BOUNDARIES AND EXTREME POINTS OF THE DUAL BALL OF A POLYHEDRAL SPACE

by

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To my parents, Rachel and Haim Livni
Abstract

A Banach space $X$ is called polyhedral if the unit ball of every finite dimensional subspace $V \subseteq X$ is a polytope. The main question in our investigation is the following

**Question 0.1.** Let $X$ be an infinite dimensional separable polyhedral Banach Space. Can $X$ be renormed so that $\text{ext}B_{X^*}$ will be countable.

This question arose from the following theorem

**Theorem 0.1 ([4]).** Every Polyhedral Banach space has a countable boundary.

By the Krein-Milman’s theorem the set of extreme points of the unit ball of the dual is always a boundary for $X$. In [9] an example of a polyhedral Banach space $X$, such that the set $\text{ext}B_{X^*}$ is uncountable was given. The spaces constructed in [9] had the following property

**Property 0.1.** The set $\text{ext}B_{X^*}$ may be covered by a sequence of norm compact sets

In [6] and [3] it was proved that if $\text{ext}B_{X^*}$ can be covered by a sequence of norm compact sets, then $X$ can be renormed in such a way that $\text{ext}B_{(X^*,\|\cdot\|)}$ is countable. The consideration above suggests the following conjecture (which was open for several years)

**Conjecture 0.1.** If $X$ is a separable polyhedral Banach space, then the set $\text{ext}B_{X^*}$ can be covered by $\bigcup_{n=1}^\infty K_n$ where each $K_n$ is norm compact.

We found a new class of separable Banach spaces without this property. In particular $\text{ext}B_{X^*}$ cannot be covered by a sequence of balls with $r_i \to 0$, $0 < r_i < 1$. Any such space is a counter example for the conjecture above. Besides we proved that there are Banach spaces $X$ such that for every subspace $M \subseteq X$ with $\text{codim} M < \infty$, $\text{ext}B_{M^*}$ is uncountable.
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Chapter 1

Introduction

1.1 Polyhedral Banach Spaces and the Set of Extreme points of the Dual Unit Ball

A convex body $C$ in a finite dimensional Banach space $V$ is called a polytope if it is the intersection of finitely many closed half planes (recall that a half plane is a set of the form \( \{ v \in V : f(v) \leq 1 \} \) where $f$ is a linear functional over $V$). A real, finite dimensional Banach space $V$ is called polyhedral if the unit ball $B_V$ is a polytope. If $B_V$ is a polytope, there is a finite set of functionals \( \{ f_1, ..., f_n \} \) such that

\[
B_V = \{ v \in V : |f_i(v)| \leq 1 \ (i = 1...n) \},
\]

and

\[
\|v\| = \max\{|f_1(v)|, .., |f_n(v)|\}.
\]

From the separation theorem it follows that $B_{V^*} = \text{conv}\{\pm f_i\}_{i=1}^n$. Hence, the set of extreme points of the unit ball of the dual must be finite. The converse is obviously true: If $\text{ext}B_{V^*}$ is finite, then $B_V$ is the intersection of finitely many half planes.

In an infinite dimensional Banach space, a convex body $C$ that is the intersection of finitely many half planes is never bounded. Hence this definition is not interesting in case of infinite dimensional Banach spaces. In [10] V. Klee introduced the following definition of a polyhedral Banach Space

**Definition 1.1.** A Banach space $X$ is called **Polyhedral** if the unit ball of every finite dimensional subspace of $X$ is a polytope.
In [10], V. Klee showed that $c_0$ is polyhedral. This is the simplest example of a polyhedral Banach space. Note that $c_0$ is a ”minimal” polyhedral Banach space, i.e. any infinite dimensional polyhedral Banach space contains $c_0$ (see [4]). One of our main concerns will be to evaluate the ”size” of the set $\text{ext}B_{X^*}$ in an infinite dimensional separable polyhedral Banach space. If $\dim X = \infty$, then by the Krein-Milman Theorem, the set $\text{ext}B_{X^*}$ is not finite. Hence, the minimal cardinality of $\text{ext}B_{X^*}$ is countable.

**Definition 1.2.** For a set $A \subset B_{X^*}$ we denote by $A'$ the set

$$A' = \{ f \in B_{X^*} : f \in w^* - \text{cl}(A/\{f\}) \}.$$ 

In [9] the following theorem was proved

**Theorem 1.1.** For a separable Banach space $X$, the following statements are equivalent

1. For some equivalent norm on $X$, the set $\text{ext}B_{X^*}'$ is contained in the interior of $B_{X^*}$. In particular $X$ is isomorphically polyhedral.

2. For some equivalent norm on $X$, the set $\text{ext}B_{X^*}$ is countable.

**Corollary 1.1 ([5]).** If $X^* = \ell_1$, then $X$ is isomorphically polyhedral.

The following question naturally arises. If $X$ is a separable polyhedral Banach space, is the set $\text{ext}B_{X^*}$ always countable? This was answered in the negative in [9]. Namely, the space $c_0$ can be renormed in such a way that the set of extreme points of the dual unit ball will be uncountable.

### 1.2 Boundary

**Definition 1.3.** Let $X$ be a Banach space. A subset $B \subseteq S_{X^*}$ is a boundary of $X$ if for any $x \in X$ there is $f \in B$ with $f(x) = \|x\|$. 

By the Krein-Milman theorem, the set $\text{ext}B_{X^*}$ is an example of a boundary. If $X$ is infinite dimensional, then the minimal cardinality of a boundary is countable. It was proved in [9] that this is the case of separable polyhedral spaces. So the following question is natural

**Question 1.1.** Let $X$ be a separable polyhedral space. Is it possible to introduce an equivalent norm $\|\cdot\|$ on $X$, such that $\text{ext}B_{(X^*,\|\cdot\|)}$ is countable?
Definition 1.4. We say that a set \( A \subseteq X^* \) is \( r \)-norming if

\[
\sup_{f \in A} |f(x)| \geq r \|x\|,
\]

for every \( x \in X \).

Definition 1.5. We say that a Banach space \( X \) has \textbf{property (\ast)} if there exists a 1-norming set \( B \subset S_{X^*} \) such that for each \( f \in B' \) and \( x \in B_X \) we have \(|f(x)| < 1\).

Note that a Banach space \( X \) has property \((\ast)\) iff it is isomorphic to a polyhedral Banach space ([12],[7]).

Definition 1.6. We shall say that a Banach space \( X \) has \textbf{property (\ast\ast)} if there exists a 1-norming set \( B \subseteq S_{X^*} \) such that \( B' \subseteq \text{int}B_{X^*} \).

If a set \( B \) is 1-norming, then from the separation theorem it follows that \( B_{X^*} = \text{conv}B'' \) and from Milman’s theorem it follows that \( \text{ext}B_{X^*} \subseteq B = B \cup B' \). Hence, property \((\ast\ast)\) implies that \( \text{ext}B_{X^*} \subseteq B \). It follows that a Banach space \( X \) has property \((\ast\ast)\) if it satisfies the statements of Theorem 1.1. We will mainly be concerned with the connection between property \((\ast)\) and property \((\ast\ast)\) in separable Banach spaces. Obviously \((\ast\ast)\) implies \((\ast)\). We investigate the following conjecture:

Conjecture 1.1. A separable Banach space has property \((\ast)\) if it is isomorphic to a Banach space with property \((\ast\ast)\).

We have the following result, in evidence of conjecture 1.1

Theorem 1.2 ([7]). If \( X \) is a separable Banach space with countable boundary \( B \), then \( X \) can be renormed to have property \((\ast)\)

If \( X \) has property \((\ast\ast)\), then \( \text{ext}B_{X^*} \subseteq B \). Hence, our conjecture can be reformulated:

Conjecture 1.2. Every separable Banach space \( X \) with a countable boundary can be renormed so that the set \( \text{ext}B_{X^*} \) is countable.

Since the set \( \text{ext}B_{X^*} \) is always a boundary, the converse of conjecture 1.2 is true and conjecture 1.1 and conjecture 1.2 are equivalent.
2.1 Minimal Boundary

Definition 2.1. Let $X$ be a Banach space. A functional $f_0 \in S_{X^*}$ is called a \textit{w*-exposed} point of $B_{X^*}$ if there is an $x_0$ in $S_X$ such that $f_0(x_0) = 1 > f(x_0)$ for each $f \in B_{X^*}$, $f \neq f_0$. Moreover $f_0 \in S_{X^*}$ is called \textit{w*-strongly exposed} if $\| \cdot \| - \lim f_n = f_0$ whenever $\{f_n\} \subset B_{X^*}$ and $\lim f_n(x_0) = 1$.

It is clear that each boundary $B$ contains the (possibly empty) set $w^*\text{-exp}B_{X^*}$ of all $w^*$-exposed points. If $B = w^* - \text{exp}B_{X^*}$ then $B$ is a minimal boundary. We also introduce the following notation. For a functional $f \in S_{X^*}$ put

$$\Gamma_f = \{x \in X : f(x) = 1\}, \quad \gamma_f = \Gamma_f \cap S_X.$$  

The following result was obtained in [4]. For an easier proof see [8].

Theorem 2.1. Let $X$ be a polyhedral Banach space with density character $w$. Then the set $B = w^* - \text{stexp}B_{X^*}$ is a (minimal) boundary for $X$ and moreover for each $f \in B$, $\text{int}_{\Gamma_f} \gamma_f \neq \emptyset$ and $\text{card}B = w$.

Recall that by Theorem [1,2] a separable polyhedral Banach space $X$ can be renormed so that $X$ has property $(\ast)$ (see [7]). For the sake of completeness we present the proof.
Proof. Let $B = \{\pm h_i\}_{i=1}^{\infty}$ be a boundary of $X$ and let $\epsilon_i \to 0$ be a sequence of positive numbers tending to zero. Define

$$V^* = \text{conv} \pm (1 + \epsilon_i)h_i^{w^*}.$$ 

$V^*$ is a $w^*$-closed convex set in $X^*$ and $B_{X^*} \subseteq V^*$. Hence, $\|x\| = \sup_{f \in V^*} |f(x)|$ is an equivalent norm on $X$ and $\text{ext} V^* \subseteq \pm (1 + \epsilon_i)h_i^{w^*}$. Let $e$ be an accumulation point of $\{\pm(1+\epsilon_i)h_i\}$, then $e$ is also an accumulation point of $B$. Hence, $\|e\| = 1$. If $e$ attains its maximum on $x \in V$ then $x \in B_X$ but then we can find some $h_j$ such that $h_j(x) = 1$. Hence, $\|x\| \geq (1 + \epsilon_j)|h_j(x)| > 1$.

2.2 A Polyhedral Banach space without $(**)$ Property

As mentioned before, in [9] an example was given of a separable polyhedral Banach space with uncountably many extreme points in the dual unit ball. This Banach space is actually isomorphic to $c_0$. We denote this Banach space by $X$ and mention some of its properties.

**Proposition 2.1.** The set $\text{ext} B_{X^*}$ can be covered by a countable union of norm compact sets. In particular the set $\text{ext} B_{X^*} = B \cup K$ where $B$ is a countable boundary of $X$ and $K$ is a norm compact set.

**Corollary 2.1.** The set $\text{ext} B_{X^*}$ can be covered by the union of a countable set of balls $f + \epsilon_i B_{X^*}$ such that $\epsilon_i \to 0$, $0 < \epsilon_i < 1$.

**Proposition 2.2.** There is a subspace $Y \subseteq X$ of finite codimension in $X$, such that $\text{ext} B_{Y^*}$ is countable.

Each of these properties may be used to prove that $X$ is isomorphic to a polyhedral Banach space with $(**)$ property.

**Theorem 2.2.** Assume that there exists a sequence of real numbers $\epsilon_i \to 0$ $0 < \epsilon_i < 1$ satisfying $\text{ext} B_{X^*} \subseteq \cup_{i=1}^{\infty} B(h_i, \epsilon_i)$. Then $X$ can be renormed so that the set $\text{ext} B_{X^*}$ is countable.

The proof is similar to the proof of theorem [12] given in [12] and [7].

**Proof.** Let $\{h_i\}$ be a countable boundary and $\epsilon_i$ a sequence of real numbers converging to zero, satisfying the requirements in the theorem. Define

$$V^* = \text{conv}(\pm((1+2\epsilon_i)h_i)_{i=1}^{\infty})^{w^*}.$$
It is easily seen that $V^*$ defines a dual norm on $X$. Denote this norm by $\|\cdot\|$.

By Milman’s theorem ([1], p. 103), $\text{ext} V^* \subseteq \{ \pm (1 + 2\epsilon_i)h_i \}_{i=1}^{\infty}$. Assume that $e$ is an extreme point of $V^*$ and $e \notin \{ \pm (1 + 2\epsilon_i)h_i \}_{i=1}^{\infty}$. Since $\epsilon_i \to 0$ we have that $e$ is in $\{ h_i \}_{i=1}^{\infty}$. Hence $e$ is in the unit ball $B_{X^*}$, hence also an extreme point of the unit ball. By the Hahn-Banach theorem we can find a function $F$ such that $\| F \| = 1 = F(e)$. But there exists an $i$ such that $\| e - h_i \| < \epsilon_i$. Hence, $F(h_i) \geq 1 - \epsilon_i$ and $F((1 + 2\epsilon_i)h_i) > 1$.

**Theorem 2.3** ([2], p.162 [6]). Assume that there is a sequence $\{ K_i \}_{i=1}^{\infty}$ of norm compact sets such that

$$\text{ext} B_{X^*} \subseteq \bigcup_{i=1}^{\infty} K_i.$$ 

Then, $X$ can be renormed so that the unit ball of the dual has countably many extreme points

**Remark 2.2.1.** Theorem 2.3 may be considered as a corollary of theorem 2.2

**Theorem 2.4.** Assume $X$ has a finite codimension subspace $Y$, such that $Y$ has property $(**)$ then $X$ can be renormed so that the unit ball of the dual has countably many extreme points

**Proof.** Recall that $Y$ is $c_0$ saturated and that if $c_0$ is contained in a separable Banach space $Y$ then by the Sobczyk Theorem, it is complemented ([11]). If $X = V \oplus Y$, $\text{dim} V < \infty$ and $Y = c_0 \oplus Z$ then,

$$X \cong V \oplus Y \cong V \oplus c_0 \oplus Z \cong c_0 \oplus Z \cong Y.$$ 

**Definition 2.2.** Let $X$ be a separable polyhedral Banach space. We say that $X$ has property $A$ if the set $\text{ext} B_{X^*}$ is uncountable. We say that $X$ has hereditary property $A$, if every subspace $V \subseteq X$ with $\text{codim} V < \infty$ has property $A$.

**Definition 2.3.** Let $X$ be a separable polyhedral Banach space and let $C \subseteq B_{X^*}$. If $C$ cannot be covered by a countable union of norm compact sets, we say that $C$ is massive.
Chapter 3

A Banach space with hereditary property $\mathcal{A}$

The main result of this chapter is the following

**Theorem 3.1.** There exists a separable polyhedral Banach space $X$ with hereditary property $\mathcal{A}$ (see definition 2.2)

### 3.0.1 The idea of construction

In [9] it was proven that if we take a sequence $\{\lambda_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} \lambda_i = 1$, $\varrho \in (0, 1)$, $a = \frac{1}{\lambda_1}$ and $a_n = \frac{a \sum_{i=1}^{\infty} \lambda_i}{1 - \varrho \sum_{i=n+1}^{\infty} \lambda_i}$. Next we define

$$A_m = \left\{ a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{i=1}^{m} \epsilon_k \lambda_k e_i^* : \epsilon_k = \pm 1 \right\}.$$  

Then

$$U^* = \text{conv}(\{e_i^*\} \cup \bigcup_{i=1}^{\infty} A_m)^w,$$

will be the dual unit ball of a polyhedral space $X$ with countable boundary yet uncountable $\text{ext} U^*$. If we take a finite codimensional subspace of $X$ this might not be the case, i.e. $X$ has a finite co-dimension subspace $V$ such that if $B_{V^*}$ is the unit ball of its dual then $\text{ext} B_{V^*}$ is countable.

**Proof.** choose $k$ such that $\sum_{i=k}^{\infty} \lambda_i < \frac{1}{3a} < \frac{1}{3}$ and take

$$V = \text{span}\{e_i\}_{i=k}^{\infty}.$$
Then,
\[ \text{ext} B_{V^*} \subseteq \{ e_i^* | V \} \cup \bigcup_{m=1}^{\infty} A_m | V \].

For each \( f \in A_m \) for some \( m \), consider \( f \) as linear functional over \( V \). We have that
\[
\| f \|_V = \| a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{i=2}^{m} \epsilon_k \lambda_k e_i^* \|_V \leq \frac{a \sum_{i=k}^{m} \lambda_i}{1 - \varrho \sum_{m+1}^{\infty} \lambda_i} < \frac{1}{3} \cdot 2 < \frac{2}{3}.
\]

Hence, \( \left\{ \bigcup_{m=1}^{\infty} A_m | V \right\}^{w*} \subseteq \frac{2}{3} B_{V^*} \) and \( \text{ext} B_{V^*} = \{ e_i^* | V \}^{i=k} \).

To construct a polyhedral space such that all its finite codimension subspaces will have uncountably many extreme points in the dual balls, we enlarge the sets \( A_m \) by ”shifting”, i.e. instead of taking \( A_m \) we shall take a countable set \( \tilde{A}_m \):
\[
\tilde{A}_m = \left\{ a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{i=1}^{m} \epsilon_k \lambda_k e_{i+n}^* : \epsilon_k = \pm 1, n \in \mathbb{N} \right\}.
\]

We claim that if \( \lambda_i \) and \( \varrho \) are chosen in appropriate way then the set
\[
a \sum_{i=1}^{\infty} \epsilon_k \lambda_k e_{i+n}^*,
\]
will be the extreme points of the dual ball, and further, in each finite codimensional subspace, uncountably many will remain extreme points.

### 3.0.2 Construction

Let \( \{ e_i \}_{i=1}^{\infty} \) be the standard basis of \( c_0 \) and \( \{ e_i^* \} \) the standard basis of \( \ell_1 \). Fix \( \varrho \in (0, \frac{1}{2}) \) and \( \lambda_i = \frac{1}{2^i} \). Let,
\[
a = \frac{1}{\lambda_1}
\]
and
\[
a_n = \frac{a \sum_{i=1}^{n} \lambda_i}{1 - \varrho \sum_{i=n+1}^{\infty} \lambda_i}.
\]

Next define:
\[
A_m = \left\{ a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{i=1}^{m} \epsilon_k \lambda_k e_{i+n}^* : \epsilon_k = \pm 1, n \in \mathbb{N} \right\}.
\]

Each \( A_m \) is countable. The set
\[
U^* = \text{conv} \bigcup_{m=1}^{\infty} A_m^{w*},
\]
is the dual unit ball of an equivalent norm $||| \cdot |||$ on $c_0$ (Note that $\{a_i e_i^* \} \subseteq A_1$.
Hence, $\lambda_1 B_{\ell_1} \subseteq U^*$). Put \( X = (c_0, ||| \cdot |||) \) and denote by \( \{h_i\}_{i=1}^\infty \) the set \( \cup_{m=1}^\infty A_m \).
To prove that \( \{h_i\} \) is a boundary, it is enough to prove that each vector in \( \{h_i\}' \) with \( ||h|| = 1 \) does not attain its maximum.

**Claim 3.0.1.** Every \( f \in \{h_i\}' \) with \( ||f|| = 1 \) does not attain its norm.

Take \( f \in (\bigcup_{m=1}^\infty A_m)' \) such that \( ||f|| = 1 \) and assume to the contrary, that there is \( x \in B_X \) such that \( f(x) = 1 \). If \( f \in A'_m \) for some fixed \( m \) then this is trivial since \( A'_m = \{0\} \).
Assume \( f \in (\bigcup_{m=1}^\infty A_m)' \) and \( f \notin A'_m \) for some fixed \( m \).
It is easy to see that \( f \) is of the form \( f = a \sum_{i=1}^\infty \epsilon_i \lambda_i e_{i+n} \),
for some set of \( \{\epsilon_i\} \), \( \epsilon_i = \pm 1 \) and \( m \in \mathbb{N} \). Choose some \( x \in c_0 \) such that \( f(x) = ||x|| = 1 \) and choose \( s > m \) so large that \( a \cdot \max\{|x_k|\}_{k=s+1} < \frac{\varrho}{2} \). Then the definition of \( ||| \cdot ||| \) implies

\[
1 = f(x) = a \sum_{k=m}^s \epsilon_k \lambda_k x_{k+m} + a \sum_{k=s+1}^\infty \epsilon_k \lambda_k x_{k+m} \\
\leq \frac{a}{a_s} \left[ a_s \left( \sum_{k=1}^s \lambda_k \right)^{-1} \sum_{k=1}^s \lambda_k |x_{k+m}| \right] \sum_{i=1}^s \lambda_i + \left( a \cdot \max_{k>s} |x_k| \right) \sum_{k=s+1}^\infty \lambda_k \\
< \frac{a}{a_s} \sum_{i=1}^s \lambda_i + \frac{a}{2} \sum_{i=n+1}^\infty \lambda_i < 1.
\]

The last inequality is a result of the following equality:

\[
\frac{a}{a_n} \sum_{i=1}^n \lambda_i + \frac{a}{2} \sum_{i=n+1}^\infty \lambda_i = 1.
\]

**Claim 3.0.2.** \( \{h_i\}_{i=1}^\infty \) is a countable boundary for \( X \) and \( X \) is polyhedral.

This is immediate from claim 3.0.1, the fact that \( \text{ext} B_{X^*} \subseteq \overline{T_i}^{w^*} \) and the fact that \( \text{ext} B_{X^*} \) is a boundary.

**Claim 3.0.3.** The set \( E = \text{ext} U^* \) is uncountable.

We shall prove that every

\[
f = a \sum_{i=1}^\infty \epsilon_i \lambda_i e_{i+n},
\]
is an extreme point. For simplification we will prove that
\[
f = a \sum_{i=1}^{\infty} \lambda_i e_{i+n}^*,
\]
is an extreme point and the general case follows from symmetry. For future use
we will prove a stronger statement

**Proposition 3.1.** Let \((\alpha_1, ..., \alpha_{n-1})\) be a set of numbers such that \(|\alpha_i| < \frac{1}{2}\). Then
\[
F = (\alpha_1, ..., \alpha_{n-1}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, ...) \in \ell_\infty.
\]
exposes \(f\)

We shall prove the proposition in steps. Denote
\[
\delta = \min_{p < n-1} \left\{ \left( \sum_{i=1}^{n-p} \lambda_n - \sum_{i=1}^{n-1-p} \lambda_i |\alpha_{i+p}| - \frac{3}{2} \lambda_{n-p} \right) \right\} > 0.
\]
It is easy to verify the \(\delta > 0\).

**Lemma 3.1.1.** \(\|F\| = |F(f)| = a\)

**Proof.** Define
\[
x_k = (\alpha_1, \alpha_2, ..., \alpha_{n-1}, \frac{3}{2}, \frac{1}{2}, ..., 0, 0, 0, 0, 0) \in c_0,
\]
and note that \(x_k \rightharpoonup F \in \ell_\infty\). It is enough to prove the \(||x_k|| \leq a\). Take \(h\) in the boundary
\[
h = a_m \left( \sum_{k=1}^{m} \lambda_k \right)^{-1} \sum_{i=1}^{m} \epsilon_k \lambda_k e_{k+1}^*,
\]
and consider three cases.

**Case 1.** \(j < n - 1\)
\[
|x_k(h)| \leq a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \left( \sum_{i=1}^{n-j} \lambda_k |\alpha_{i+j}| + \lambda_{n-j} \frac{3}{2} + \frac{1}{2} \sum_{i=n+1-j}^{m} \lambda_k \right) \leq a_m (1-\delta).
\]

**Case 2.** \(j > n - 1\)
\[
|x_k(h)| \leq a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{i=1}^{m} \lambda_k \frac{1}{2} < a_m \frac{1}{2}.
\]
case 3 \( j = n - 1 \)

\[
|x_k(h)| \leq a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \left( \frac{3}{2} \lambda_1 + \frac{1}{2} \sum_{i=2}^{m} \lambda_i \right) =
\]

\[
= a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \left( \lambda_1 + \lambda_2 + \sum_{i=3}^{m+1} \lambda_i \right) = a_m \frac{2^m}{2^m - 1} \frac{2^{m+1} - 1}{2^m - 1}
\]

Recall

\[
a_n = \frac{a \sum_{i=n+1}^{\infty} \lambda_i}{1 - \theta \sum_{i=n+1}^{\infty} \lambda_i} = \frac{2(1 - \frac{1}{2^n})}{1 - \theta \frac{1}{2^n}} < \frac{2(1 - \frac{1}{2^n})}{1 - \frac{1}{2^{n+1}}} = 4 \cdot \frac{2^n - 1}{2^{n+1} - 1}.
\]

Hence,

\[
a_m \frac{2}{2^m - 1} \left( \frac{2^{m+1} - 1}{2^m - 1} \right) < \frac{2^m - 1}{2^{m+1} - 1} \left( \frac{2^{m+1} - 1}{2^m - 1} \right) = 2 = a.
\]

Lemma 3.1.2. Assume \( g \in B_{X^*} \) and \( |F(g)| = a \) then

\[
g \in \overline{\text{conv} B_{n-k}^{w^*}},
\]

where

\[
B_j = \{a_m \sum_{i=1}^{m} \epsilon_k \lambda_k e_{k+j} : \epsilon_k = \pm 1, m \geq 1 \}.
\]

Proof. Define

\[
H = \overline{\text{conv} \bigcup_{j=1}^{n-k} B_j}^{w^*}
\]

\[
N = \overline{\text{conv} \bigcup_{j=n}^{\infty} B_j}^{w^*}
\]

\[
V = \overline{\text{conv} B_{n-1}^{w^*}}.
\]

By using the definition of \( U^* = B_{X^*} \) it is easy to see that \( \text{conv}(H \cup N \cup V) = U^* \) therefore each \( g \in U^* \) may be represented in the form

\[
g = \lambda_1 h + \lambda_2 n + \lambda_3 v,
\]

where \( \lambda_i > 0, \lambda_1 + \lambda_2 + \lambda_3 = 1, h \in H, n \in N \) and \( v \in V \). By our prior calculations we have the following estimates:
1. \(|F(H)| < a(1 - \delta)\)

2. \(|F(N)| < \frac{a}{2}\)

So

\[|F(\lambda_1 h + \lambda_2 n + \lambda_3 v)| \leq \lambda_1 |F(h)| + \lambda_2 |F(n)| + \lambda_3 |F(v)| \leq \lambda_1 (a - \delta) + \lambda_2 \frac{a}{2} + \lambda_3 |F(v)| = a.\]

Hence, \(\lambda_1 = \lambda_2 = 0\) and \(g \in V\)

\[\text{Lemma 3.1.3.} \text{ If } g \in B_{X^*} \text{ and } |F(g)| = a \text{ then } g \in \text{conv } \bigcup_{m=s+1}^{\infty} (A_m \cap B_{n-1})^w.\]

\[\text{for each } s \in \mathbb{N}\]

\[\text{Proof.} \text{ Define } P_s = \text{conv } \bigcup_{m=1}^{s} (A_m \cap B_{n-1})^w, \quad Q_s = \text{conv } \bigcup_{m=s+1}^{\infty} (A_m \cap B_{n-1})^w.\]

Choose \(p_s \in P_s, q_s \in Q_s\) and \(\mu_s \in [0, 1]\) such that \(g = \mu_s p_s + (1 - \mu_s) q_s\). Then,

\[a = |F(g)| \leq \mu_s |F(p_s)| + (1 - \mu_s) |F(q_s)| \leq a\]

Now

\[|F(p_s)| \leq \max\{|F(h)| : h \in A_m \cap B_{n-1} \text{ } m \leq s\}.\]

For each \(h \in A_m \cap B_{n-1}\)

\[|F(h)| \leq a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \left( \frac{3}{2} \lambda_1 + \frac{1}{2} \sum_{i=2}^{m} \lambda_i \right) < a.\]

The last inequality was evaluated in proposition 3.0.2 case 3. It follows that \(\mu_s = 0\).\]

\[\text{Proof of proposition 3.1.} \text{ Let } g \in B_{X^*} \text{ be such that } F(g) = a. \text{ From proposition 3.0.2 it is immediate that } g_m = 0 \text{ for each } m < n. \text{ We claim that } g_{k+n} \leq a \lambda_k \text{ for all } k \in \mathbb{N}. \text{ If not, there exists } k \in \mathbb{N} \text{ such that }\]

\[g_{k+n} - a \lambda_k =: 2 \epsilon > 0.\]

Since \(a_n \to a\) there exists \(s \in \mathbb{N}\) such that

\[|a - a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1}| < \epsilon \text{ for each } m > s.\]
Then, for each $m > s$, we have
\[
g_{k+n} - a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \lambda_k \geq g_{k+n} - a\lambda_k - a - a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \lambda_k
\]
\[
> 2\epsilon - \epsilon\lambda_k > \epsilon.
\]
In other words, $a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \lambda_k < g_{k+n} - \epsilon$. But this implies that the $k+n$-th coordinate of each element of $\bigcup_{m=s+1}^{\infty} A_m \cap B_{n-1}$ is smaller then $g_{k+n} - \epsilon$, which is in contradiction with $g \in Q_s$.

Since 
\[
\frac{3}{2} g_n + \sum_{n+1}^{\infty} \frac{1}{2} g_k = F(g) = \frac{3}{2} a\lambda_1 + \sum_{n=2}^{\infty} \frac{1}{2} a_\lambda k,
\]
R. But this implies that $g_{k+n} = a\lambda_k$ and $g = f$

**Corollary 3.1.** Every finite codimension subspace, $M$ of $X$, the unit ball of its dual has uncountably many extreme points

**Proof.** Take $\{v_1, .., v_n\} \in B_{\ell_1}$ linearly independent such that

\[
M = \cap_{i=1}^{n} \ker v_i.
\]

Since $\{v_i\}_{i=1}^{n}$ are linearly independent WLOG we assume that there is a set of coordinates $j_1, .., j_n$ such that $v_{j_i} = \delta_{i,k}$. Choose $m$ sufficiently large so that

\[
\frac{3}{2} |v_{j_i}^m| + \sum_{n=m+1}^{\infty} \frac{1}{2} |v_{j_i}^n| < \frac{1}{2},
\]

for each choice of $i \leq n$. Define $F = (\alpha_k) \in \ell_\infty$ as follows: for each $i \leq n$

\[
\alpha_{j_i} = - \left( \frac{3}{2} v_{j_i}^m + \sum_{n=m+1}^{\infty} \frac{1}{2} v_{j_i}^n \right),
\]

and $\alpha_k = 0$ for every $k \neq j_i$ for some $i \leq n$. Then by proposition 3.1

\[
F = (\alpha_1, .., \alpha_{m-1}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, ...),
\]

exposes $f = a \sum_{i=1}^{\infty} \lambda_i e_{i+m}^*$. Since $F(v_i) = 0$, $F \in M^{**}$ and it exposes $f_{|M}$ in $M^*$. Similarly each vector $f = \sum_{i=1}^{\infty} c_i \lambda_i e_{i+m}^*$ is exposed.
In this chapter we prove our main result which is the following

**Theorem 4.1.** There exists a separable polyhedral Banach space $X$, such that for each sequence $\epsilon_i \to 0$, $0 < \epsilon_i < 1$ and a sequence of balls $B(x_i, \epsilon_i)$. It holds that $\text{ext} B_{X^*} \not\subseteq \bigcup_{i=1}^{\infty} B(x_i, \epsilon_i)$

Fix $\varrho \in (0, \frac{1}{2})$ and a series of positive numbers $\{\lambda_i\}$ such that $\lambda_i = \frac{1}{2^i}$,

$$ a = \frac{1}{\lambda_1}, $$

and

$$ a_n = \frac{a \sum_{i=1}^{n} \lambda_i}{1 - \varrho \sum_{i=n+1}^{\infty} \lambda_i}. $$

Denote by $G_m$ the set of injective, non-decreasing mappings from $\{1, ..., m\}$ to $\mathbb{N}$ and $G_\infty$ the set of injective, non-decreasing mappings from $\mathbb{N}$ to $\mathbb{N}$.

Next define:

$$ A_m = \left\{ a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=1}^{m} \epsilon_k \lambda_k e^*_{g(k)} : \epsilon_k = \pm 1, g \in G_m \right\}. $$

Each $A_m$ is countable. The set

$$ U^* = \text{conv} \bigcup_{m=1}^{\infty} A_m^*, $$
is the dual unit ball of an equivalent norm $\| \cdot \|$ on $c_0$ (Note that $\{a_1e_i^*\} \subseteq A_1$ hence $a_1B_{\ell_1} \subseteq U^*$). Put $X = (c_0, \| \cdot \|)$ and denote by $\{h_i\}_{i=1}^\infty$ the set $\bigcup_{m=1}^\infty A_m$. To prove that $\{h_i\}$ is a boundary, it is enough to prove that each vector $f$ in $\{h_i\}'$ with $\|f\| = 1$, doesn’t attain its supremum on $U$.

**Claim 4.0.4.** Every $f \in \{h_i\}'$ with $\|f\| = 1$ doesn’t attain its norm.

Take $f \in (\bigcup_{m=1}^\infty A_m)'$ such that $\|f\| = 1$. If $f \in A_m'$ for some fixed $m$ then

$$f = a_m \left( \sum_{i=1}^m \lambda_i \right)^{-1} \sum_{k=1}^n \epsilon_k \lambda_k e_{g(k)}^*,$$

for some $n < m$ and $g \in G_n$.

$$\|f\| = \|a_m \left( \sum_{i=1}^m \lambda_i \right)^{-1} \sum_{k=1}^n \epsilon_k \lambda_k e_{g(k)}^*\| = \frac{a m \left( \sum_{i=1}^m \lambda_i \right)^{-1}}{a_n \left( \sum_{i=1}^n \lambda_i \right)^{-1}} \| \sum_{k=1}^n \epsilon_k \lambda_k e_{g(k)}^*\| \leq \frac{1 - \epsilon \sum_{i=m+1}^n \lambda_i}{1 - \epsilon 2 \sum_{i=n+1}^\infty \lambda_i} < 1.$$

Assume $f \in (\bigcup_{m=1}^\infty A_m)'$ and $f \notin A_m'$ for some fixed $m$. It is easy to see that $f$ is of the form

$$f = a \sum_{k=1}^\infty \epsilon_k \lambda_k e_{g(k)}^*,$$

for some set of $\{\epsilon_k\}$ $\epsilon_k = \pm 1$ and $g \in G_\infty$ or

$$f = a \sum_{k=1}^n \epsilon_k \lambda_k e_{g(k)}^*,$$

for some $g \in G_n$. If $f$ is of the first form, the same argument as the one used in claim 3.0.1 may be used to prove the $f$ doesn’t attain its norm. If $f$ is of the later form, then $\|f\| < 1$.

**Claim 4.0.5.** $\{h_i\}_{i=1}^\infty$ is a countable boundary for $X$ and $X$ is polyhedral.

This is immediate from claim 4.0.4, the fact that $extB_{X^*} \subseteq \overline{h_i}$ and the fact that $extB_{X^*}$ is a boundary.

**Claim 4.0.6.** The set $E = extU^*$ is uncountable.

We shall prove that

$$f = a \sum_{i=1}^\infty \lambda_i e_{g(k)}^*,$$

for some $g \in G_\infty$ is an extreme point.
**Definition 4.1.** Given \( g \in G_\infty \) and \( f = a \sum_{i=1}^\infty \lambda_i e_{g(i)}^* \). We define

\[
U_j = \left\{ d_m \left( \sum_{i=1}^{m+1} \lambda_i \right)^{-1} \sum_{i=1}^m \epsilon_k \lambda_i e_{g,i}^* : \forall i \leq j g'(i) = g(i) \right\}.
\]

Before proving the claim we prove a proposition.

**Proposition 4.1.** If \( f = \frac{1}{2} f_1 + \frac{1}{2} f_2 \) and \( f_1, f_2 \in U^* \) then \( f_2, f_1 \in \bigcap_{j=1}^\infty \text{conv} U_j^{w^*} \).

**Proof.** Denote 
\[
\delta = \inf_{m \geq 0} \left\{ 2 - \frac{\sum_{i=1}^{m+1} \lambda_i}{\sum_{i=1}^m \lambda_i} \right\} > 0.
\]

The proof is by induction. Assume \( f_2, f_1 \in \text{conv} U_{n-1}^{w^*} \). Define

\[
H = \text{conv} \cup_{j=1}^{g(n)-1} B_j^{w^*} \cap \text{conv} U_{n-1}^{w^*},
\]

\[
N = \text{conv} \cup_{j=g(n)+1}^{\infty} B_j^{w^*} \cap \text{conv} U_{n-1}^{w^*}
\]

and

\[
V = \text{conv} B_{g(n)}^{w^*} \cap \text{conv} U_{n-1}^{w^*} = \text{conv} U_n^{w^*},
\]

where

\[
B_j = \left\{ d_m \left( \sum_{i=1}^{m+1} \lambda_i \right)^{-1} \sum_{i=1}^m \epsilon_k \lambda_i e_{g,i}^* : \epsilon_k = \pm 1, m \in \mathbb{N} \text{ and } g_x \in G_m \land g_x(n) = j \right\}.
\]

Note that \( U_{n-1} \subseteq H \cup N \cup V \). Let \( \{ c_i \}_{i=1}^\infty \) be the sequence of natural numbers, not in the image of \( g \), ordered by their natural order. Define \( F = (F)_j \in \ell_\infty \) as follows

\[
(F)_j = \begin{cases} 
0 & j = c_i \\
0 & j = g(k), k < n \\
\frac{3}{2} & j = g(n) \\
\frac{1}{2} & \text{else}
\end{cases}
\]

Take \( h \in U_{n-1} \),

\[
h = a_m \left( \sum_{k=1}^m \lambda_i \right)^{-1} \sum_{i=1}^m \epsilon_k \lambda_i e_{g,i}^*.
\]

and consider three cases
Case 1 \( h \in H \)

\[
|F(h)| \leq a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \left( \frac{3}{2} \lambda_{n+1} + \frac{1}{2} \sum_{i=n+2}^{m} \lambda_i \right) \leq a_m \lambda_n \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \left( \frac{3}{2} \lambda_1 + \frac{1}{2} \sum_{i=2}^{m} \lambda_i \right) =
\]

\[
a_m \lambda_n \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \left( \lambda_1 + \lambda_2 + \sum_{i=3}^{m+1} \lambda_i \right) \leq a_m \lambda_n (2 - \delta) \leq a_m \lambda_n - 2(1 - \delta) \leq \frac{a}{2^n - 1} - \delta \frac{a}{2^n}.
\]

Case 2 \( h \in N \)

\[
|F(h)| \leq a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=n}^{m} \lambda_k \frac{1}{2} < a_m \lambda_{n-1} \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=1}^{m} \lambda_k \frac{1}{2} \leq \frac{1}{2} \frac{a}{2^n - 1}.
\]

Case 3 \( h \in V \)

\[
|F(h)| \leq a_m \left( \sum_{k=1}^{m} \lambda_k \right)^{-1} \left( \frac{3}{2} \lambda_n + \frac{1}{2} \sum_{k=1}^{m} \lambda_{n+k} \right) =
\]

\[
a_m \lambda_{n-1} \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \left( \lambda_1 + \lambda_2 + \sum_{i=3}^{m} \lambda_i \right) = a_m \lambda_{n-1} \frac{2m}{2m-1} \frac{2m+1-1}{2m+1} =
\]

\[
a_m \left( \frac{2m+1-1}{2m-1} \right) < \frac{a}{2^n - 1} \frac{2m-1}{2m+1-1} \left( \frac{2m+1-1}{2m-1} \right) = 2 = a.
\]

Recall

\[
a_n = \frac{a}{1 - \frac{a}{2^n - 1} - \lambda_i} = \frac{2(1 - \frac{1}{2^n})}{1 - \frac{1}{2^n + 1} - \lambda_i} < \frac{2(1 - \frac{1}{2^n})}{1 - \frac{1}{2^n + 1}} = 4 \cdot \frac{2^n - 1}{2^n + 1}.
\]

Hence,

\[
a_m \left( \frac{2m+1-1}{2m-1} \right) < \frac{2m-1}{2m+1-1} \left( \frac{2m+1-1}{2m-1} \right) = 2 = a.
\]

Note that \( F(f) = \frac{a}{2^n - 1} \). Since \( U_{n-1} = \text{conv}(H \cup N \cup V) \), then by our estimates

\[
\sup_{g \in \text{conv}(U_{n-1})^*} F(g) \leq F(f). \text{ Hence, } f_1(F) = f_2(F) = \frac{a}{2^n - 1}. \text{ Next choose three positive numbers } \alpha_1 + \alpha_2 + \alpha_3 = 1, h \in H, n \in N \text{ and } v \in V \text{ such that }
\]

\[
f_1 = \alpha_1 h + \alpha_2 n + \alpha_3 v.
\]

Again by our prior estimates

\[
|F(H)| < \frac{a}{2^n - 1} - \delta \frac{a}{2^n}
\]

and

\[
|F(N)| < \frac{1}{2} \frac{a}{2^n - 1}.
\]

Hence, \( \alpha_1 = \alpha_2 = 0 \) and \( f_1 \in \text{conv} V^w \).

\[\square\]
Proof of claim. Now assume \( f = \frac{1}{2} f_1 + \frac{1}{2} f_2 \). By our prior result, \( f_1 \in \cap_{i=1}^{\infty} \text{conv} U_i^{w^*} \). It is easy to see that \( e_{g(n)} \left( \text{conv} U_m^{w^*} \right) \leq a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \lambda_{g(n)} \). Hence \( f_1(e_{g(n)}) \leq a \lambda_{g(n)} \) and the same holds for \( f_2 \). Since \( f(e_{g(n)}) = a \lambda_{g(n)} \) it holds that \( f_1(e_{g(n)}) = f_2(e_{g(n)}) = f(e_{g(n)}) \). It is also easy to see that for every \( c_i f_1(e_{c_i}) = f_2(e_{c_i}) = 0 \) and \( f_1 = f_2 = f \).

\[ \square \]

Claim 4.0.7. The set \( \text{ext} U^* \) cannot be covered by a countable union of balls \( B_{X^*}(x_i, \epsilon_i) \) with \( 0 < \epsilon_i < 1 \) tending to zero.

Assume otherwise. Denote \( E = \left\{ a \sum_{i=1}^{\infty} \lambda_i e_{g(i)}^* : g \in G \right\} \) and let

\[ E \subseteq \cup B_{X^*}(x_i, \epsilon_i) \quad \epsilon_i \to 0. \]

Since \( B_{X^*} \subseteq 2B_{\ell_1} \) it holds that

\[ E \subseteq \cup B_{\ell_1}(x_i, 2\epsilon_i). \]

For each ball \( i \) choose a representative \( y_i \in B_{\ell_1}(x_i, 2\epsilon_i) \). Next note that for any two vectors \( y_1, y_2 \in E \), if \( g_{y_1}(n) \neq g_{y_2}(n) \) then \( \| y_1 - y_2 \| > \frac{1}{2} \). Choose \( m_0 \) sufficiently large so that for \( m_0 < m \) it holds that \( 2\epsilon_m < \frac{1}{4} \). Choose \( n_0 \) sufficiently large so that if \( g_{x}(1) > n_0 \) then

\[ \max\{2\epsilon_1, \ldots, 2\epsilon_{m_0}\} < \| x - y_j \|, \]

for each \( j \leq m_0 \) (Note that this is possible since \( 2\epsilon_i < 2 \) and \( E \subseteq 2S_{\ell_1} \)). Denote by \( G_0 \) the set \( \{1, 2, \ldots, n_0\} \). Choose \( m_1 > m_0 \) sufficiently large such that if \( m_1 < m \) then \( 2\epsilon_m < \frac{1}{8} \). Denote by \( G_1 \) the set \( \{g_{y_{m_0+1}}(1), \ldots, g_{y_{m_1}}(1)\} \). By our remark above, if \( x \in E \) and \( g_{x}(1) \notin G_1 \) then \( \| x - y_j \| > \frac{1}{2} \) for \( m_0 < j \leq m_1 \). Hence, \( x \notin \cup_{i=m_0}^{m_1} B_{\ell_1}(x_i, 2\epsilon_i) \). Continue to choose inductively \( m_n \) and \( G_n \) such that

1. for every \( m_n < m, \epsilon_m < \frac{1}{2^{m+1}} \).

2. \( G_n \) is finite.

3. if \( g_x(n) \notin G_n \) then \( x \notin \cup_{i=m_{n-1}+1}^{m_n+1} B_{\ell_1}(x_i, 2\epsilon_i) \).

Choose \( m_{n+1} \) so that for \( m_{n+1} < m \) it holds that \( 2\epsilon_m < \frac{1}{2^{m+2}} \). Denote by \( G_{n+1} \) the set \( \{g_{y_{m_{n+1}+1}}(n + 1), \ldots, g_{y_{m_{n+1}}}(n + 1)\} \). For every \( x \in E \) and \( m_n < j \leq m_{n+1} \) if \( g_x(n+1) \notin G_{n+1} \) then \( \| x - y_j \| > \frac{1}{2^{m+3}} > 4\epsilon_j \) and \( x \notin \cup_{j=m_{n+1}+1}^{m_{n+1}+1} B_{\ell_1}(x_i, 2\epsilon_i) \). Define \( a_1 = \max G_0 \cup G_1 + 1 \) and Define \( a_n \) to be \( \max \{\cup_{i=0}^{n} G_n, a_1, \ldots, a_{n-1}\} + 1 \). Next define \( g \in G_{\infty} \) to be \( g(n) = a_n \), then \( x = \sum_{i=1}^{\infty} \lambda_i e_{g(i)}^* \notin \cup_{i=1}^{\infty} B_{\ell_1}(x_i, 2\epsilon_i) \).
Bibliography


