Lecture 5

Unique Games Conj, Global Corr Rounding

2-CSP, Constraint Satisfaction Problems

$x_1, \dots, x_n \to$ take values in finite alphabet : $[q]$.

$\{(C_i, S_i)\}_{1 \le i \le m}$: m-constraints

$C_i : (x_{i_1}, x_{i_2}) \in S_i \subseteq [q] \times [q]$

is sat

Goal: find an assignment that satisfies as many constraints as possible

Ex: 1) Max 2-SAT : $x_1 \dots x_n$

each assignment $\{0,1\}$ for x_i

$x_i \vee x_j$ is satisfied if $\{(0,1), (1,0), (1,1)\}$

2) Max-Cut : $x_1 \dots x_n$

for each edge $\{i,j\} \in G$,

"$x_i \neq x_j$": $(x_i, x_j) \in \{(0,1), (1,0)\}$

$\in \{R, B, G\}$

3) Max 3-Coloring

given a graph G find a coloring of vertices with R, B, G s.t. for as many edges as possible, end points are colored differently.

$x_1 \ldots \ldots x_n$

$\{i,j\} \in G$:

$(x_i, x_j) \in \{ (R,B), (B,R), (R,G), (G,R), (G,B), (B,G) \}$

Promise Problem: $0 \le s \le c \le 1$

(c,s) – CSP is the problem where the input is a CSP instance & the goal is to decide if

1. $\exists$ an assignment that satisfies a $\geq c$-fraction of constraints
2. $\forall$ assignments, the frac of constraints sat is at most s.

$m \equiv$ # constraints

Observation 1: $(1, 1 - 1/m)$ – CSP

$\equiv$ checking of satisfiability

Observation 2: for $q \geq 3$,

[illegible] q CSP is NP-Hard.

$(1, 1-1/m) - 2CSP$ ...

Unique 2-CSP (Unique Game) : (C,S)

A q-size alphabet, 2-variable constraint is said to be "unique" if for every possible assignment to one of variables $\exists$ exactly one assignment to the other variable that is satisfying.

Equivalent to saying that every constraint $S = \{(i, \pi(i)) \mid i \in [q]\}$

$\equiv$ permutation $\pi : [q] \to [q]$

A 2-CSP is said to be a U.G. if all its constraints are unique.

Ex. Max-Cut is a U.G. (boolean U.G.).

Max 2SAT is NOT a U.G.

Max-2LIN $(\mathbb{F}_p)$: $x_i - x_j = b_{i,j} \bmod p$

"2 variable lin equation"

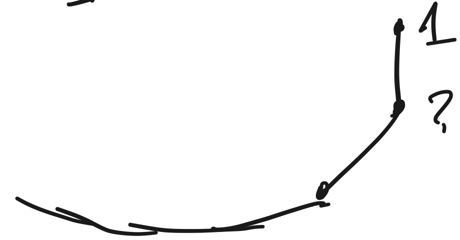
3-Coloring: Not a unique game

Observation: $\forall q \geq 0$ There's poly time algo for $(1, 1-1/m)$-U-G.

Proof: G : graph of pairs that appear in constraints.

Suppose, G is connected (WLOG)

Start from arbitrary vertex 1 $\leftarrow$ color 1



"**Propagation**"

Khot'2002. (Unique Games Conjecture)

For $\varepsilon > 0$, there's a large enough $q = q(\varepsilon)$ s.t. $(1-\varepsilon, \varepsilon)$-UG is NP-Hard.

(c, s)-CSP

c $\rightarrow$ completeness

s $\leftarrow$ soundness

For $c = 1$ cannot be NP-Hard

Remarks: ① random assignment satisfies $1/q$ fraction of constraints!

$\varepsilon > 1/q$ or $q \gg 1/\varepsilon$

→ PCP Theorem

$\exists$ constant $c > 0$ s.t.

$(1, 1-c)$-3-SAT is NP Hard.

→ Parallel Repetition Thm (Raz'95)

→ Håstad '99

$\forall \varepsilon > 0$, $(1, 7/8+\varepsilon)$- 3SAT is NP Hard.

→ Max 3LIN $x_i + x_j + x_k = b_{i,j,k} \bmod 2$

$(1, 1/2+\varepsilon)$-3LIN is NP-Hard

Max-Cut

algo for $(1-\varepsilon, 1-O(\sqrt{\varepsilon}))$ -Max-Cut.

$(1-\rho_{GW}, \alpha_{GW})$ - Max-Cut

α_{GW}: 0.878 $(1-\rho_{GW})$

Max 2LIN

→ $(1-\varepsilon, 1-O(\sqrt{\varepsilon}))$ -Max 2LIN

Vertex Cover

G

2-approx. algo

Khot'02: UGC ⇒ $\forall \varepsilon$, $(1-\varepsilon, 1-\varepsilon^t)$ - 2-LIN is NP-Hard

$\forall$ $t > 1/2$

Khot-Regev'03: $\forall \varepsilon > 0$. UGC ⇒ $(2-\varepsilon)$-VC is NP-Hard

Khot-Kindler-Mossell-O'Donnell '04:

UGC $\overset{\forall \varepsilon > 0}{\Rightarrow}$ $(1-\rho_{GW}, \alpha_{GW}+\varepsilon)$-Cut is NP-Hard.

Surprising connections between gaussian rounding of SDPs and 2-CSPs.

UGC $\Rightarrow$

O'Donnell-Wu' 06: beating the guarantee of SDP + "simple rounding" is NP-Hard.

Raghavendra' 08: $\forall$ CSP $\exists$ a natural SDP (deg 2 SoS) + natural rounding (analog of RPR²) that is optimal under UGC.

Is the UGC true?

CSP

If there's a $> \frac{s}{c}$ approx. algo for then (c, s)-CSP is easy.

<u>Khot'02:</u> <u>trivial:</u> $\frac{1}{q}$ frac approx. $\underline{\forall \varepsilon}$

$$(1-\varepsilon,\ \underline{1-O(q^2 \varepsilon^{1/5}\sqrt{\log 1/\varepsilon})})\text{-UG}$$

is solvable in poly time.

"for ε small enough $\gg 1/q$ approx."

all this is saying: $q \overset{\sim}{\gg} 1/q^{5/2}$

<u>Charikar-Makarychev-Makarychev '07</u>

$(1-\varepsilon, 1-O(\sqrt{\varepsilon \cdot \log q}))$-UG has a poly time algo.

<u>KKMO</u> :

UGC $\Rightarrow \forall \varepsilon > 0$

$$\Big(1-\varepsilon, 1-\underbrace{\sqrt{\tfrac{2}{\pi}}\sqrt{\varepsilon \cdot \log q}}+\varepsilon\Big) \text{ is NP-Hard}$$

enough to improve CMM algo to <u>beat</u> <u>UGC</u>

... the best polynomial time

These were results about polynomial time algorithms using degree 2 SoS.

2010: Arora-Barak-Steurer "Sub-exponential" algo

$\forall q, \varepsilon_1$: $\exists$ a $2^{q^2 n^{O(\varepsilon^{1/3})}}$ time algorithm to find a $\frac{1}{2}$ SAT assignment for any $1-\varepsilon$ satisfiable instance of UG.

$\underbrace{(1-\varepsilon_1, \tfrac{1}{2})\text{-UG}}_{\text{algo.}}$ has a $2^{q^2 \cdot n^{O(\varepsilon^{1/3})}}$ time

$2^{o(n)}$ $q^2 \cdot n^{O(\varepsilon^{1/3})} \to n^{o(1)}$ $\ll q^n$

Exponential Time Hypothesis (IKW'01)

3-SAT doesn't have a $2^{o(n)}$ algo.

$\Downarrow$

3-Coloring doesn't have a $2^{o(n)}$ algo.

If UG has a $2^{n^{o(1)}}$ algo then ETH $\Rightarrow$ UGC is false.

UGC is true n^{10}

$\Rightarrow$ 3SAT $\xrightarrow{\text{poly}(n)}$ U-G. $\quad 2^{n^{o(1)}}$ time algo.

$n \qquad n^{\varepsilon^{-1/3}}$

$2^{n^{o(1)}}$ algo for 3SAT $\to$ Contradicts ETH.

Theorem (2018): Dinur-Khot-Kindler-Minzer-Safra

$\forall \varepsilon$: $(\frac{1}{2}, \varepsilon)$-UG is NP-Hard. } a "halfway" there?

$(1-\varepsilon, \varepsilon)$-UGC is NP-Hard (UGC)

$\hookrightarrow \nexists\ 2^{n^{o(1)}}$ algo for $(\frac{1}{2}, \varepsilon)$-UG (if ETH is true)

Next Time:

Global Correlation Rounding

$\hookrightarrow$ Arora-Barak-Steurer

than

HW: Max-Bisection

$$\frac{1}{n^2} \sum_{i,j} |\mathbb{E}\, x_i x_j| \leq \delta$$

$\Rightarrow$ $0.878 - O(\sqrt{\delta})$ approx.

(we'll make this "assumption free" next time)

Rag'08: Integrality gap of deg 2 SoS relaxation is achieved by ~~Rn~~ analog of RPR^2 rounding.

G: $opt(G)$, f_G:

$\exists$ a μ p.d. of deg 2, $\tilde{\mathbb{E}}_\mu f_G \approx$ SDP-OPT

$$\max_G \left\{ \frac{\text{SDP-OPT}}{opt(G)} \right\} = \text{integrality gap}$$

"RPR^2" roundings achieve integ gap $\forall$ graphs.

UGC $\Rightarrow$ Cannot beat integrality gap of deg 2 SoS.

Rag'08: G: integrality gap α for deg 2 SoS for Max-Cut

then you can $UG \longrightarrow$ Max-Cut that proves a α factor hardness.

10 vertex graph that exhibits Integrality gap

$\downarrow$ gadget reduction where the "gap" graph is a gadget

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Fix a graph G: $\frac{SDP(G)}{OPT(G)}$