Homework 1

Out: *Oct 15* Due: *Nov 9*

You can collaborate with your classmates, but be sure to list your collaborators with your answer. If you get help from a published source (book, paper etc.), cite that.

You are encouraged to use your own judgment for the appropriate use of classmates/sources to get the most out of the homework. The homework is intended to be challenging so that you can understand the material better, but not stressful. If you use Piazza for help, I strongly recommend making posts public so that the class can help answer.

§1 Vickrey-Clarke-Groves. Consider a combinatorial auction with n bidders and m items.

(a) Say that $n = 1$. Design a welfare-maximizing algorithm that runs in poly-time. Design a payment rule to make it dominant strategy truthful.

(b) Say that $m = 1$. Design a welfare-maximizing algorithm. Design a payment rule to make it dominant strategy truthful.

(c) Say that you are given an algorithm $\mathcal{A}$, which takes as input n valuation functions $v_1, \dots, v_n$ and outputs a partition maximizing the welfare. Design a dominant strategy truthful mechanism that makes $n + 1$ black-box calls to $\mathcal{A}$, and poly(n) value queries to each $v_i(\cdot)$ and maximizes the welfare (assume that the bidders can indeed "report" their valuations without concern for communication/computational constraints).

§2 Classes of valuation functions.

(a) Prove that every additive function is gross substitutes.

(b) Prove that every gross substitutes function is submodular.

(c) Prove that every submodular function is fractionally subadditive.

(d) Prove that every fractionally subadditive function is subadditive.

(e) Prove that XOS and fractionally subadditive are equivalent.
Hint: Find a way to use strong LP duality.

(f) Prove that the two provided definitions of submodular are equivalent.

(g) Define $v_j(\cdot)$ to be $v(S \cup \{j\}) - v(\{j\})$. Prove that $v_j(\cdot)$ is submodular whenever $v(\cdot)$ is submodular.

§3 Pointwise approximation.

DEFINITION 1 *Say that a class of valuations $\mathcal{V}$ pointwise β-approximates a class $\mathcal{W}$ if for all $w \in \mathcal{W}$, $S \subseteq [m]$ there exists a $v \in \mathcal{V}$ such that:*

- $v(S) \geq \beta w(S)$.
- $v(T) \leq w(T)$ *for all* T.

(a) Prove that additive functions pointwise 1-approximate XOS functions.

(b) Prove that if $\mathcal{V}$ pointwise β_1-approximates $\mathcal{W}$, and $\mathcal{W}$ pointwise β_2-approximates $\mathcal{U}$, then $\mathcal{V}$ pointwise $\beta_1\beta_2$-approximates $\mathcal{U}$.

(c) Prove that XOS valuations pointwise H_m-approximate subadditive functions.

(d) Prove that Binary additive functions pointwise H_m-approximate subadditive functions. Recall that a binary additive function is an additive function where all non-zero item values are the same.