COS 445 — Final

Due online Monday, May 14th at 11:59 pm

• All problems on this exam are no collaboration problems.
• You may not discuss any aspect of any problems with anyone except for the course staff.
• You may not consult any external resources, the Internet, etc.
• You may consult the course lecture notes on Ed, any of the five course readings, past Ed discussion, or any notes directly linked on the course webpage (e.g. the cheatsheet, or notes on linear programming).
• You may discuss the test with the course staff, but we will only answer clarification questions and will not give any guidance or hints. You should feel free to ask any questions and let us judge whether or not to answer, but just know that we may choose to politely decline to answer. We may choose to answer questions with a response of “I’m sorry, but I’m not comfortable answering that question,” or “it is within the scope of the exam for you to answer that question yourself” (or some variant of these).
• If you choose to ask a question on Ed, ask it privately. We will maintain a pinned FAQ for questions that are asked multiple times (please also reference this FAQ).
• Please upload each problem as a separate file via codePost, as usual.
• You may not use late days on the exam. You must upload your solution by May 14th at 11:59pm. If you are working down to the wire, upload your partial progress in advance. There is no grace period for the exam. In case of a true emergency where you cannot upload, email me (smweinberg@princeton.edu) your solutions asap.
• If you miss the deadline, university policy prohibits me from grading your exam without explicit permission from your dean. Please make sure you have something submitted by the deadline, and take into account that the server may be overloaded or sluggish near the end. For example, you may wish to treat 11:50pm on May 14th as a “pencils down”, to leave yourself enough time to safely upload (or email your solutions if codePost crashes).
• There are no exceptions, extensions, etc. to the exam policy (again, in case of a truly exceptional circumstance, you should quickly get in touch with your dean).
Problem 1: COS 445 Speedrun (120 points)

For each of the 12 problems below: unless otherwise specified, you do not need to show any work and can just state the answer. However, if you simply state an incorrect answer with no justification, we cannot award partial credit. You are encouraged to provide a very brief outline/justification in order to receive partial credit in the event of a tiny mistake. For example, we will award very significant partial credit if you clearly execute the correct outline, but make a mistake in implementation.

Part a: Stable Matching (10 points)

Find the matching output by student-proposing deferred acceptance in the following example.

Alice: Princeton $\succ$ Yale $\succ$ Harvard
Bob: Princeton $\succ$ Yale $\succ$ Harvard
Charlie: Princeton $\succ$ Harvard $\succ$ Yale
Princeton: Alice $\succ$ Bob $\succ$ Charlie
Harvard: Alice $\succ$ Charlie $\succ$ Bob
Yale: Bob $\succ$ Charlie $\succ$ Alice

Part b: Voting (10 points)

Definition 1 (Copeland Rule) For every pair of candidates $a, b$, give one point to whichever candidate a majority of voters prefer, tie-breaking in favor of the lexicographically-first candidate. Output the candidate with the most points, again tie-breaking in favor of the lexicographically-first candidate.

Find the candidate (the candidates are the foods — Alice, Bob, and Charlie are voting to pick a restaurant) output by the Copeland rule in the following example.

Alice: Tacos $\succ$ Seafood $\succ$ Pasta
Bob: Tacos $\succ$ Seafood $\succ$ Pasta
Charlie: Pasta $\succ$ Seafood $\succ$ Tacos

Part c: Game Theory (10 points)

Find a Nash equilibrium of the following game and state the expected payoff for both players.

Player $X$, the row player, chooses between actions $x_1$ and $x_2$. Player $Y$, the column player, chooses between actions $y_1$ and $y_2$. The first number in each box denotes the payoff to $X$, and the second number is the payoff to $Y$. For example, if $X$ plays action $x_1$ and the column player plays action $y_1$, then $X$ gets payoff 4 and $Y$ gets payoff 2.

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>(4,2)</td>
<td>(0,-1)</td>
</tr>
<tr>
<td>$x_2$</td>
<td>(-3,6)</td>
<td>(7,7)</td>
</tr>
</tbody>
</table>
Part d: Extensive Form Games (10 points)

Consider the extensive form game in Figure 1.

There are two players (named 1 and 2) and three rounds. First player 1 plays, then player 2, then player 1 again. The numbers on the leaves denote the payoffs to the first and second players, respectively (as labeled at the internal nodes of the tree). The labels on the edges denote the names of the actions they can play at that turn.

(i) Find a subgame-perfect Nash equilibrium for this game.

(ii) Find a pure Nash equilibrium such that both players receive strictly higher payoff than in the subgame-perfect Nash equilibrium from (i).

Recall that for both parts, a complete strategy lists a pure action for every internal node. For example, \{L, S\} is not a complete strategy for player 1, nor is \(a\) a complete strategy for player 2. But \{L, S, U, W, Y\} is a complete strategy for player 1, as is \{a, c\} for player 2.

Part e: Linear Programming (10 points)

Write the dual of the following LP. You do not need to solve the LP. You only need to write the dual.

Maximize \(3x + 2y\), such that:

\[
\begin{align*}
4x + 4y & \leq 1, \\
3x + 7y & \leq 3, \\
x, y & \geq 0.
\end{align*}
\]

\footnote{If otherwise specified, you should follow the otherwise specifications.}
Part f: Scoring Rules (10 points)

Suppose you are asked to predict tomorrow’s weather. There’s four possible outcomes: it will be sunny, rainy, cloudy or snowy. You will be paid according to Brier’s scoring rule (if even \( i \) happens, and you report distribution \( \vec{x} \), your payment is \( S(\vec{x}, i) = x_i - \sum_{j \in \{1,2,3,4\}} x_j^2/2 \)). What is your expected payoff if your true belief is that each outcome is equally likely, and this is the belief you report?

Part g: Welfare-maximizing Auctions (10 points)

There are three bidders, and two ad slots. The first ad slot has a click-through rate of 1, and the second has a click-through rate of 1/2. The three bidders submit bids of \( b_1 = 10, b_2 = 6, b_3 = 4 \). The auctioneer is running a VCG auction (to assign each bidder at most one slot, and each slot to at most one bidder).

For each of the three bidders, state the slot they win and their payment (state their total payment, not their payment per-click).

Part h: Revenue-maximizing Auctions

Suppose you are selling a pen to a single buyer. The buyer’s value is drawn uniformly from \([3, 10]\). What is the revenue-optimal auction (menu) for you to sell the pen? What is the expected revenue you achieve?

Note: You may use without proof that the PDF of the uniform distribution on \([3, 10]\) is equal to \( \frac{1}{10-3} \) on the entire interval \([3, 10]\). You may also use without proof that the CDF \( F(\cdot) \) of the uniform distribution on \([3, 10]\) satisfies \( F(x) = \frac{x-3}{10-3} \) when \( x \in \[3, 10]\).

Part i: Price of Anarchy (10 points)

Consider the network in Figure 2. There are two nodes, \( s \) and \( t \), and one unit of flow traveling from \( s \) to \( t \). There are two directed edges from \( s \) to \( t \), one with cost \( c(x) = 2 \) and the other with cost \( c(x) = 1 + x \). Compute the Price of Anarchy of this graph.

![Figure 2: A routing network.](image)

Part j: Cake cutting (10 points)

There is a single cake, the unit-interval \([0, 1]\). Alice, Bob, and Charlie all have normalized, additive valuations (that is, \( v([0, 1]) = 1, v(\emptyset) = 0, \) and \( v(X \cup Y) = v(X) + v(Y) \) whenever \( X \cap Y = \emptyset \),
and \( v(X) \geq 0 \) for all \( X \). Alice’s valuation satisfies \( v_A([0, 1/4]) = 1 \), distributed uniformly. Bob’s satisfies \( v_B([0, 2/3]) = 1 \), distributed uniformly. Charlie’s satisfies \( v_C([1/2, 5/6]) = 1 \), distributed uniformly.

Consider the allocation which awards Alice the interval \([0, 1/3]\), Bob the interval \([1/3, 2/3]\), and Charlie the interval \([2/3, 1]\). Is the allocation proportional? Is it envy-free? Is it equitable?

**Part k: Behavioral Economics (10 points)**

Recall that a utility function \( f(\cdot) \) maps deterministic outcomes to a utility. Say that there are three possible outcomes, \( A, B, C \). Define a utility function such that an expected utility maximizer with utility function \( f(\cdot) \) prefers the randomized outcome which is \( A \) with probability 1/2, \( B \) with probability 1/4, and \( C \) with probability 1/4 to the randomized outcome which is \( A \) with probability 1/3, \( B \) with probability 1/3, and \( C \) with probability 1/3.

**Part ℓ: Time-Inconsistent Planning (10 points)**

In the planning graph of Figure 3, what path would be taken by a naive planner with present bias \( b = 2 \) to get from \( s \) to \( t \)? What about a sophisticated planner with present bias \( b = 2 \) (also to get from \( s \) to \( t \))? What is the shortest path from \( s \) to \( t \)? **Note that this problem asks for you to provide three paths from \( s \) to \( t \).**

![Figure 3: A planning graph.](image-url)
Problem 2: Switching Stable Matchings (40 points)

Recall the following definitions regarding stable matchings. Note that these are all the same standard definitions that you’re familiar with from earlier in the course, but they are repeated here in an extra-precise manner so that the question being asked is clear.

**Definition 2 (Stable Matching Preferences)** A stable matching instance has $n$ students and $n$ universities, and defines a strict preference ordering $\succ_s$ for each student $s$ and $\succ_u$ for each university $u$. We use $\succ$ to refer to the list of all $2n$ preferences, and call $\succ$ “the preferences.”

**Definition 3 (Blocking pair)** A pair $(s, u)$ is a blocking pair for matching $M$ under preferences $\succ$ if $s \succ_u M(u)$ and $u \succ_s M(s)$. That is, $(s, u)$ is a blocking pair for matching $M$ under preferences $\succ$ if $s$ strictly prefers $u$ to their partner in $M$, and $u$ strictly prefers $s$ to their partner in $M$ as well.

**Definition 4 (Stable Matching)** A matching $M$ is stable for preferences $\succ$ if there are no blocking pairs for $M$ under $\succ$. That is, for all pairs $(s, u)$, $(s, u)$ is not a blocking pair for $M$ under $\succ$.

For any stable matching preferences $\succ$, and any two matchings $M, M'$, let $A$ denote the set of students who strictly prefer their match in $M$ to their match in $M'$ (that is, $A$ is the set of students for whom $M(s) \succ_s M'(s)$), and let $B$ denote the set of universities who strictly prefer their match in $M'$ to their match in $M$ (that is, $B$ is the set of universities for whom $M'(u) \succ_u M(u)$).

Prove that for all preferences $\succ$, and all $M, M'$ that are both stable for $\succ$, that in both $M$ and $M'$, every student in $A$ is matched to a university in $B$.

**Note:** This problem asks you to prove two claims (one for $M$ and one for $M'$). For full credit, your solution must clearly provide the logical steps so that a grader can easily follow the logic for both claims. If you feel that this can be accomplished by writing one proof, then copy/pasting it and changing a few words, that is OK. If you feel like that can be accomplished by writing one proof, and instructing the grader to “reread the first proof but swap X and Y everywhere,” (or some other short/precise/clear instructions) that is also OK.

But it must be easy for the grader to follow the logic for both claims. This is likely not accomplished by just saying “The second claim follows similarly to the first.” Of course, you are also allowed to write two completely different proofs.
Problem 3: Revenue vs. Welfare (60 points)

For a distribution with CDF $F$, consider the following three definitions. Recall that a CDF $F$ is defined so that $F(p) = \Pr_{x \leftarrow F}[x < p]$.\footnote{The distinction between $<$ and $\leq$ is not relevant for this problem, and the solution is correct either way. But the notation required for mathematical rigor is simpler using $<$ instead of $\leq$.} Observe that this means that $1 - F(p) = \Pr_{x \leftarrow F}[x \geq p]$.

- **REVENUE**$(F) := \max_p \{ p \cdot (1 - F(p)) \}$. That is, REVENUE$(F)$ denotes the maximum expected revenue that a seller with a single item for sale could achieve by selling to a single buyer whose value is drawn from $F$. Put another way, for a price $p$, $p \cdot (1 - F(p))$ is the expected revenue the seller achieves by setting price $p$ (price times probability the buyer purchases). REVENUE$(F)$ takes the maximum over all $p$.

- **VALUE**$(F) := \mathbb{E}_{v \leftarrow F}[v] = \int_0^\infty (1 - F(x)) dx$. That is, VALUE$(F)$ denotes the expected value of a single draw $v$ from $F$.

- For a number $H > 1$, a distribution with CDF $F$ is supported on $[1, H]$ if a value $v$ drawn from that distribution is always in $[1, H]$. For simplicity of notation, we’ll define $\mathcal{F}_H$ to be the set of all CDF’s which are supported on $[1, H]$. The three bullet points below are a mathematically formal way to describe a CDF $F$ for a distribution supported on $[1, H]$, in case you find it helpful.
  - $F(x) = 0$ for all $x \leq 1$.
  - $F(x) = 1$ for all $x > H$.
  - $F(x)$ is monotone non-decreasing on $[1, H]$ (which, by the two bullets above, implies it is monotone non-decreasing on $(0, \infty)$ as well).

For a number, $H > 1$, define RATIO$(H)$, to be the maximum over all distributions $F$ supported on $[1, H]$ of $\frac{\text{VALUE}(F)}{\text{REVENUE}(F)}$. Put in pure math, RATIO$(H) := \max_{F \in \mathcal{F}_H} \{ \frac{\text{VALUE}(F)}{\text{REVENUE}(F)} \}$.

**Note:** When writing your solution, it is OK not to use fancy formatting. If you want to use the formatting style above, you can write VALUE using the command ‘\textsc{Value}’.

**Part a (15 points)**

Prove that RATIO$(H) \leq H$. That is, prove that for all $F$ supported on $[1, H]$, VALUE$(F) \leq H \cdot \text{REVENUE}(F)$.

**Hint:** Try to separately prove that REVENUE$(F) \geq 1$, and also that VALUE$(F) \leq H$. Be sure to explicitly state why this implies that RATIO$(H) \leq H$.

**Part b (15 points)**

When $H \geq 2$, prove that RATIO$(H) \geq 3/2$. That is, find an $F$ supported on $[1, H]$ so that VALUE$(F) \geq 1.5 \cdot \text{REVENUE}(F)$.

**Hint:** Try the uniform distribution on $[1, H]$, which has CDF $F(x) = (x - 1)/(H - 1)$ on $[1, H]$ and PDF $f(x) = 1/(H - 1)$ on $[1, H]$. Compute both VALUE$(F)$ and REVENUE$(F)$ and see what you get. Be sure to explicitly state why this implies that RATIO$(H) \geq 3/2$.\footnote{The notation required for mathematical rigor is simpler using $<$ instead of $\leq$.}
Problem 3c is challenging! If you don’t have the energy to give it a shot, just skip this page. Recall that you can still compete for 274.5 points while completely skipping 3c, and that 256.5 points is a 100%.

Part c (30 points)

Find $\text{RATIO}(H)$ for all $H > 1$, and prove that your answer is correct (that is, your answer should be given as a function of $H$). Note that a complete proof requires both an example distribution $F$ achieving $\frac{\text{VALUE}(F)}{\text{REVENUE}(F)} = \text{RATIO}(H)$, and a proof that no distribution $\overline{F}$ supported on $[1, H]$ can have $\frac{\text{VALUE}(\overline{F})}{\text{REVENUE}(\overline{F})} > \text{RATIO}(H)$.

You will receive partial credit for providing suboptimal upper bounds on $\text{RATIO}(H)$ (i.e. improving part a), or suboptimal lower bounds on $\text{RATIO}(H)$ (i.e. improving part b).

Note: If you believe that your solution to part c resolves parts a and b, you may write “see part c” for parts a and b. But if you do this and your solution to part c is incorrect, you will also lose points on part a and b.

Hint: While solving this problem, you should expect to be stuck for a bit while you “play around with things.” Don’t be scared if that happens! If you’re still stuck when you’re ready to stop working, try to find some concrete fully-proved improvement to part a, and/or some concrete fully-proved improvement to part b.

Hint: To improve part a, try asking “if $\text{REVENUE}(F) = x$, and $F \in \mathcal{F}_H$, how big can $\text{VALUE}(F)$ be?”

Hint: To improve part b, try playing around with other distributions. There exist distributions with simple CDFs that will improve part b. A full solution will require some calculus, but will not be calculus-intensive.
Problem 4: Almost-Envy-Free (80 points)

There are \( n \) players and \( m \) discrete items (i.e. you are dividing appliances, not cakes. You cannot cut a toaster to share it). Each player \( i \) has value \( v_{ij} \geq 0 \) for item \( j \), and value \( v_i(S) = \sum_{j \in S} v_{ij} \) for a set \( S \) of items (that is, the valuations are additive). Your goal is to fairly allocate the items to the players. Like in Lecture, you should assume the players participate honestly. An allocation is a partition of the items into \( S_1, \ldots, S_n \) (\( S_i \) is the set of items allocated to player \( i \)) such that \( S_i \cap S_{i'} = \emptyset \) for all \( i \neq i' \) and \( \bigcup_i S_i = [m] \) (that is, every item is allocated exactly once).

An allocation is envy-free if for all pairs of players \( i, i' \), player \( i \) (weakly) prefers \( S_i \) to \( S_{i'} \) (that is, \( v_i(S_i) \geq v_i(S_{i'}) \) for all \( i, i' \)). An allocation is almost-envy-free if for all pairs of players, \( i, i' \): \( v_i(S_i) \geq v_i(S_{i'}) \) or there exists an item \( k \in S_{i'} \) such that \( v_i(S_i) \geq v_i(S_{i'} \setminus \{k\}) \). That is, for all \( i, i' \), there exists an item \( k \) such that player \( i \) (weakly) prefers \( S_i \) to \( S_{i'} \) after removing item \( k \) (or, player \( i \) already prefers \( S_i \) to \( S_{i'} \)).

Part a (10 points)

Let there be two players and five items, and \( v_{1j} = 1 \) for all \( j \), and \( v_{2j} = j \), for all \( j \). For each of the following allocations, state whether they are (i) envy free (or not), and (ii) whether they are almost-envy-free or not, and provide a very brief justification.

- \( S_1 = \{1, 3, 5\}, S_2 = \{2, 4\} \).
- \( S_1 = \{3, 4, 5\}, S_2 = \{1, 2\} \).
- \( S_1 = \{1, 2, 3\}, S_2 = \{4, 5\} \).

Part b (20 points)

Prove that, for all \( m > 0 \), and all \( n > 0 \), and any \( \vec{v} \), the following protocol produces an almost-envy-free allocation.

1. **Input:** \( v_{ij} \), for all \( i \in [n], j \in [m] \).
2. **Initialize** \( i := 1, R = [m] \) (\( R \) denotes the set of remaining items), \( S_i = \emptyset \) for all \( i \).
3. **While** \( R \neq \emptyset \):
   (a) Ask player \( i \) what is their favorite item \( \ell \) in \( R \) (i.e., arg max_{j \in R} \{v_{ij}\} \), tie-breaking arbitrarily).
   (b) Remove item \( \ell \) from \( R \), and add it to \( S_i \) (i.e. give item \( \ell \) to player \( i \), and item \( \ell \) is no longer remaining).
   (c) Increase \( i \) by one (unless \( i = n \), then update \( i := 1 \)).
4. **Output:** The partition \( S_1, \ldots, S_n \).

**Note:** The protocol works for all \( m, n > 0 \). However, if you find that it helps simplify notation in your proof, you may assume that \( m \) is an integer multiple of \( n \).
Part c (20 points)

Now, it is no longer the case that the valuations are additive. The only assumptions you should make on the valuations for this part are that the valuations are monotone (that is, \(v_i(S \cup T) \geq v_i(S)\) for all \(S\)) and normalized (that is, \(v_i(\emptyset) = 0\)).

Prove that, for all \(m\) and \(n = 2\), and any \(v_1(\cdot), v_2(\cdot)\), an almost-envy-free allocation always exists. That is, provide a protocol that takes as input \(v_1(\cdot), v_2(\cdot)\) and outputs \(S_1, S_2\) such that \(S_1 \cap S_2 = \emptyset\) and \(S_1 \cup S_2 = [m]\), and is almost-envy-free.

**Hint:** Try a protocol that is similar to cut-and-choose from Lecture. It may be tricky to get the details correct.

Part d (30 points)

Again, it is no longer the case that the valuations are additive. The only assumptions you should make on the valuations for this part are that the valuations are monotone (that is, \(v_i(S \cup T) \geq v_i(S)\) for all \(S\)) and normalized (that is, \(v_i(\emptyset) = 0\)). Prove that, for all \(m\) and all \(n\), and any \(v_1(\cdot), \ldots, v_n(\cdot)\), the following protocol produces an almost-envy-free allocation.

1. **Input:** \(v_i(\cdot)\), for all \(i \in [n]\).
2. \(S_i = \emptyset\) for all \(i\).
3. At all times, let \(G\) denote the following graph: there is a node for each player \(i \in [n]\), put a directed edge from player \(i\) to player \(j\) if \(v_i(S_i) < v_i(S_j)\) (that is, put a directed edge from \(i\) to \(j\) if player \(i\) strictly prefers player \(j\)'s current set to their own).
4. For \(j := 1\) to \(m\):
   a. While \(G\) contains a directed cycle:
      i. Pick such a cycle arbitrarily, and trade along that cycle. That is, pick an arbitrary cycle of players \(i_1, i_2, \ldots, i_\ell\). For all \(k \in \{1, \ldots, \ell\}\), simultaneously give player \(i_k\) the set formerly held by player \(i_{k+1}\). Very formally, update \(S_{i_k}\) to now be the former \(S_{i_{k+1}}\) (and update \(S_{i_{\ell}}\) to now be the former \(S_{i_1}\)).
      ii. After trading, immediately update \(G\).
   b. Pick any player \(i\) that does not have an incoming edge in \(G\). Add item \(j\) to \(S_i\).
5. **Output:** \(S_1, \ldots, S_n\).

**Note:** If you believe that your solution to part d resolves part c, you may write “see part d” for part c. But if you do this and your solution to part d is incorrect, you will also lose points on part c. Note that your solution to part d cannot possibly solve part b, because you are analyzing a different algorithm.

**Note:** Note that for full credit, you should establish that the algorithm terminates in finite time (e.g. that the while loop will eventually break) and that the algorithm is well-defined (e.g. that the player \(i\) required in step 4b exists). It is OK if proofs of these claims are extremely brief, but you should at least (extremely briefly) address them.