Online Matroid Intersection: Beating Half for Random Arrival

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Carnegie Mellon University

26th June, 2017

Outline

Edge arrival

- Bipartite graph: Intersection of **partition matroids**

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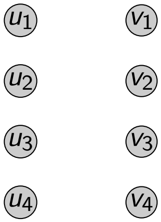

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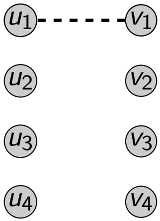

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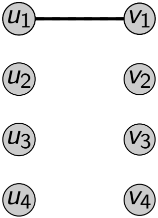

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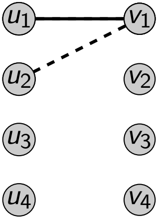

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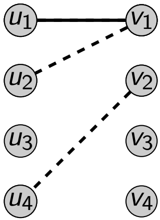

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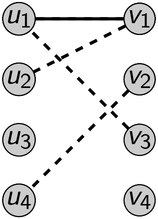

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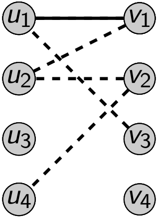

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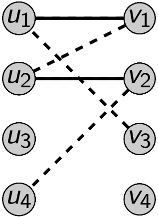

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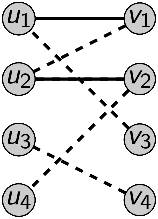

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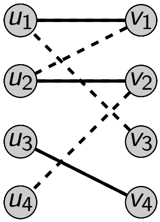

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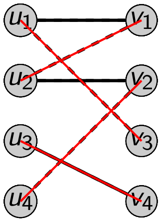

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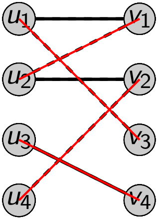

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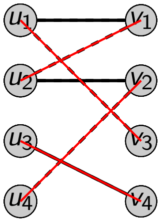

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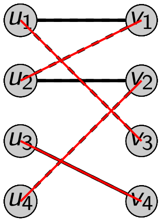

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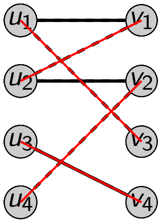

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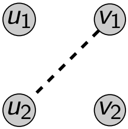

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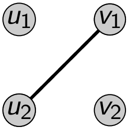

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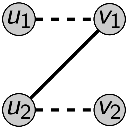

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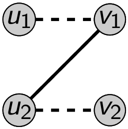

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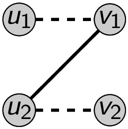

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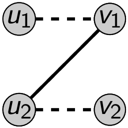

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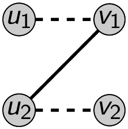

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- Now, is better than $\frac{1}{2}$ possible?

Online Matroid Intersection

- Two **unknown** matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$
- Elements revealed one-by-one: Adversarial/Random arrival
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Theorem

There exists a $(\frac{1}{2} + \epsilon)$-competitive algorithm when the elements are revealed in a **random order**, where $\epsilon > 10^{-5}$.

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Comparison to Vertex Arrival

- **Adversarial arrival (KVV algo.[1]):** $1 - \frac{1}{e} \approx 0.63$
  (a) Give a random rank to $\{u_1, u_2, \ldots, u_n\}$
  (b) Match v_i to lowest available u_j

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Adversarial	≈ 0.63	3

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Faster Algorithms

Offline Algorithms

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- Linear time $(1-\epsilon)$-approx max cardinality matching[4]

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- Even for exact matroid intersection, only linear time lower bounds known[6]

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Other Edge Arrival Models

- **Edge Weighted Bipartite Matching**
  - (a) Maximize weight of matching
  - (b) No constant approx possible for adversarial arrival
  - (c) For random arrival, constant approx possible[7]

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- **Semi-Streaming Models**
  (a) Decisions for $\tilde{O}(n)$ edges can be postponed
  (b) For edge-weighted, $1/2 - \epsilon$ recently shown[8]
  (c) For unweighted, $1/2 + \epsilon$ known when edges arrive randomly[9]

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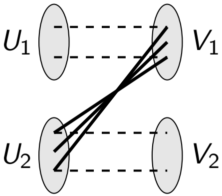

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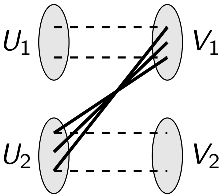

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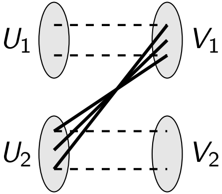

- Only $\frac{1}{2} + o(1)$ approx – **bad graph**
- Regular graphs > 0.63 approx

Can assume GREEDY is bad

- Design **ALG** that gives $\frac{1}{2} + \epsilon$ for '**bad**' graphs

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	Good graphs	**Bad Graphs**
GREEDY	$\geq \frac{1}{2} + \epsilon$ (= 50.1%)	$\geq \frac{1}{2}$
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  and **ALG** w.p. ϵ $(= 0.1\%)$

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- Run GREEDY w.p. $1 - \epsilon$ (= 99.9%)
  and **ALG** w.p. ϵ (= 0.1%)
- Now, $\mathbb{E}[Good] \geq (1/2 + \epsilon)(1 - \epsilon) + 0 = 1/2 + \epsilon/2 - \epsilon^2$
  and $\mathbb{E}[Bad] \geq 1/2(1 - \epsilon) + \epsilon(1/2 + \epsilon) = 1/2 + \epsilon^2$.

Prior work

- **Hastiness Lemma [Konrad-Magniez-Mathieu[10]]**:
  If GREEDY is **bad** then whatever it picks, it picks quickly

[10] *Maximum matching in semi-streaming with few passes.*, APPROX '12

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$$\text{If } \mathbb{E}[\textsc{Greedy}\ (100\%)] < \frac{1}{2} + \epsilon \quad (50.1\%)$$

$$\text{then } \mathbb{E}[\textsc{Greedy}\ (10\%)] \geq \frac{1}{2} - 10\epsilon \quad (49\%)$$

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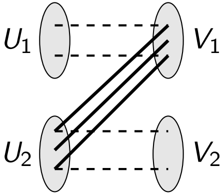

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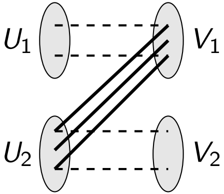
- Suppose GREEDY for first 10% edges – close to half



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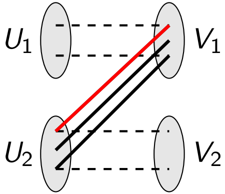

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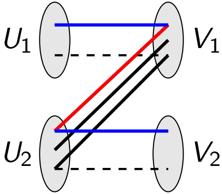

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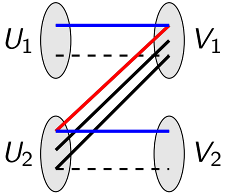

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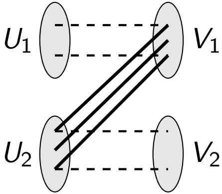
- What edges are augmentable?

Two Phase Algorithm **ALG**

(a) GREEDY for 10% edges



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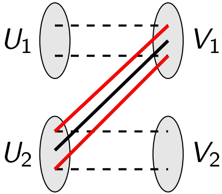

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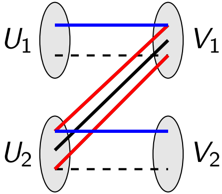

(b) Try **augmenting** marked

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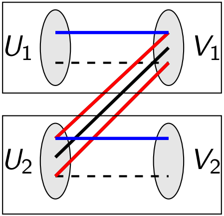

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Run GREEDY (U_1, V_1) and GREEDY (U_2, V_2)

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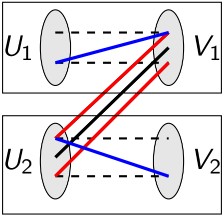

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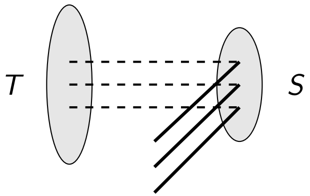
- Augmentations kill each other?

Random sampling



- Bip. graph (T, S) with S-perfect matching

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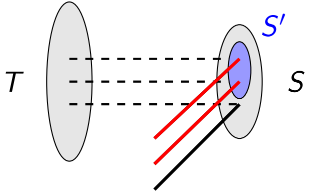

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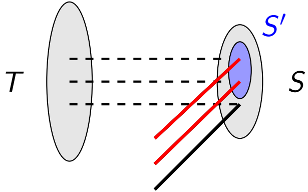

- Bip. graph (T, S) with S-perfect matching
- $S' \subseteq S$ with sampling prob 0.2
- $\mathbb{E}[\textsc{Greedy}\ (T, S')]$: Better than $\mathbb{E}[|S'|]\ (\frac{1}{2})$?

Sampling Lemma

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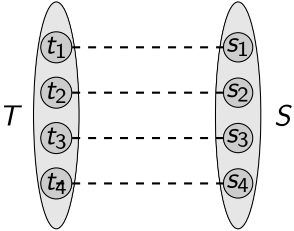

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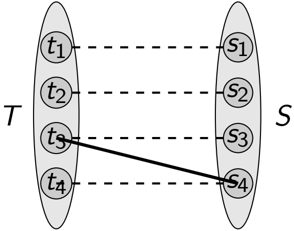

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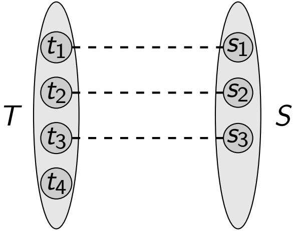

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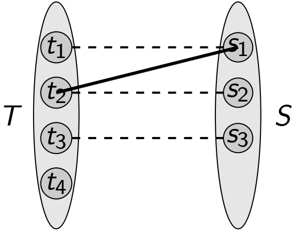

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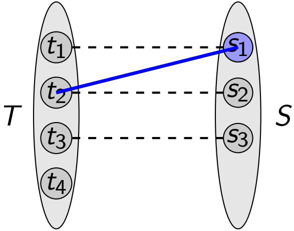

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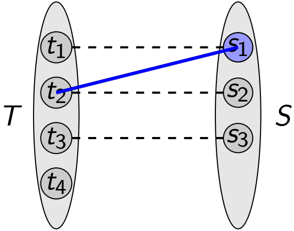
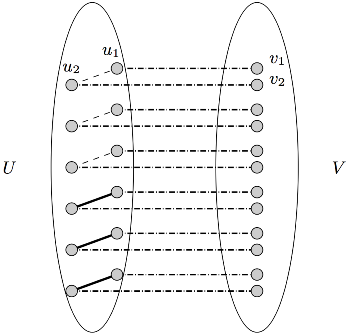


- Note s_2 marked w.p. only 0.2

Outline

General Matching

Assume GREEDY is bad



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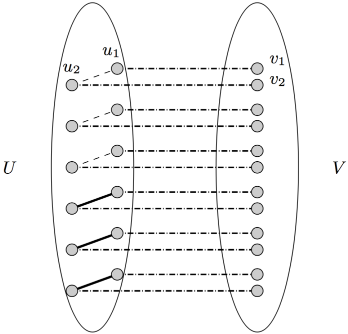

- U denotes vertices matched by GREEDY (in Phase (a))
- Reduces to bipartite matching problem

Matroid Intersection

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Open Problems

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Is there a linear time $(1-\epsilon)$-approximation algorithm for offline matroid intersection?

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Question 3

For OMI, can we "significantly" improve the $(\frac{1}{2}+\epsilon)$-competitive ratio?

Conclusion

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