

- Suppose that the $i^{th}$ INCREMENT operation flips $t_i$ bits from 1 to 0.
- The actual cost $c_i \leq t_i + 1$.  
  \[ \text{operation flips at most one bit from 0 to 1} \]
  \[ \text{(no bits flipped to 1 when counter overflows)} \]
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
  \[ \leq c_i + 1 - t_i \]
  \[ \text{potential decreases by 1 for } t_i \text{ bits flipped from 1 to 0} \]
  \[ \text{and increases by 1 for bit flipped from 0 to 1} \]
  \[ \leq 2. \]
- Total actual cost of $n$ operations $\leq$ sum of amortized costs $\leq 2n$.  

\[ \text{potential method theorem} \]
Famous potential functions

**Fibonacci heaps.** \( \Phi(H) = 2 \text{trees}(H) + 2 \text{marks}(H) \)

**Splay trees.** \( \Phi(T) = \sum_{x \in T} \lfloor \log_2 \text{size}(x) \rfloor \)

**Move-to-front.** \( \Phi(L) = 2 \text{inversions}(L, L^*) \)

**Preflow-push.** \( \Phi(f) = \sum_{v : \text{excess}(v) > 0} \text{height}(v) \)

**Red–black trees.** \( \Phi(T) = \sum_{x \in T} w(x) \)

\[
w(x) = \begin{cases} 
0 & \text{if } x \text{ is red} \\
1 & \text{if } x \text{ is black and has no red children} \\
0 & \text{if } x \text{ is black and has one red child} \\
2 & \text{if } x \text{ is black and has two red children}
\end{cases}
\]
Amortized Analysis

- binary counter
- multi-pop stack
- dynamic table

Section 17.4
**Multipop stack**

**Goal.** Support operations on a set of elements:
- \( \text{PUSH}(S, x) \): add element \( x \) to stack \( S \).
- \( \text{POP}(S) \): remove and return the most-recently added element.
- \( \text{MULTI-POP}(S, k) \): remove the most-recently added \( k \) elements.

\[
\text{MULTI-POP}(S, k) \\
\text{FOR } i = 1 \text{ TO } k \\
\text{POP}(S).
\]

**Exceptions.** We assume \( \text{POP} \) throws an exception if stack is empty.
**Multipop stack**

**Goal.** Support operations on a set of elements:
- \( \text{PUSH}(S, x) \): add element \( x \) to stack \( S \).
- \( \text{POP}(S) \): remove and return the most-recently added element.
- \( \text{MULTI-POP}(S, k) \): remove the most-recently added \( k \) elements.

**Theorem.** Starting from an empty stack, any intermixed sequence of \( n \) \( \text{PUSH}, \text{POP}, \) and \( \text{MULTI-POP} \) operations takes \( O(n^2) \) time.

**Pf.**
- Use a singly linked list.
- \( \text{POP} \) and \( \text{PUSH} \) take \( O(1) \) time each.
- \( \text{MULTI-POP} \) takes \( O(n) \) time.

![Diagram of a linked list](image.png)
Multipop stack: aggregate method

Goal. Support operations on a set of elements:
- \( \text{PUSH}(S, x) \): add element \( x \) to stack \( S \).
- \( \text{POP}(S) \): remove and return the most-recently added element.
- \( \text{MULTI-POP}(S, k) \): remove the most-recently added \( k \) elements.

Theorem. Starting from an empty stack, any intermixed sequence of \( n \) \( \text{PUSH} \), \( \text{POP} \), and \( \text{MULTI-POP} \) operations takes \( O(n) \) time.

Pf.
- An element is popped at most once for each time that it is pushed.
- There are \( \leq n \) \( \text{PUSH} \) operations.
- Thus, there are \( \leq n \) \( \text{POP} \) operations
  (including those made within \( \text{MULTI-POP} \)).  ■
Multipop stack: accounting method

Credits. 1 credit pays for either a \textsc{Push} or \textsc{Pop}.

Invariant. Every element on the stack has 1 credit.

Accounting.

- \textsc{Push}(S, x): charge 2 credits.
  - use 1 credit to pay for pushing \( x \) now
  - store 1 credit to pay for popping \( x \) at some point in the future
- \textsc{Pop}(S): charge 0 credits.
- \textsc{MultiPop}(S, k): charge 0 credits.

Theorem. Starting from an empty stack, any intermixed sequence of \( n \) \textsc{Push}, \textsc{Pop}, and \textsc{Multi-Pop} operations takes \( O(n) \) time.

Pf.

- Invariant \( \Rightarrow \) number of credits in data structure \( \geq 0 \).
- Amortized cost per operation \( \leq 2 \).
- Total actual cost of \( n \) operations \( \leq \) sum of amortized costs \( \leq 2n \).  

\vspace{-1cm}

\text{accounting method theorem}
Multipop stack: potential method

Potential function. Let $\Phi(D) =$ number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 1: push]

- Suppose that the $i^{th}$ operation is a PUSH.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$. 
**Multipop stack: potential method**

**Potential function.** Let $\Phi(D) = \text{number of elements currently on the stack.}$

- $\Phi(D_0) = 0.$
- $\Phi(D_i) \geq 0$ for each $D_i.$

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ `PUSH`, `POP`, and `MULTI-POP` operations takes $O(n)$ time.

**Pf.** [Case 2: pop]

- Suppose that the $i^{th}$ operation is a `POP`.
- The actual cost $c_i = 1.$
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0.$
Multipop stack: potential method

**Potential function.** Let $\Phi(D) =$ number of elements currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ `PUSH`, `POP`, and `MULTI-POP` operations takes $O(n)$ time.

**Pf.** [Case 3: multi-pop]
- Suppose that the $i^{th}$ operation is a `MULTI-POP` of $k$ objects.
- The actual cost $c_i = k$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = k - k = 0$. □
**Potential function.** Let $\Phi(D) =$ number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ \textsc{Push}, \textsc{Pop}, and \textsc{Multi-Pop} operations takes $O(n)$ time.

**Pf.** [putting everything together]

- Amortized cost $\hat{c}_i \leq 2$. \textcolor{Red}{2 for push; 0 for pop and multi-pop}
- Sum of amortized costs $\hat{c}_i$ of the $n$ operations $\leq 2n$.
- Total actual cost $\leq$ sum of amortized cost $\leq 2n$.  

\[\text{potential method theorem}\]
Amortized Analysis

- binary counter
- multi-pop stack
- dynamic table

Section 17.4
Dynamic table

Goal. Store items in a table (e.g., for hash table, binary heap).

- Two operations: `INSERT` and `DELETE`.
  - too many items inserted $\Rightarrow$ expand table.
  - too many items deleted $\Rightarrow$ contract table.
- Requirement: if table contains $m$ items, then space $= \Theta(m)$.

Theorem. Starting from an empty dynamic table, any intermixed sequence of $n$ `INSERT` and `DELETE` operations takes $O(n^2)$ time.

Pf. Each `INSERT` or `DELETE` takes $O(n)$ time. □
Dynamic table: insert only

- When inserting into an empty table, allocate a table of capacity 1.
- When inserting into a full table, allocate a new table of twice the capacity and copy all items.
- Insert item into table.

<table>
<thead>
<tr>
<th>insert</th>
<th>old capacity</th>
<th>new capacity</th>
<th>insert cost</th>
<th>copy cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>2</td>
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<td>4</td>
<td>4</td>
<td>4</td>
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<td>–</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
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<tr>
<td>8</td>
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<td>1</td>
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</tr>
<tr>
<td>9</td>
<td>8</td>
<td>16</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
</tr>
</tbody>
</table>

Cost model. Number of items written (due to insertion or copy).
Dynamic table: insert only (aggregate method)

**Theorem.** [via aggregate method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $c_i$ denote the cost of the $i^{th}$ insertion.

$$c_i = \begin{cases} i & \text{if } i - 1 \text{ is an exact power of 2} \\ 1 & \text{otherwise} \end{cases}$$

Starting from empty table, the cost of a sequence of $n$ INSERT operations is:

$$\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j$$

$$< n + 2n$$

$$= 3n \quad \blacksquare$$
Dynamic table demo: insert only (accounting method)

Insert. Charge 3 credits (use 1 credit to insert; save 2 with new item).
Invariant. 2 credits with each item in right half of table; none in left half.

insert N

capacity = 16

| A | B | C | D | E | F | G | H | I | J | K | L | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

![Diagram of dynamic table]

42
Dynamic table: insert only (accounting method)

**Insert.** Charge 3 credits (use 1 credit to insert; save 2 with new item).

**Invariant.** 2 credits with each item in right half of table; none in left half.

**Pf.** [induction]
- Each newly inserted item gets 2 credits.
- When table doubles from $k$ to $2k$, $k/2$ items in the table have 2 credits.
  - these $k$ credits pay for the work needed to copy the $k$ items
  - now, all $k$ items are in left half of table (and have 0 credits)

**Theorem.** [via accounting method] Starting from an empty dynamic table, any sequence of $n$ `INSERT` operations takes $O(n)$ time.

**Pf.**
- Invariant $\Rightarrow$ number of credits in data structure $\geq 0$.
- Amortized cost per `INSERT` = 3.
- Total actual cost of $n$ operations $\leq$ sum of amortized cost $\leq 3n$.  

slight cheat if table capacity = 1
(can charge only 2 credits for first insert)

accounting method theorem
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
</table>

- size = 6
- capacity = 8
- $\Phi = 4$
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) \textsc{insert} operations takes \( O(n) \) time.

**Pf.** Let \( \Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i) \).

- \( \Phi(D_0) = 0 \).
- \( \Phi(D_i) \geq 0 \) for each \( D_i \).

**Case 0.** [first insertion]

- Actual cost \( c_1 = 1 \).
- \( \Phi(D_1) - \Phi(D_0) = (2 \text{size}(D_1) - \text{capacity}(D_1)) - (2 \text{size}(D_0) - \text{capacity}(D_0)) = 1 \).
- Amortized cost \( \hat{c}_i = c_i + (\Phi(D_1) - \Phi(D_0)) = 1 + 1 = 2 \).
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Case 1.** [no array expansion] $\text{capacity}(D_i) = \text{capacity}(D_{i-1})$.

- Actual cost $c_i = 1$.
- $\Phi(D_i) - \Phi(D_{i-1}) = (2 \text{size}(D_i) - \text{capacity}(D_i)) - (2 \text{size}(D_{i-1}) - \text{capacity}(D_{i-1}))$
  
  $$= 2.$$  

- Amortized cost $\hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1}))$
  
  $$= 1 + 2$$
  
  $$= 3.$$
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) \textsc{insert} operations takes \( O(n) \) time.

**Pf.** Let \( \Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i) \).

- \( \Phi(D_0) = 0 \).
- \( \Phi(D_i) \geq 0 \) for each \( D_i \).

**Case 2.** [array expansion] \( \text{capacity}(D_i) = 2 \text{capacity}(D_{i-1}) \).

- Actual cost \( c_i = 1 + \text{capacity}(D_{i-1}) \).
- \( \Phi(D_i) - \Phi(D_{i-1}) = (2 \text{size}(D_i) - \text{capacity}(D_i)) - (2 \text{size}(D_{i-1}) - \text{capacity}(D_{i-1})) \)
  
  \[ = 2 - \text{capacity}(D_i) + \text{capacity}(D_{i-1}) \]
  
  \[ = 2 - \text{capacity}(D_{i-1}). \]

- Amortized cost \( \hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1})) \)
  
  \[ = 1 + \text{capacity}(D_{i-1}) + (2 - \text{capacity}(D_{i-1})) \]
  
  \[ = 3. \]
Dynamic table: insert only (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $\Phi(D_i) = 2 \text{size}(D_i) - \text{capacity}(D_i)$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

[putting everything together]
- Amortized cost per operation $\hat{c}_i \leq 3$.
- Total actual cost of $n$ operations $\leq$ sum of amortized cost $\leq 3n$. $\blacksquare$
Dynamic table: doubling and halving

Thrashing.
- **INSERT**: when inserting into a full table, double capacity.
- **DELETE**: when deleting from a table that is $\frac{1}{2}$-full, halve capacity.

Efficient solution.
- When inserting into an empty table, initialize table size to 1; when deleting from a table of size 1, free the table.
- **INSERT**: when inserting into a full table, double capacity.
- **DELETE**: when deleting from a table that is $\frac{1}{4}$-full, halve capacity.

Memory usage. A dynamic table uses $\Theta(n)$ memory to store $n$ items.
**Pf.** Table is always between 25% and 100% full. □
Dynamic table demo: insert and delete (accounting method)

**Insert.** Charge 3 credits (1 to insert; save 2 with item if in right half).

**Delete.** Charge 2 credits (1 to delete; save 1 in empty slot if in left half).

**Invariant 1.** 2 credits with each item in right half of table.

**Invariant 2.** 1 credit with each empty slot in left half of table.

delete M

capacity = 16

<p>| | | | | | | | | | | | | |</p>
<table>
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</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>E</td>
<td>F</td>
<td>G</td>
<td>H</td>
<td>I</td>
<td>J</td>
<td>K</td>
<td>L</td>
<td>M</td>
</tr>
</tbody>
</table>

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![Image of dynamic table demo with credits and slots]
Dynamic table: insert and delete (accounting method)

Insert.  Charge 3 credits (1 to insert; save 2 with item if in right half).
Delete.  Charge 2 credits (1 to delete; save 1 in empty slot if in left half).

Invariant 1.  2 credits with each item in right half of table.
Invariant 2.  1 credit with each empty slot in left half of table.

Theorem.  [via accounting method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

Pf.
- Invariants $\Rightarrow$ number of credits in data structure $\geq 0$.
- Amortized cost per operation $\leq 3$.
- Total actual cost of $n$ operations $\leq$ sum of amortized cost $\leq 3n$.  □
Dynamic table: insert and delete (potential method)

**Theorem.** [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

**Pf sketch.**

- Let $\alpha(D_i) = \text{size}(D_i) / \text{capacity}(D_i)$.

- Define $\Phi(D_i) = \begin{cases} 2 \text{size}(D_i) - \text{capacity}(D_i) & \text{if } \alpha(D_i) \geq 1/2 \\ \frac{1}{2} \text{capacity}(D_i) - \text{size}(D_i) & \text{if } \alpha(D_i) < 1/2 \end{cases}$

- $\Phi(D_0) = 0$, $\Phi(D_i) \geq 0$. [a potential function]
- When $\alpha(D_i) = 1/2$, $\Phi(D_i) = 0$. [zero potential after resizing]
- When $\alpha(D_i) = 1$, $\Phi(D_i) = \text{size}(D_i)$. [can pay for expansion]
- When $\alpha(D_i) = 1/4$, $\Phi(D_i) = \text{size}(D_i)$. [can pay for contraction]

...