Lecture 6: Global Correlation Rounding

Gaussian Rounding : (p.d. degree 2)

G.R + Squared triangle inequalities (p.d. of deg 4)

today: one-way to use higher degree SOS.

Running application: Max-Cut problem on certain restricted class of graphs

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Recalling basic Spectral Graph Theory

G: d-regular graph on [n] vertices

Normalized Adj Matrix : $A(i,j) = \begin{cases} 1/d & \text{if } \{i,j\} \in G \\ 0 & \text{o/w} \end{cases}$

$\equiv$ Random Walk matrix of G

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigs of A

Obs^n 1: $\lambda_1 = 1$ (all 1s vector is the corr eig-vector)

Obs^n 2: $\lambda_2 = 1$ iff G is disconnected.

G is disconnected: C_1 & C_2 are conn components

$\mathbb{1}_S$ vector c_1 ,

$\mathbb{1}_S$ vector c_2

$v: \langle v, \mathbb{1} \rangle = 0$.

if $Av = v$ then $v_i = v_j$ if $i \sim j$

the only way $v \neq c \cdot \mathbb{1}$ is if G is disconnected.

<u>G is expander</u> : $\Phi_G \geq \delta$

$\lambda_2 \leq 1 - O(\delta^2)$

<u>G is a random d-reg graph</u> :

$$\lambda_2(A) = \frac{2\sqrt{d-1}}{d} + o(1)$$

<u>Def (Threshold Rank)</u>

Let G be a d-reg graph. $\lambda_1 \geq \cdots \geq \lambda_n$

The ρ- threshold rank of G <u>$\text{rank}_\rho(G)$</u>

$= \#$ eigs of $G \geq \rho$.

<u>Ex</u>: Suppose G is an expander graph.

$\rho : 1 - o(1)$

What is $\text{rank}_\rho(G) = 1$

Bounded threshold rank ≡ "generalized expanders"

Running example / Main Goal.

Theorem: Let G be a d-reg. graph on $[n]$.

$$\text{Let } f_G = \frac{1}{4} \sum_{\{i,j\} \in E} (x_i - x_j)^2$$

Let $r = \text{rank}_{\delta^2}(G) / \delta^4$. Then given a deg $r+2$ p.d. μ, can find a dist^n x' s.t.

$$\mathbb{E}\, f_G(x') \geq \underbrace{\tilde{\mathbb{E}}_{\mu}\, f_G}_{} - O(\delta) \cdot m$$

" $n^{O(r)}$ time algo to compute additive approx."

$\equiv$ $n^{O(r)}$ time algo to compute $(1 - O(\delta))$ multiplicative approx. to max-cut".

Global Correlation Rounding

We'll first study some basic props of higher degree pseudo-dist^{n}s.

<u>Def</u>: "Marginal of a p.d. on a set S of deg r."

Let μ be a p.d. on $\{\pm 1\}^n$. For any $S \subseteq [n]$, $\underbrace{\mu|_S}$: $\{\pm 1\}^S \to \mathbb{R}$ is the <u>following f^n</u>:

$$\mu|_S(y) = \sum_{x : x|_S = y} \mu(x).$$

<u>Remark</u>: $\forall\, U \subseteq S$, $\tilde{\mathbb{E}}_{\mu|_S} x_U = \tilde{\mathbb{E}}_{\mu}\, x_U$.

<u>Thus</u>, $\forall$ f of degree r, f only depends on variables in S,

$$\tilde{\mathbb{E}}_{\mu|_S} f = \tilde{\mathbb{E}}_{\mu} f.$$

<u>Prop</u>: Let μ be a p.d. on $\{\pm 1\}^n$ of deg r. Let $S \subseteq [n]$ of size at most $r/2$ ($|S| \leq \frac{r}{2}$). Then $\mu|_S$ is an actual prob. distn

<u>Proof</u>: μ is a p.d. of deg r

$\Downarrow$

$\mu|_S$ is also a p.d. of deg r [Note all expectations of $f : \{\pm 1\}^S \to \mathbb{R}$

$\mu|_S$ is ... are the same)

<u>Since $|S| \le \frac{r}{2}$</u>: using that $\succeq 2r$ deg p.d. on $\{\pm 1\}^S$ are actual dist^n → D.

<u>Def/Notation/Name</u>: $\forall |S| \le \frac{r}{2}$, $\mu|_S$ the "local dist^n" of μ on S.

<u>Def (Reweighting)</u>:

Let μ be a p.d. of deg r. Suppose p is an SoS polynomial of degree $r' \le r$. (Note: $\tilde{\mathbb{E}}_\mu p \succeq 0$). Assume $\tilde{\mathbb{E}}_\mu p > 0$.

<u>Then</u>, μ' is said to be <u>reweighting of μ by</u> p if

$$\mu'(x) = \frac{\mu(x)\cdot p(x)}{\tilde{\mathbb{E}}_\mu p}.$$

If μ' has to be a p.d.

$\sum_x \mu'(x) = 1$

$\sum_x \mu(x)\cdot p(x)\cdot Z = 1$

$Z \cdot \tilde{\mathbb{E}}_\mu p = 1$

$\Rightarrow Z = \frac{1}{\tilde{\mathbb{E}}_\mu p}$

<u>Remark</u>:

$$\tilde{\mathbb{E}}_{\mu'} f = \sum_x \mu'(x)\cdot f(x)$$

$$= \sum_x \frac{\mu(x) \cdot p(x) \cdot f(x)}{\tilde{\mathbb{E}}_\mu \cdot p}$$

$$= \boxed{\frac{\tilde{\mathbb{E}}_\mu \cdot p \cdot f}{\tilde{\mathbb{E}}_\mu \cdot p}}$$

<u>Prop</u>: Suppose μ is a deg r p.d. on $\{\pm 1\}^n$. Let p be an SOS polynomial of deg $\leq r'$ & $\tilde{\mathbb{E}}_\mu \cdot p > 0$. Then the reweighting $\mu' = \frac{\mu(x) \cdot p(x)}{\tilde{\mathbb{E}}_\mu \cdot p}$ is also a pseudo-distn of deg $\geq r - r'$.

<u>Proof</u>: 1) $\tilde{\mathbb{E}}_{\mu'} 1 = 1$?

2) $\tilde{\mathbb{E}}_{\mu'} \cdot f^2 \geq 0 \quad \forall f : \deg(f) \leq \frac{r - r'}{2}$.

$$\tilde{\mathbb{E}}_{\mu'} 1 = \frac{\tilde{\mathbb{E}}_\mu \cdot p \cdot 1}{\tilde{\mathbb{E}}_\mu \cdot p} = 1 \quad \checkmark$$

$$\tilde{\mathbb{E}}_{\mu'} f^2 = \tilde{\mathbb{E}}_\mu \, p \cdot f^2 \geq 0$$

$\tilde{\mathbb{E}}_\mu \cdot p$

Claim: $p \cdot f^2$ is SoS of deg $\le r$

$\square$

Special, interesting case

Conditioning

Let μ be a p.d. of deg r.

Let $|S| \le r'/2$, $S \subseteq [n]$

Let $\alpha \in \{\pm 1\}^S$.

Then $\mu|_{x_S = \alpha}$ = reweighting of μ by the polynomial f_α^2 ✓

where: $f_\alpha(x) = \begin{cases} 1 & \text{if } x_{|S} = \alpha \\ 0 & \text{o/w.} \end{cases}$

assuming $\tilde{\mathbb{E}}_\mu f_\alpha > 0$.

$\deg(f_\alpha) \le |S|$

Goal: Cond^n on the event $x_{|S} = \alpha$

Note: $\mu|_{x_S=\alpha}$ is a p.d. of deg $\ge r - r'$

... pseudo-dist^{n}s ?

how to round high deg pseudo

here's a simple "high-deg" rounding.

Def (Independent Rounding)

Let μ be a p.d. of deg $= r$.
Let x' be the following actual distn.

Set $x_i' = \begin{cases} 1 & \text{w.p. } \frac{1}{2} + \frac{1}{2}\tilde{\mathbb{E}}_\mu x_i \\ -1 & \text{o/w} \end{cases}$ independently for $1 \leq i \leq n$.

$\Downarrow$

$\mathbb{E} x_i' = \tilde{\mathbb{E}}_\mu x_i$

"round each variable independently"

When is it good?

Prop: Let x' be indep rounding of a p.d. μ of deg ≥ 2. Then.

$$\mathbb{E} f_G(x') = \tilde{\mathbb{E}}_\mu f_G(x) + \sum_{\{i,j\}} \widetilde{\mathrm{Cov}}(x_i, x_j)$$

$\{1/j\}$GE 2

Def (Pseudo-covariance):

$$\tilde{\mathbb{E}}_{\mu}(x_i - \tilde{\mathbb{E}}_{\mu} x_i)(x_j - \tilde{\mathbb{E}}_{\mu} x_j) \overset{def}{=} \widetilde{Cov}(X_i, X_j)$$

Proof:

$$\mathbb{E}(x_i' - x_j')^2 = \mathbb{E} x_i'^2 + \mathbb{E} x_j'^2 - 2 \cdot \mathbb{E} x_i' \cdot x_j'$$

$$= 1 + 1 - 2 \cdot (\mathbb{E} x_i')(\mathbb{E} x_j')$$

$$\tilde{\mathbb{E}}_{\mu}(x_i - x_j)^2 = 2 - 2 \cdot \tilde{\mathbb{E}}_{\mu} x_i x_j$$

$$\frac{1}{4}\mathbb{E}(x_i' - x_j')^2 = \frac{1}{4}\tilde{\mathbb{E}}_{\mu}(x_i - x_j)^2 + \frac{1}{2}\left(\tilde{\mathbb{E}}_{\mu} x_i x_j - \mathbb{E} x_i' \cdot \mathbb{E} x_j'\right)$$

$$= \frac{1}{4} \cdot \tilde{\mathbb{E}}_{\mu}(x_i - x_j)^2 + \frac{1}{2} \cdot \left(\tilde{\mathbb{E}}_{\mu} x_i x_j - \tilde{\mathbb{E}}_{\mu} x_i \tilde{\mathbb{E}}_{\mu} x_j\right)$$

$$= \frac{1}{4} \cdot \tilde{\mathbb{E}}_{\mu}(x_i - x_j)^2 + \frac{1}{2} \widetilde{Cov}(X_i, X_j)$$

done by linearity.

<u>Main insight</u>: Start from a high deg p.d. μ.
and produce a <u>low average covariance</u> p.d.; μ_1
by conditioning μ. while ensuring that

$$\tilde{\mathbb{E}}_{\mu} f_G \simeq \tilde{\mathbb{E}}_{\mu_1} f_G$$

Then can independently round on μ_1.

<u>Low average covariance</u> $= \underbrace{\mathbb{E}_{\{i,j\} \in E} |\widetilde{\mathrm{Cov}}(X_i, X_j)|}_{\leq \delta}$

<u>Then</u>, $\mathbb{E} f_G(x') \gtrsim \tilde{\mathbb{E}}_{\mu_1} f_G - \delta \cdot m$

$$\approx \tilde{\mathbb{E}}_{\mu} f_G - \delta \cdot m$$

$\mu \longrightarrow \mu_2$ Under Symmetrization, 2nd moments are preserved, covariances are not

$$\mu_2(x) = \frac{\mu(x) + \mu(-x)}{2}$$

not covariance

$\mathbb{E}_{\mu_2} X = 0$, $\underline{\mathbb{E}_{\mu_2} X X^T} = \underline{\mathbb{E}_{\mu} X X^T}$

<u>Def</u> (Local Correlation of a pseudodistⁿ)

Let μ be a p.d. on $\{\pm 1\}^n$.

Let G be any graph on $[n]$, $G([n], E)$.

<u>Then</u>, the <u>local correlation</u> of μ <u>under G</u>

$$= \mathbb{E}_{\{i,j\} \in E} |\widetilde{Cov}(X_i, X_j)|$$

$$\boxed{\text{Summary of Discussion: If Indep rounding fails to give additive } \delta \text{ approx. then } \mathbb{E}_{\{i,j\} \in E} |\widetilde{Cov}(X_i, X_j)| \geq 2 \cdot \delta}$$

Can't expect to find p.d. of low local corr for arbitrary G. Instead we'll define a "global Correlation" & relate G.C. to L.C. for "nice" graphs.

Basic Information Theory

$[q]^n \equiv$ Space of assignments to CSPs with alphabet of size 10

X: be a r.v. on $[q]$.

$$H(X) \stackrel{\text{def}}{=} -\sum_{i \in [q]} \text{prob}(X=i) \cdot \log(\Pr(X=i))$$

$$\mathbb{I}(X;Y) = \sum_{i,j \in [q]} \Pr[X=i, Y=j] \cdot \log \frac{\Pr(X=i, Y=j)}{\Pr(X=i) \cdot \Pr(Y=j)}$$

(mutual information)

$$H(X|Y) = \sum_{i \in [q]} \Pr(Y=i)\, H(X|Y=i)$$

(conditional entropy)

<u>Note</u>: If X & Y are indep. then $H(X|Y) = H(X)$.

$$\mathbb{I}(X;Y) = H(X) - H(X|Y).$$
$$= H(Y) - H(Y|X)$$

<u>Note</u>: $\mathbb{I}(X;Y) = 0$ when X & Y are indep.

"measure of how far from indep are X & Y"

<u>Fact</u>: Let X, Y be $[q]$-valued r.v.s.

<u>Then</u> $|\mathrm{Cov}(X,Y)| \lesssim O_q\left(\sqrt{\mathbb{I}(X,Y)}\right)$.

<u>Def (Global Correlation)</u> Let μ be a p.d. of deg ≥ 4.

$$GC(\mu) \overset{def}{=} \mathop{\mathbb{E}}_{1 \le i,j \le n} \underbrace{\mathbb{I}_{\mu_{|\{i,j\}}}(X_i ; X_j)}_{\text{"well-defined"}}.$$

<u>Lemma</u>: Let μ be a p.d. on $\{\pm 1\}^n$ of degree $\frac{2}{\eta}+\ell$. Then there's a reweighting μ' <u>of deg $\geq \ell$</u> with $\underbrace{\text{degree } \frac{2}{\eta} \text{ SoS polynomial}}$. s.t. averaging

$GC(\mu') \leq \eta$, $\underbrace{\mathbb{E}}_{\text{random choices}}\tilde{\mathbb{E}}_{\mu'} f_G \geq \underbrace{\mathbb{E}}_{\text{random choices}}\tilde{\mathbb{E}}_{\mu} f_G - O(\eta)$

<u>Proof</u>: We'll build a sequence of reweightings $\mu_0 = \mu, \mu_1, \ldots, \mu_{t-1}, \ldots, \mu_{1/\eta}$. each μ_t will be a "degree 2" reweighting of μ_{t-1}. & one of them will satisfy the conclusion

1. Sample a uniformly random $1/\eta$-tuple of variables from $[n]$: $i_1, \ldots, i_{1/\eta}$.

For $1 \le t \le 1/\eta$ ⓐ sample $x_{i_t} \sim \alpha_t$ from local dist^n on i_t under μ_{t-1}.

ⓑ Set $\mu_t = \mu_{t-1} \cdot \underbrace{\frac{\mathbb{1}(X_{i_t} = \alpha_t)}{\mathbb{E}_{\mu_{t-1}} \mathbb{1}^2(X_{i_t} = \alpha_t)}}$ "Condition on $x_{i_t} = \alpha_t$"

ⓒ Stop if $GC(\mu_t) < \eta$.

Proof that one of μ_is works:

<u>Idea</u>: Potential Function: $\mathbb{E}_i H_{\mu_t}(X_i)$

Assume for $t \le 1/\eta$, <u>High G.C.</u>: $\mathbb{E}_{1 \le i,j \le n} \mathbb{I}_{\mu_t}(X_i ; X_j) \ge \eta$

<u>"typical" i</u>: $\boxed{\underbrace{\mathbb{E}_j \mathbb{I}_{\mu_t}(X_i ; X_j) \ge \eta}}$

What happens when I condition on $X_i = \alpha_i$?

$H_{\mu_t}(X_j) = H \quad (X_j \mid X_i)$

$$H_{\mu_t}(X_j) = H_{\mu_{t-1}}(X_j | \ldots$$

$$\mathbb{E}_j\left(H_{\mu_{t-1}}(X_j) - H_{\mu_t}(X_j)\right)$$

$$= H_{\mu_{t-1}}(X_j) - H_{\mu_{t-1}}(X_j | X_i)$$

$$= \mathbb{E}_j \mathbb{I}_{\mu_{t-1}}(X_i ; X_j) > \eta$$

$\Rightarrow$ Potential has fallen by $>\eta$

<u>OTOH</u>. $\mathbb{E}_i H_{\mu_t}(X_i) \lesssim 1$

<u>Thus</u> if all of $\mu_1 \text{---} \mu_{1/\eta}$ fail then

<u>contradiction</u>.

This analysis is an <u>expectation</u> over the

$i_1, \ldots, i_t$ & $\alpha_1 \text{---} \alpha_t$

By averaging:

$\Rightarrow \exists$ ... assignment to $i_1 \text{---} i_{1/\eta}$ that

$\Rightarrow$ $\exists$ an assignment ... works.

$\Rightarrow$ $n^{1/\eta}$ time algorithm to "brute force" Search for such a set of vertices & their assignment

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What's the Connection betⁿ Local & Global Correlations? No conn. for arbitrary graphs. will prove relationship when th-rank(G) is Small.

Connection Lemma: Let G be a graph on $[n]$.

Let μ be a p.d. on $\{\pm 1\}^n$.

Then, If $L.C._G(\mu) \geq \beta$

then $\forall \rho \leq \beta^2$, $G.C.(\mu) \geq \left(\frac{\beta^2 - \rho}{\text{rank}_\rho(G)}\right)^2$

Cor: $G.C.(\mu) \geq \left(\frac{\beta^2}{\underbrace{2 \cdot \text{rank}_{\beta/2}(G)}}\right)^2$ ✓

Cor: If $GC(\mu) < \eta$ ✓

$\Rightarrow \quad \beta^4 \leq 4 \cdot \mathrm{rank}_{\frac{\beta}{2}}(G)^2 \cdot \eta.$

For our application, choose $\beta = \delta$.

& $\eta = \dfrac{\delta^4}{4\,\mathrm{rank}_{\delta/2}(G)^2}$.

This lemma roughly tells us that starting with

$\dfrac{16\,\mathrm{rank}_{\frac{\delta}{2}}(G)^2}{\delta^4} + 2$ deg p.d. is enough to get rounding additive error $\leq \delta m$.

Let G be a graph on (n).

Technical Lemma: Let M be a $n \times n$ psd matrix s.t. $\mathrm{tr}(M) \leq n$, $|M(i,j)| \leq 1 \;\forall i,j$.

Then $\boxed{\mathop{\mathbb{E}}_{\{i,j\} \in E(G)} M(i,j) \geq \beta}$ (local correlation)

$$\Rightarrow \mathop{\mathbb{E}}_{1 \leq i,j \leq n} M(i,j)^2 \geq \left(\frac{\beta - \rho}{\mathrm{rank}_\rho(G)}\right)^2.$$

Proof of Conn. Lemma using Technical lemma:

Assume $L.C._G(\mu) \geq \delta$ - otherwise indep rounding works.

$\overline{X_i} \overset{def}{=} X_i - \tilde{\mathbb{E}} X_i$

$\overline{X_j} \overset{def}{=} X_j - {}^{\mu}\tilde{\mathbb{E}} X_j$

$$\Rightarrow \mathbb{E}_{\{i,j\}\in E} \left|\tilde{\mathbb{E}}_{\mu} \bar{x}_i \cdot \bar{x}_j\right| \geq \delta$$

Jensen's: $$\mathbb{E}_{\{i,j\}\in E} \left(\tilde{\mathbb{E}}_{\mu} \bar{x}_i \cdot \bar{x}_j\right)^2 \geq \delta^2 \quad *$$

$$\geq \left(\mathbb{E}_{\{i,j\}\in E} \left|\tilde{\mathbb{E}}_{\mu} \bar{x}_i \bar{x}_j\right|\right)^2$$

Let $M(i,j) = \left(\tilde{\mathbb{E}}_{\mu} \cdot \bar{x}_i \cdot \bar{x}_j\right)^2$.

Then $M \succeq 0$.

Proof of $M \succeq 0$: $\tilde{\mathbb{E}}_{\mu} \cdot \bar{X} \cdot \bar{X}^T$

$$M = \widetilde{Cov} \circ \widetilde{Cov}$$

Hadamard product

$$*: \mathbb{E}_{\{i,j\}\in E} M(i,j) \geq \delta^2$$

Apply technical lemma

$$\Rightarrow \mathbb{E}_{1\leq i,j\leq n} M(i,j)^2 \geq \left(\frac{\delta^2 - \rho}{\text{rank}_\rho(G)}\right)^2$$

Note: $M(i,j)^2 \leq |M(i,j)| \quad \forall i,j$

Thus: $$\mathbb{E}_{1\leq i,j\leq n} M(i,j) \geq \left(\frac{\delta^2 - \rho}{\text{rank}_\rho(G)}\right)^2$$

$$\Rightarrow \mathbb{E}_{i,j} \mathrm{Cov}(X_i, X_j)^2 \gtrsim \left(\frac{\delta^2 - \rho}{\mathrm{rank}_\rho(G)}\right)^2 \text{ --**}$$

Info theory fact:

$$\mathrm{Cov}(X_i; X_j)^2 \leq O(\mathbb{I}(X_i; Y))$$

$$** \Rightarrow \mathbb{E}_{i,j} \mathbb{I}(X_i; X_j) \gtrsim \Omega\left(\frac{\delta^2 - \rho}{\mathrm{rank}_\rho(G)}\right)^2$$

$$\Downarrow$$

$$G.C.(\mu) = \Omega\left(\frac{\delta^2 - \rho}{\mathrm{rank}_\rho(G)}\right)^2$$

Choose $? = \delta^2/4$ in our application

Proof of the techical lemma

$$\mathbb{E}_{\{i,j\} \in E} M(i,j) = \beta, \quad \mathbb{E}_{1 \leq i,j \leq n} M(i,j)^2 \gtrsim \left(\frac{\beta - \rho}{\mathrm{rank}_\rho(G)}\right)^2$$

$$\beta = \mathbb{E}_{\{i,j\} \in E} M(i,j)$$

Facts:

1. $A = \sum \lambda_i v_i v_i^T$

$$= \frac{1}{n} \sum_i \underbrace{A_{ij}} \cdot M(i,j)$$

v_i are orthonormal

$\|v_i\|_2 = 1$

$\sum_i v_i^T M v_i = tr(M)$

$$= \frac{1}{n} \sum_i \lambda_i \cdot v_i^T M \cdot v_i$$

$$\leq \frac{1}{n} \sum_{i : \lambda_i \geq \rho} \lambda_i \cdot \underbrace{v_i^T M v_i} + \underbrace{\rho \cdot \frac{1}{n} \sum_{i=1}^{n} \underbrace{v_i^T M v_i}}$$

(M is psd

$v_i^T M v_i \geq 0$

$$\leq \frac{1}{n} \sum_{i : \lambda_i \geq \rho} \lambda_i \cdot v_i^T M v_i + \rho \cdot 1$$

<u>Rearranging:</u> $\beta - \rho \leq \frac{1}{n} \sum_{i : \lambda_i \geq \rho} \lambda_i \cdot v_i^T M v_i$

at most $r = \mathrm{rank}_\rho(G)$ terms

<u>$\exists\, v_i :$</u> $\quad \lambda_i \cdot v_i^T M v_i \geq \left(\frac{\beta - \rho}{r} \right) \cdot n$

$$\boxed{r \stackrel{def}{=} \mathrm{rank}_\rho(G)}$$

$\Downarrow$

$$v_i^T M v_i \geq \left(\frac{\beta-\rho}{\gamma}\right)\cdot n.$$

$$\max_{v:\|v\|_2=1} v^T M v \overset{def}{=} \|M\|_2 \geq \left(\frac{\beta-\rho}{\gamma}\right)\cdot n$$

$$\sum M(i,j)^2 = \sum_i \lambda_i^2(M) \overset{def}{=} \|M\|_F \geq \left(\frac{\beta-\rho}{\gamma}\right) n$$

$$\Rightarrow \sum_{i,j} M(i,j)^2 \geq \left(\frac{\beta-\rho}{\gamma}\right)^2 \cdot n^2$$

$$\Rightarrow \frac{1}{n^2}\cdot\sum_{i,j} M(i,j)^2 \geq \left(\frac{\beta-\rho}{\gamma}\right)^2$$

Have we designed an algo for Max-Cut on low threshold rank graphs?

<u>Roadmap</u>

Start from μ. If $LC_G(\mu) < \delta$, done.

If not $GC(\mu) \geq \frac{\delta^4}{16\,\mathrm{rank}_{\frac{\delta}{2}}(G)^2}$

$\downarrow$

$GC(\mu') < \eta$. for $\eta = \frac{\delta^4}{16\,\mathrm{rank}_{\frac{\delta}{2}}(G)^2}$

$\downarrow$ $\implies LC_G(\mu) \leq \delta.$

$L.C._G(\mu') < \delta$

$\downarrow$

Independently Round

Need:
$\tilde{\mathbb{E}}_{\mu'} f_G \gtrsim \tilde{\mathbb{E}}_{\mu} f_G - \delta$

1. manage by adding constraint $\{f_G = opt\}$
2. or by averaging argument in GC lowering lemma

Diderot: ① Constrained p.d.

② Arora-Barak-Steurer algorithm

More Applications in HW3