Homework 5

Out: *Nov 18* Due: *Dec 9*

1. Prove that the VCG reduction doesn't work for the following greedy approximation algorithm:
   - Initialize $S_i = \emptyset$ for all bidders i.
   - For $j = 1$ to n, let bidder $i_j := \arg\max_i\{v_i(S_i \cup \{j\}) - v_i(S_i)\}$ be the bidder who gets the most marginal benefit from item j. Award item j to bidder i (update $S_i := S_i \cup \{j\}$) and continue.

   Specifically, prove that VCG with this algorithm is *not* Dominant Strategy Truthful: provide two valuations $v_1(\cdot), v_2(\cdot)$ over two items such that bidder 1 is strictly better off by lying to the VCG mechanism that uses the above approximation algorithm when their value is $v_1(\cdot)$ and bidder 2 reports $v_2(\cdot)$.

2. Prove that the integrality gap of the configuration LP is at least $\Omega(\sqrt{m})$ (recall that in class we proved that the integrality gap is at most $\sqrt{2m}$). That is, for all m, pick an n. Define n valuation functions $v_1(\cdot), \ldots, v_n(\cdot)$ such that the welfare of the optimal deterministic allocation is X, but there exists a feasible fractional allocation for the configuration LP that has value $\Omega(\sqrt{m}) \cdot X$.

3. Design a *randomized* communication protocol for Equality. That is, assume that Alice and Bob have access to an infinite stream of shared random bits (and accessing these bits doesn't count towards the communication of the protocol). Design a communication strategy where Alice and Bob each output only $O(1)$ bits, such that:
   - If Alice and Bob have equal inputs, they will certainly output "yes."
   - If Alice and Bob have unequal inputs, they will output "no" with probability at least $2/3$ (where the probability is over the randomness in the shared random stream).

Extra Credit:

1. (Extra Credit) Prove that the simple greedy algorithm described in problem one is a 2-approximation for welfare maximization for any number of bidders and items as long as each $v_i(\cdot)$ is submodular.

2. (Extra Credit) Consider the following variant on the secretary problem: an adversary puts the elements into any order they desire. Then, instead of being randomly permuted, the elements are revealed either in order, or in reverse order, each with probability $1/2$ (everything else is the same: upon seeing an element, you must immediately and irrevocably accept or reject). Prove that no algorithm can guarantee acceptance of the heaviest element with probability $> 1/n$ when there are n elements.

3. (Extra Credit) Consider the following variant on prophet inequalities: instead of each X_i being independently drawn, there is a joint distribution over $(X_1, \dots, X_n)$ (everything else is the same: you know the joint distribution, the random variables X_i are revealed to you in order, and you must immediately accept/reject upon seeing). Prove that no algorithm can guarantee better than $\mathbb{E}[\max_i X_i]/n$.

4. (Extra Credit) A *non-deterministic* communication protcol for $f(\cdot, \cdot)$ has the following properties (similar to non-deterministic algorithms):

   - Alice decides what to say in round i deterministically as a function of her own input, $A \in \{0,1\}^n$, an advice string, S, and the transcript during rounds 1 thru $i-1$.
   - Bob decides what to say in round i deterministically as a function of his own input, $B \in \{0,1\}^n$, an advice string, S, and the transcript during rounds 1 thru $i-1$.
   - If $f(A,B) = 1$, then there exists an advice string S such that Alice and Bob will output 1.
   - If $f(A,B) = 0$, then for all advice strings S, Alice and Bob will output 0.
   - $|S|$ counts towards the amount of communication.

   Design a non-deterministic algorithm for NotEquality (i.e. $f(A,B) = 1$ if and only if $A \neq B$), and another for NotDisjointness (i.e. $f(A,B) = 1$ if and only if $A \cap B \neq \emptyset$), each using total communication $O(\log n)$. Prove that every non-deterministic algorithm for Equality and Disjointness require communication n.