A Model of Type Theory with Directed Univalence in Bicubical Sets

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Directed Type Theory

- Riehl-Shulman defines a type theory for $\infty$-categories with a model bisimplicial sets
  1. Begin with HoTT
  2. Add Hom-types
  3. $\infty$-categories (Segal types) and univalent $\infty$-category (Rezk types) given internally as predicates on types
  4. Predicate isCov($B : A \to U$) for covariant discrete fibrations
  5. Cavallo, Riehl and Sattler have also (externally) defined the universe of covariant fibrations (the $\infty$-category of spaces and continuous functions) and shown

\[ \text{Directed Univalence: } \text{Hom}_{\text{UCov}} A B \simeq A \to B \]
Constructive(?) Directed Type Theory

• Can we make this constructive?
  1. Begin with Cubical Type Theory
  2. Use a second cubical interval to define Hom-types
  3. Use LOPS to define universe of covariant fibrations and construct directed univalence internally...

• ...unfortunately, directed univalence is a bit trickier than expected
Let's see how far the techniques from cubical type theory get us!
# Defining Bicubical Directed Type Theory

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Defining Bicubical Directed Type Theory

Cubical Type Theory
(in the style of Orton-Pitts)

2. Add an interval: $\mathbb{I}$

\[
\begin{align*}
\mathbb{I} &: \text{Type} \\
\hline
0_{\mathbb{I}} &: \mathbb{I} \\
1_{\mathbb{I}} &: \mathbb{I}
\end{align*}
\]

i.e. generators for the Cartesian cubes

Directed Type Theory

2. Add an interval: $\mathbb{2}$

\[
\begin{align*}
\mathbb{2} &: \text{Type} \\
\hline
0_{\mathbb{2}} &: \mathbb{2} \\
1_{\mathbb{2}} &: \mathbb{2}
\end{align*}
\]

\[
\begin{align*}
x &: \mathbb{2} \\
y &: \mathbb{2}
\end{align*}
\]

\[
\begin{align*}
x \land y &: \mathbb{2} \\
x \lor y &: \mathbb{2}
\end{align*}
\]

and equations...

i.e. generators for the Dedekind cubes
The Directed Interval

- Why Dedekind cubes instead of Cartesian?
  \[ x \leq y := x = x \land y \]

- We also add the following axioms:
  
  - \( p : \mathbb{I} \to \mathbb{2} \) is constant (\( \Pi x \ y : \mathbb{I}, \ p \ x = p \ y \) )
  
  - \( p : \mathbb{2} \to \mathbb{2} \) is monotone (\( \Pi x \ y : \mathbb{2}, \text{ if } x \leq y \text{ then } p \ x \leq p \ y \) )
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**Cubical Type Theory**  
*(in the style of Orton-Pitts)*

1. Begin with MLTT
2. Add an interval: \( \mathbb{I} \)
3. Specify gen. cofibrations for \( \mathbb{I} \)
4. Define filling problem for Kan fibrations
5. Define universe of Kan fibrations
6. Construct univalence

**Directed Type Theory**

1. Begin with Cubical Type Theory
2. Add an interval: \( \mathcal{2} \)
3. Specify gen. cofibrations for \( \mathcal{2} \)
4. Define filling problem for covariant fibrations
5. Define universe of covariant fibrations
6. Construct directed univalence
### Defining Bicubical Directed Type Theory

**Cubical Type Theory**  
*in the style of Orton-Pitts*

#### 3. Specify gen. cofibrations for \(\mathbb{I}\)

\[
\text{isCof} : \Omega \to \Omega \\
\text{Cof} := \Sigma \phi : \Omega . \text{isCof} \phi
\]

Cof closed under \(\wedge, \vee, \bot, \top\)

\[
\begin{array}{c}
x : \mathbb{I} \quad y : \mathbb{I} \\
\_ : \text{isCof} (x = y)
\end{array}
\]

\[
\phi : \mathbb{I} \to \text{Cof} \\
\_ : \text{isCof} (\Pi x : \mathbb{I} . \phi x)
\]

---

**Directed Type Theory**

#### 3. Specify gen. cofibrations for \(\mathcal{2}\)

\[
\begin{array}{c}
x : \mathcal{2} \quad y : \mathcal{2} \\
\_ : \text{isCof} (x = y)
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\[
\phi : \mathcal{2} \to \text{Cof} \\
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# Defining Bicubical Directed Type Theory

## Cubical Type Theory
*(in the style of Orton-Pitts)*

4. Define filling problem for Kan fibrations

\[
\text{hasCom} : (\mathbb{I} \to U) \to U
\]

\[
\text{hasCom} \ A = \prod \ i \ j : \mathbb{I} . \\
\quad \prod \alpha : \text{Cof} . \\
\quad \prod \ t : (\prod \ x : \mathbb{I} . \alpha \to A \ x) \\
\quad \prod \ b : (A \ i)[\alpha \mapsto t \ i] . \\
\quad (A \ j)[\alpha \mapsto t \ j ; \ i = j \mapsto b]
\]

\[
\text{relCom} : (A : U) \to (A \to U) \to U
\]

\[
\text{relCom} \ A \ B = \prod \ p : \mathbb{I} \to A . \\
\quad \text{hasCom} \ (B \circ p)
\]

## Directed Type Theory

4. Define filling problem for covariant fibrations

\[
\text{hasCov} : (\mathcal{2} \to U) \to U
\]

\[
\text{hasCov} \ A = \prod \alpha : \text{Cof} . \\
\quad \prod \ t : (\prod \ x : \mathcal{2} . \alpha \to A \ x) \\
\quad \prod \ b : (A \ 0_2)[\alpha \mapsto t \ 0_2] . \\
\quad (A \ 1_2)[\alpha \mapsto t \ 1_2]
\]

\[
\text{relCov} : (A : U) \to (A \to U) \to U
\]

\[
\text{relCov} \ A \ B = \prod \ p : \mathcal{2} \to A . \\
\quad \text{hasCov} \ (B \circ p)
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## Defining Bicubical Directed Type Theory

### Cubical Type Theory
*(in the style of Orton-Pitts)*

5. Define universe of Kan fibrations

- $U_{\text{Kan}}$ given by LOPS construction for relCom

### Directed Type Theory

5. Define universe of covariant fibrations

- $U_{\text{Cov}}$ given by LOPS construction for relCov.

**Lemma**: $\text{relCov}$ is in $U_{\text{Kan}}$
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### Cubical Type Theory (in the style of Orton-Pitts)

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4. Define filling problem for Kan fibrations
5. Define universe of Kan fibrations
6. **Construct univalence**

### Directed Type Theory

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Glue Types

\[
\text{Glue} \left[ \alpha \mapsto (T, f) \right] B :=
\]

\[
\alpha \vdash T
\]

\[
\alpha \vdash f
\]

\[
\alpha \vdash Glue \left[ \alpha \mapsto (T, f) \right] B \equiv T
\]

\[
g : \text{Glue} \left[ \alpha \mapsto (T, f) \right] B
\]

\[
\text{un glue} \ g : B
\]

\[
\alpha \vdash \text{glue} \ t \ b \equiv t
\]

\[
\alpha \vdash \text{un glue} \ (\text{glue} \ t \ b) \equiv f \ t
\]

\[
\text{glue} \ g \ (\text{un glue} \ g) \equiv g
\]
Defining Directed Univalence

dua i A B f := Glue [ i = 0 ↦ (A, f : A → B), i = 1 ↦ (B, id) ] B : Hom(u A B)
Naive Directed Univalence

- dua is Kan + covariant, and thus lands in $U_{Cov}$
- $U_{Cov}$ itself is Kan
- Path univalence holds in $U_{Cov}$

These allow us to define the following for $U_{Cov}$:

- $dcoe : (\text{Hom } A \to B) \to (A \to B)$
- $dua : (A \to B) \to \text{Hom } A \to B$
- $dua_\beta : \Pi f : A \to B . \text{Path } f (dcoe (dua f))$
- $dual_\eta_{\text{fun}} : \Pi p : \text{Hom } A \to B . \Pi i : \mathbb{2} . p i \to (dua (dcoe p)) i$
Naive Directed Univalence

• We're thus left with the following picture:

• To complete directed univalence, we need $\text{dua}_{\eta_{\text{fun}}}^{-1}$

• Agda: https://github.com/dlicata335/cart-cube
What next?

• Cavallo, Riehl and Sattler's proof of directed univalence contains the precise lemma we need to finish.

• New goal: use any techniques available to confirm directed univalence holds at all in a cubical setting.

• Note: We would love any/all feedback on the math that follows.
What next?

• The proof in the bisimplicial model relies on simplices being a Reedy category
  
  • specifically: weak equivalences in the model are level-wise weak equivalences of simplicial sets
  
  • Dedekind cubes are not Reedy...
Our New Goal

- Find a setting that...
  1. is cubical set valued presheaves of a Reedy category
  2. interprets the axioms from our internal language
  3. allows for the LOPS construction of universes
     - tiny interval
What are Reedy Categories?

• The Idea: Categories permitting inductive constructions of presheaves and their morphisms (akin to cell complexes)

• (informal/incomplete) Definition: A generalized Reedy category is a category $C$ along with a degree function $\delta : \text{ob } C \rightarrow \mathbb{N}$ such that every morphism (that isn't an iso) factors through an object of strictly smaller degree
The Dedekind Cubes

- Free Cartesian category on an interval generated by:
  - face maps (+)
  - diagonals (+)
  - degeneracies (-)
  - connections (-)
The Dedekind Cubes

\[(x, y) \mapsto (x, y, y) \mapsto (x \land y, y)\]
The Dedekind Cubes

$\langle x, y \rangle \mapsto \langle x, y, y \rangle \mapsto \langle x \land y, y \rangle$
The Dedekind Cubes

\[(x, y) \mapsto (x, y, y) \mapsto (x \land y, y)\]

up by a diagonal
The Dedekind Cubes

\[(x, y) \mapsto (x, y, y) \mapsto (x \land y, y)\]

down by a connection
The Dedekind Cubes

\[(x, y) \mapsto (x \wedge y, y)\]
The Dedekind Cubes

\[(x, y) \mapsto (x \land y, y)\]
The Dedekind Cubes

\[(x, y) \mapsto (x \land y, y)\]
The Dedekind Cubes

$$(x, y) \mapsto (x \land y, y)$$
The Dedekind Cubes

\[(x, y) \mapsto (x \land y, y)\]
The Image Closure

- The Idea: formally add image objects for every morphism

- The Construction: Given a small Category $C$, the image closure $\text{Im}(C)$ is the full subcategory of $[C^{\text{op}}, \text{Set}]$ containing, for each morphism $f$ in $C$, the coimage of $f$.

- Useful Lemma: We can build a topology $J_{\text{im}}$ (the image covering) on $\text{Im}(C)$ such that $[C^{\text{op}}, \text{Set}] \cong \text{Sh}(\text{Im}(C), J_{\text{im}})$.
  - Inspired by Kapulkin and Voevodsky
  - The Comparison Lemma: [SGA 4, The Elephant]
The Prism Category

• Definition: The prism category is the image closure of the Dedekind cube category.

• Lemma (with Christian Sattler): The prism category is equivalent to the full subcategory of simplicial sets containing subobjects of the Dedekind cubes \((\Gamma, \phi)\) generated by the following formulae:
  • \(\top\) : true
  • \(x \leq y\) : the equalizer of the degeneracy map \(x\) and connection \(x \land y\)
  • \(\phi \land \psi\) : the pullback of the subobjects \((\Gamma, \phi)\) and \((\Gamma, \psi)\)
  • \(\phi \lor \psi\) : the pushout of the pullback for \((\Gamma, \phi \land \psi)\)
The Prism Category

• The Prism category
  • is a finite product category...
  • ...and thus the Yoneda embedding of its interval is tiny...
Prisms are Reedy

- Theorem: The prism category is a generalized Reedy category.
  - The down maps are those that are regular epis in the presheaf category
  - The up maps are the monos
  - The Reedy factorization is the image factorization

- Corollary: The opposite of the prism category is also generalized Reedy

- Question: For which categories $C$ is $\text{Im}(C)$ Reedy?
Model Category One: Prismatic Cubical Sets

- Reedy model structure on $[\text{Prism}^{\text{op}}, [\text{Cube}^{\text{op}}, \text{Set}]]$, starting with model structure on Cartesian cubes [Sattler, Awodey]

- The lemma missing from the bicubical internal language now is provable in the same way as in bisimplicial sets.

- As our internal language axioms interpret into this model, we get a model with directed univalence!

- Can we make this even more cubical?
Model Category Two: Bicubical Sets

- Sheafification gives us an adjunction between prismatic cubical sets and bicubical sets

- We can transfer the model structure along the adjunction to bicubical sets
  - Left Induced Model Structure: [Hess-Kedziorek-Riehl-Shipley, Garner-Kedziorek-Riehl]
  - Path Object Argument: [Quillen]

- Our internal language axioms still interpret after the transfer

- The lemma that finished directed univalence is still true after the transfer