Competitive Analysis

Beyond Worst Case Analysis

Worst-case analysis.

- Analyze running time as function of worst input of a given size.

Average case analysis.

- Analyze average running time over some distribution of inputs.
- Ex: quicksort.

Amortized analysis.

- Worst-case bound on sequence of operations.
- Ex: splay trees, union-find.

Competitive analysis.

- **Make quantitative statements about online algorithms.**
- **Ex: paging, load balancing.**

Online Algorithm and Competitive Analysis

Paging problem: Given two-level store consisting of fast memory (cache) that can hold k pages, and slow memory that can store infinitely many pages.

- Sequence of page requests p:
  - if page p already in cache, no cost incurred
  - otherwise, eject some other page q from cache and replace with p, and pay unit cost for page fault.
- If p not in cache, which page q should you evict?
- Most fundamental and practically important online problem in CS.

Online Algorithm and Competitive Analysis

Competitive analysis. **(Sleator-Tarjan)**

- **Algorithm A is ρ-competitive if there exists some constant b such that for every sequence of inputs σ :**

$$\text{cost}_A(\sigma) \leq \rho\, \text{cost}_{OPT}(\sigma) + b.$$

where OPT is optimal offline algorithm.

- **OPT = MIN: evict page whose next access is furthest away.**

- **A = LRU: evict page whose most recent access was earliest**
  - **Traditional analysis completely uninformative.**
  - **We show LRU is k-competitive.**
- **A = LIFO: evict page brought in most recently.**
  - **LIFO can have arbitrarily bad competitive ratio.**

- **Fact: no online paging algorithm is better than k-competitive.**

Online Algorithm and Competitive Analysis

Theorem. **LRU is k-competitive.**

Proof: **Let τ be a subsequence of σ on which LRU faults exactly k times, and τ does not contain fist access in σ. Let p denote page requested just before τ.**

- **Case 1: LRU faults in sequence τ on p.**
  - **τ requests at least k+1 different pages $\Rightarrow$ MIN faults at least once**

- **Case 2: LRU faults on some page, say q, at least twice in τ.**
  - **τ requests at least k+1 different pages $\Rightarrow$ MIN faults at least once**

Online Algorithm and Competitive Analysis

Theorem. LRU is k-competitive.

Proof: Let τ be a subsequence of σ on which LRU faults exactly k times, and τ does not contain fist access in σ. Let p denote page requested just before τ.

- Case 3: LRU does not fault on p, nor on any page more than once.
  - k different pages are accessed and faulted on, none of which is p
  - p is in MIN's cache at start of $\tau \Rightarrow$ MIN faults at least once