COS 445 - Midterm

Due online Monday, March 13th at 11:59 pm

- All problems on this exam are no collaboration problems.
- You may not discuss any aspect of any problems with anyone except for the course staff.
- You may not consult any external resources, the Internet, etc.
- You may consult the course lecture notes on Ed, any of the five course readings, past Ed discussion, or any notes directly linked on the course webpage (e.g. the cheatsheet, or notes on linear programming).
- You may discuss the test with the course staff, but we will only answer clarification questions and will not give any guidance or hints. You should feel free to ask any questions and let us judge whether or not to answer, but just know that we may choose to politely decline to answer. We may choose to answer questions with a response of “I’m sorry, but I’m not comfortable answering that question,” or “it is within the scope of the exam for you to answer that question yourself” (or some variant of these).
- If you choose to ask a question on Ed, ask it privately. We will maintain a pinned FAQ for questions that are asked multiple times (please also reference this FAQ).
- Please upload each problem as a separate file via codePost, as usual.
- You may not use late days on the exam. You must upload your solution by March 13th at 11:59 pm. If you are working down to the wire, upload your partial progress in advance. There is no grace period for the exam. In case of a true emergency where you cannot upload, email smweinberg@princeton.edu your solutions asap.
- If you miss the codePost deadline, we will not completely ignore your submission, but we will apply a substantial deduction before grading it. Please make sure you have something submitted by the deadline, and take into account that the server may be overloaded or sluggish near the end.
- There are no exceptions, extensions, etc. to the exam policy (again, in case of a truly exceptional circumstance, you should reach out to your residential dean and have them contact us).
Problem 1: Matching, Voting, Games, and LPs (40 points)

For each of the 4 problems below: unless otherwise specified1 you do not need to show any work and can just state the answer. However, if you simply state an incorrect answer with no justification, we cannot award partial credit. You are encouraged to provide a very brief outline/justification in order to receive partial credit in the event of a tiny mistake. For example, we will award very significant partial credit if you clearly execute the correct outline, but make a mistake in implementation.

Part a: Stable Matchings (10 points)

Four students Alice, Bob, Claire and David are applying to summer internships at Apple, Bell Labs, Capital One and Dell (all of which need exactly one intern). Here are their preferences (listed in decreasing order).

- Alice: Capital One ≻ Bell Labs ≻ Apple ≻ Dell.
- Bob: Apple ≻ Bell Labs ≻ Dell ≻ Capital One.
- Claire: Capital One ≻ Dell ≻ Apple ≻ Bell Labs.
- David: Dell ≻ Apple ≻ Bell Labs ≻ Capital One.

and the companies preferences:

- Apple: David ≻ Alice ≻ Claire ≻ Bob.
- Bell Labs: Claire ≻ Bob ≻ Alice ≻ David.
- Capital One: David ≻ Claire ≻ Alice ≻ Bob.
- Dell: Bob ≻ David ≻ Claire ≻ Alice.

Find the stable matching that results from student-proposing deferred acceptance. A reminder of the Deferred Acceptance algorithm is the Lecture Stable Matchings I.

Part b: Voting Rules (10 points)

A town of 20 voters is holding an election between candidates Alice, Bob and Carol.

- 8 of the voters prefer Alice ≻ Bob ≻ Carol
- 7 of the voters prefer Carol ≻ Bob ≻ Alice
- 5 of the voters prefer Bob ≻ Carol ≻ Alice

State the winning candidate selected by each of the following voting rules: Borda, IRV, Plurality. A reminder of these three voting rules is in Lecture Voting Theory I.

1If otherwise specified, you should follow the otherwise specifications.
Part c: Game Theory (10 points)

Find a Nash equilibrium of the following game and state the expected payoff for both players. A definition of Nash equilibrium can be found in Lecture Game Theory II.

Player $X$, the row player, chooses between actions $x_1$ and $x_2$. Player $Y$, the column player, chooses between actions $y_1$ and $y_2$. The first number in each box denotes the payoff to $X$, and the second number is the payoff to $Y$. For example, if $X$ plays action $x_1$ and the column player plays action $y_1$, then $X$ gets payoff 3 and $Y$ gets payoff 2.

\[
\begin{array}{c|cc}
  & y_1 & y_2 \\
\hline
x_1 & (3,2) & (1,8) \\
x_2 & (0,6) & (4,4)
\end{array}
\]

Part d: Linear Programming (10 points)

Write the dual of the following LP. You do not need to solve the LP. You only need to write the dual. A reminder of LP duality is in Lecture Linear Programming.

Maximize $6x + 2y$, such that:

- $4x + 3y \leq 10$.
- $7x + 2y \leq 6$.
- $x, y \geq 0$. 
Problem 2: Build your own game (40 points)

In this problem, you are asked to design a game with several properties. You will receive partial credit for designing a game that satisfies any subset of the properties (but you may only submit one game. I.e. you cannot design 4 games, each satisfying a single property). Recall also the following definitions:

Definition 1 (Weakly Dominate) A strategy $a$ weakly dominates strategy $b$ for player $i$ if strategy $a$ always yields at least as much payoff as $b$ for player $i$, no matter what strategy the other players use, and also there exists a strategy profile for the other players such that strategy $a$ yields strictly more payoff than strategy $b$ against this strategy profile. A strategy is weakly dominant for player $i$ if it weakly dominates all other strategies for player $i$.

Definition 2 (Iterated Deletion of Dominated Strategies) Start from an arbitrary game with $n$ players and $m$ actions each. Let strategy $a$ weakly dominate strategy $b$ for player $i$. Then one deletion of a dominated strategy simply deletes the action $b$ from player $i$. Now player $i$ has only $m-1$ available actions, and all definitions of domination are updated (since player $i$ can no longer play strategy $b$). Intuitively, think of this as all players reasoning that player $i$ will never play $b$ since it is weakly dominated).

Iterated deletion of dominated strategies is simply repeating this operation until no weakly dominated strategies remain. If each player has only a single remaining strategy, we say that the game is solvable by iterated deletion of dominated strategies, and we say that iterated deletion of dominated strategies predicts that each player will play their only remaining response.

Finally, if the game is solvable by iterated deletion of dominated strategies, and the resulting strategy for each player $i$ is $s_i$, we say that $(s_1, \ldots, s_n)$ results from iterated deletion of dominated strategies.

Specify the payoff matrix for a game with two players and two actions each such that:

(i) The row player does not have a weakly dominant pure strategy. (10 points)

(ii) The column player has a weakly dominant pure strategy. Furthermore, after deleting the column player’s weakly dominated pure strategy, the row player now has a weakly dominant pure strategy. (In other words, the game is solvable by iterated deletion of weakly dominated strategies). (10 points)

(iii) The game has exactly two pure Nash equilibria. (10 points)

(iv) The pure Nash equilibrium resulting from iterated deletion of dominated strategies is strictly worse for both players than the other pure Nash equilibrium (which does not result from iterated deletion of dominated strategies). That is, each player receives strictly less payoff in the pure Nash equilibrium resulting from iterated deletion of dominated strategies than in the other pure Nash equilibrium.(10 points)

For each property, provide a brief justification of why your game satisfies it.

Note: It is OK if your game has a mixed Nash equilibrium that is not pure, and also OK if your game does not have a mixed Nash equilibrium that is not pure (and also OK if your game has any number of mixed Nash equilibria). You do not have to reason about mixed Nash equilibria at all in this problem for full credit.

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You do not need to provide any flavor text to explain the game. You just need to design the payoff matrix.
Problem 3: Implications of Strategyproofness (50 points)

Note: You may NOT cite any results from the course materials for this problem. Your proof for this problem must be entirely self-contained (that is, you must prove any claim that you make, and may not cite the fact that it is stated/proved in a lecture/textbook).

Recall the following definitions:

**Definition 3** A voting rule $F$ is strategyproof if no voter can ever be strictly happier by lying. Formally, $F$ is strategyproof if for all voters $i$, all true preferences $\succ_i$, and all possible lies $\succ'_i$, and all possible votes of other voters $\succ'_{-i}$, $F(\succ_i; \succ'_{-i}) \succeq_i F(\succ'_i; \succ'_{-i})$.

**Definition 4 (Unanimous)** A voting rule $F$ is unanimous if for all candidates $a$, whenever all voters select $a$ as their favorite candidate, $F$ outputs $a$. Formally, $F$ is unanimous if whenever there exists a candidate $a$ such that $a \succ_i b$ for all $i$ and all $b \neq a$, then $F(\succ_1, \ldots, \succ_n) = a$.

**Part a: Super Monotonicity (30 points)**

Let $F$ be a strategyproof voting rule. Prove that $F$ is Super Monotone, defined below:

**Definition 5** A voting rule $F$ is super monotone if for all preference lists $\succ_1, \ldots, \succ_n$, $\succ'_1, \ldots, \succ'_n$ such that (1) $F(\succ_1, \ldots, \succ_n) = a$, and (2) for all voters $i$ and all candidates $b \neq a$, $a \succ_i b \Rightarrow a \succ'_i b$ (for all voters $i$ and candidates $b$, if they preferred $a$ to $b$ under $\succ_i$, they still prefer $a$ to $b$ under $\succ'_i$), then (3) $F(\succ'_1, \ldots, \succ'_n) = a$ as well.

**Part b: No Losers Allowed (20 points)**

Let $F$ be a strategyproof and unanimous voting rule. Prove that $F$ is loser-free, defined below:

**Definition 6** A candidate $b$ is a loser for votes $\succ_1, \ldots, \succ_n$ if there exists another candidate $a$ such that $a \succ_i b$ for all $i$. A voting rule $F$ is loser-free if for all $\succ_1, \ldots, \succ_n$, $F(\succ_1, \ldots, \succ_n)$ is not a loser for $\succ_1, \ldots, \succ_n$ (that is, $F$ never outputs a loser).

If you would like to, you can use part a in your proof of part b, even if you did not solve part a. If your solution to part b is correct, subject to this, you will receive full credit on part b.

**Reminder**: You may not cite any results from the course materials for this problem, and your solution must be self-contained. The only “exception” to this is that you may use part a in your solution to part b, even if you did not solve part a.

The reason for this policy is because the problem becomes significantly simpler if you cite the Gibbard-Satterthwaite theorem. In case you are truly stuck on the problem, solving either part while citing the Gibbard-Satterthwaite theorem will give you partial credit equal to the lower “concrete partial progress” mark on the generic rubric.

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3Here, we have used the notation $x \succeq_i y$ to denote “either $x \succ_i y$ or $x = y$.” This is the same as saying $x \succeq_i y$ whenever $y \not\succ_i x$.

4If you remember PSet two, this is saying that $b$ is a loser if there exists a candidate such that $a$ dominates $b$. If you don’t remember PSet two, you should ignore this footnote (it is not a hint, just clarifying the definition).

5If you remember PSet two, this is saying that $F$ never outputs a dominated candidate. Again, if you don’t remember PSet two, just ignore this footnote (it is again not a hint, and just clarifying the definition).
Problem 4: Matching with Linear Programs (70 points)

There are \( n \) students \( S \) and \( n \) universities \( U \), each with capacity one and each with full, strict preferences over the other side. For this problem, it will also be more convenient to represent a matching \( M \) using the following notation: \( M_{su} = 1 \) if and only if \( s \) is matched to \( u \) in \( M \), and \( M_{su} = 0 \) otherwise.

This proof will guide you through the first steps of a proof that stable matchings exist based on LP duality (versus appealing to the Gale-Shapley algorithm).

Part a: An Integer Program (20 points)

Consider the following linear constraints, where the first three constraints don’t depend on the preferences \( \succ \), but the final constraint does:

- \( \sum_{u \in U} M_{su} = 1 \), for all \( s \in S \).
- \( \sum_{s \in S} M_{su} = 1 \), for all \( u \in U \).
- \( M_{su} \geq 0 \) for all \( s \in S, u \in U \).
- \( M_{su} + \sum_{s' \in S, \text{such that } s' \succ_u s} M_{s'u} + \sum_{u' \in U, \text{such that } u' \succ_s u'} M_{su'} \leq 1 \), for all \( s \in S, u \in U \).

Prove that a \{0, 1\}-matrix \( M \) (with \( n \) rows and \( n \) columns, all entries are either 0 or 1) corresponds to a stable matching for preferences \( \succ \) if and only if \( M \) satisfies all of the constraints above.

Part b: A Linear Program (20 points)

Take the dual of the following linear program, and very briefly justify that you’ve taken the dual correctly. For full credit, your solution should be easy for a grader to parse (there should be roughly as many symbols/notation in your dual LP as in the primal LP below).

To match notation for the hint in part c, you should use the variable \( \alpha_s \) as the dual variable for constraints one, \( \beta_u \) as the dual variable for constraints two, and \( \gamma_{su} \) as the dual variable for constraints four. You are free to ignore this suggestion, but the hints for part c may not fit perfectly if you do.

Minimize \( \sum_{s,u} M_{su} \)

Such that:

- \( \sum_{u \in U} M_{su} \geq 1 \), for all \( s \in S \).
- \( \sum_{s \in S} M_{su} \geq 1 \), for all \( u \in U \).
- \( M_{su} \geq 0 \) for all \( s \in S, u \in U \).
- \( -M_{su} - \sum_{s' \in S, \text{such that } s' \succ_u s} M_{s'u} - \sum_{u' \in U, \text{such that } u' \succ_s u'} M_{su'} \geq -1 \), for all \( s \in S, u \in U \).
Part c: Its own Dual (30 points)

Prove that all feasible solutions to the LP from part b is are also optimal solutions to the LP from part b. That is, prove that all matrices $M \in \mathbb{R}^{n''}$ satisfying the above four constraints have the same value of $\sum_{s,u} M_{su}$.

Hint: For any feasible $M$, consider the dual solution: $\gamma_{su} = M_{su}$, $\alpha_s = \sum_{u \in U} M_{su}$, $\beta_u = \sum_{s \in S} M_{su}$ and find a way to invoke weak duality (or strong duality, if you prefer).