Lecture 4
Approximating Graph Expansion

$G:(V,E)$, $n=|V|$, d-regular graphs

$$f_G(x) = \frac{1}{4}\cdot\sum_{\{i,j\}\in E} (x_i-x_j)^2 \quad , x\in\{\pm1\}^n$$

for any $S\subseteq V$, $E(S,\bar{S}) = \{\{i,j\} \mid i\in S, j\in\bar{S}\}$

$$f_G(\mathbb{1}_S) = |E(S,\bar{S})|$$

$\min\limits_{x\in\{\pm1\}^n} f_G(x) \equiv$ min-cut ✓

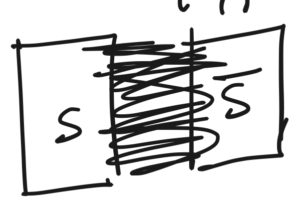
$\max\limits_{x\in\{\pm1\}^n} f_G(x) \equiv$ max-cut (0.878)

min-normalized-cut expansion $\equiv$ Uniform Sparsest Cut

$\boxed{\text{Expansion}}$ of set S

Def: $$\varphi_G(S) = \frac{|E(S,\bar{S})|}{\frac{d}{n}\cdot|S|\cdot\underbrace{(n-|S|)}_{|\bar{S}|}}$$

$G(n,\frac{d}{n})$.



$\frac{d}{n}|S|\cdot(n-|S|)$.

Expansion of G

Def: $\min\limits_{S\subseteq V} \varphi_G(S) := \varphi_G$

$S \subseteq V$
$0 < |S| < n$

→ expansion of the graph.

* random walk on G from a random vertex in S. $\equiv$ $\underline{\varphi_G(S)}$.

* application to Partitioning graphs, divide & conquer algos

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* **Goal**: given G, compute φ_G. (NP-Hard)

[Chawla-Krauthgamer-Kumar, Rabani-Sivakumar]
UGC $\Rightarrow$ $\nexists$ no constant factor approx. for this problem.

What does happen to a random cut?

Exercise: $\mathbb{E}_S \, \varphi_G(S) \geq 1/2$

Approximating Expansion via deg 2 SoS : Cheeger's Inequality

Cheeger '70
Dodziuk '84
Alon-Millman '85
Alon '86

$\forall$ d reg G, $|V| = n$

ASIDE: Formulation

$|E(S, \bar{S})|$

$$\varphi_G = \min_{\substack{S \subseteq V \\ 0<|S|<1}} \frac{|E(S,\bar{S})|}{\frac{d}{n}\cdot \underbrace{|S|(n-|S|)}}$$

ASIDE

$$= \min_{x \in \{\pm 1\}^n} \frac{f_G(x)}{\frac{d}{n}\left(\sum_{i,j}(x_i-x_j)^2\right)^{1/4}} \checkmark$$

$\rightarrow$ rational function $\checkmark$

$$\min_x \frac{P(x)}{Q(x)} = \alpha \equiv \underbrace{P(x) - \alpha \cdot Q(x) \geq 0}$$

Theorem (Cheeger's Inequality) — Proof in HW3

$\forall$ d reg G, $|V| = n$

$$f_G(x) - C \cdot \frac{d}{n} \cdot \sum_{i,j}(x_i - x_j)^2$$

has a deg 2 SoS certificate for $C = \frac{1}{2}\cdot\varphi^2(G)$

Further, given any p.d. μ of deg ≥ 2, s.t.

$$\underbrace{\tilde{\mathbb{E}}_\mu \cdot f_G(x)} - C\cdot \underbrace{\tilde{\mathbb{E}}_\mu \frac{d}{n}\cdot\sum_i (x_i-x_j)^2} \leq 0$$

Can find a set S with expansion $\varphi_G(S) = O(\sqrt{C})$

ASIDE:

Algo: Find a p.d. μ. s.t. $\min C$ $\quad \tilde{\mathbb{E}}_\mu \cdot f_G(x) - C \cdot \tilde{\mathbb{E}} \frac{d}{n} \sum_i (x_i - x_j)^2$

$\mathbb{E}_{\mu}\, n_{i,j}$

≤ 0

when $\varphi(G)$ is large $\rightarrow$ expander graphs.

when $\underline{\varphi(G) = o(1)}$ $\rightarrow$ <u>weak</u>. TIGHT (HW 3!)

Improving on Cheeger via higher deg SoS

<u>88</u>: Leighton-Rao (Using linear programming)

<u>Given</u> G, Can find in poly time a set

S: $\varphi_G(S) = O(\log(n)) \cdot \varphi(G)$.

improves on Cheeger when $\varphi(G) \ll 1/\log(n)$.

<u>2004</u> (Arora-Rao-Vazirani)

Can find a set S: $\varphi_G(S) = O(\sqrt{\log(n)}) \cdot \varphi_G$

Meta observations

1. degree 4 SoS helps.

2. So far: local analysis.

<u>ARV</u>: "global analysis" truly helps.

3. Leads to better algos for approx coloring Vertex Cover, Coloring Graphs, Max-Cut Gain ...

ALSO ON HW3

Theorem (ARV) G d reg, $|V|=n$

there's a deg 4 SoS cert. for the polynomial

$$f_G(x) - \frac{\varphi_G(x)}{O(\sqrt{\log(n)})} \frac{d}{n} \cdot \frac{1}{4} \sum_{i,j} (x_i - x_j)^2$$

Further, given any deg 4 p.d. μ on $\{\pm 1\}^n$, can find a $S \subseteq V$ in poly time, $\varphi_G(S) \leq O(\sqrt{\log(n)}) \cdot \frac{\tilde{\mathbb{E}}_\mu f_G}{\underbrace{\tilde{\mathbb{E}}_\mu \frac{d}{n} \cdot \sum_{i,j} (x_i - x_j)^2}}$

Proof: $\boxed{\text{Pseudo-prob as distances}}$

$0 \leq \frac{1}{4} \tilde{\mathbb{E}}_\mu (x_i - x_j)^2 \equiv$ "pseudo" prob that i & j are separated.

WHY?

$\leq \underbrace{1}$ define $D(i,j)$

WHY?

HW2: $\frac{1}{4} \tilde{\mathbb{E}}_\mu (x_i - x_k)^2 \leq \frac{1}{4} \cdot \tilde{\mathbb{E}}_\mu (x_i - x_j)^2 + \frac{1}{4} \cdot \tilde{\mathbb{E}}_\mu (x_j - x_k)^2$

"SQUARED TRIANGLE INEQUALITY"

$|||$

DOESN'T HOLD $\forall$ GAUSSIANS

$\forall i,j,k$: $D(i,k) \leq D(i,j) + D(j,k)$

HOLDS $\forall$

CAUTION: Usual triangle inequality

$$\sqrt{\tilde{\mathbb{E}}_\mu (x_i - x_j)^2} \le \sqrt{\tilde{\mathbb{E}}_\mu (x_i - x_j)^2} + \sqrt{\tilde{\mathbb{E}}_\mu (x_j - x_k)^2}$$

GAUSSIAN

true if deg 2 p.d. ✓

to prove it do gaussian rounding & work with actual $dist^n$

→ we have a metric on n vertices D

SIMPLE GOAL: (STEP 1): Solving Min-Cut

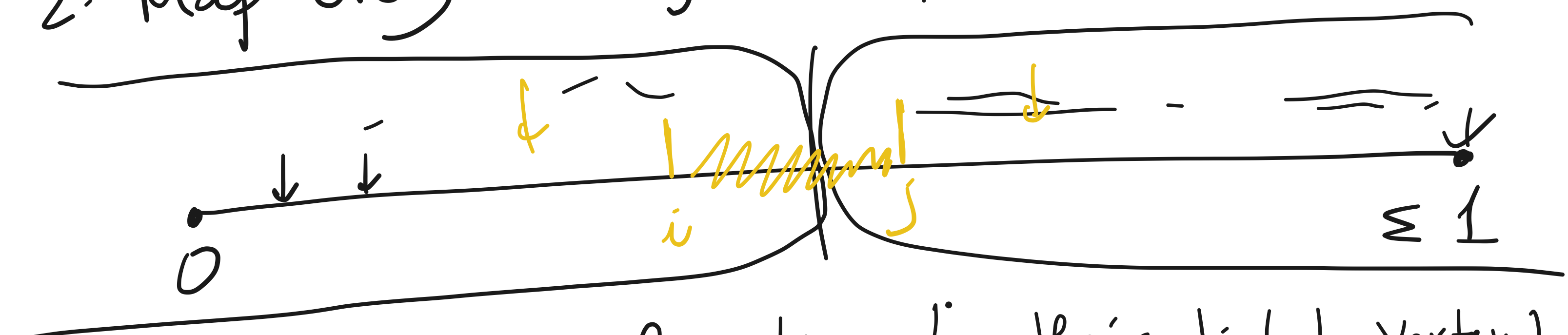
Let's first round to a $dist^n$ on cuts so that $\mathbb{E}\, f_G(x') \le \tilde{\mathbb{E}}_\mu f_G(x)$

rounded dist | pseudo dist

Sledgehammer to crack a nut

Rounding:

1. Label the vertices 1, 2, - - -, n.
2. Map every vertex j to the point D(1, j)



1D Embedding of vertices by their dist to vertex 1.

3. Choose $t \in [0, D(1,n)]$
4. Output $S = \{ i \mid D(1,i) \le t \}$

1D What's the chance that any edge: {i, j} is cut?

What's the ...

$$|D(1,j) - D(1,i)| \leq D(i,j).$$

$$\equiv \quad |D(1,j) - D(1,i)|$$

$$D(1,j) - D(1,i) \leq D(i,j)$$

We have shown (using linearity of expectation)

$$\mathbb{E}_S f_G(S) \leq \tilde{\mathbb{E}}_\mu f_G(x)$$

$\square$

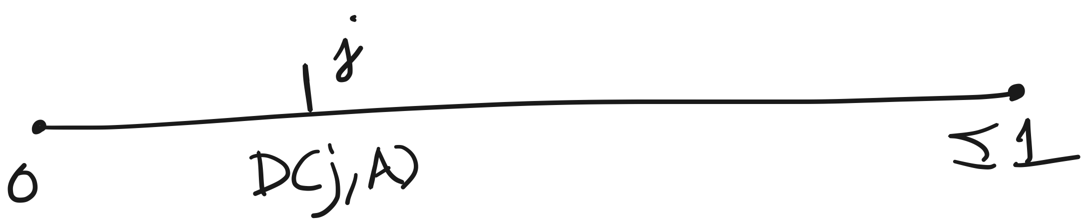
CAUTION: Why is this cut non-trivial? HW 3

Generalizing procedure

Let $A \subseteq V$. Use dist from A as 1D embedding.

Def: $D(i,A) = \min\limits_{j \in A} D(i,j)$

Claim: the same analysis works if you start from the "1D embedding". $D(j,A)$.



Proof: Prob edge $\{i,j\}$ is cut?

$$|D(i,A) - D(j,A)| \leq D(i,j)$$

triangle inequality

CAUTION: Why does this rounding give a non-trivial cut?

SIMPLIFIED GOAL 2: $O(\log n)$ approx. for expansion

Find a dist^n $\longrightarrow$ $x' \in \{\pm 1\}^n$

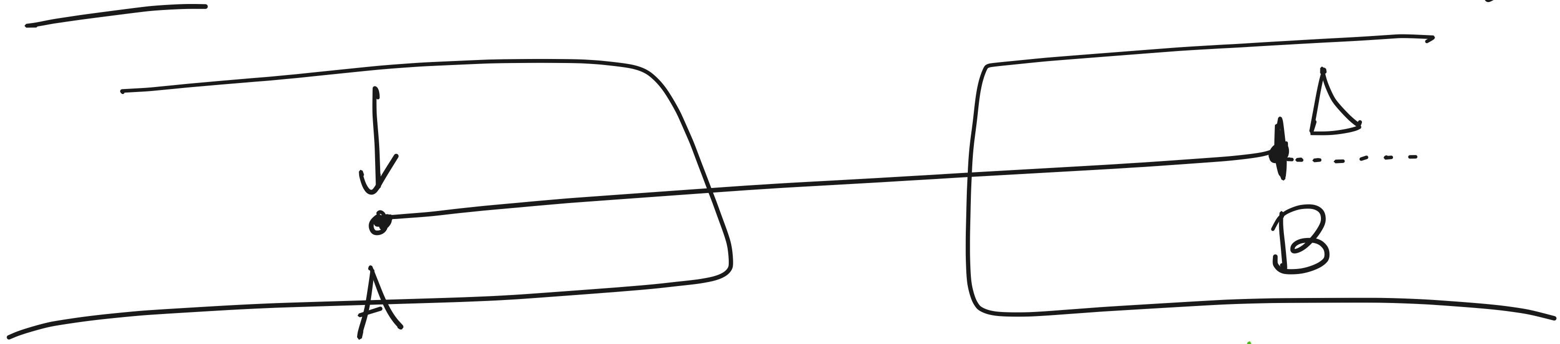
So that
$$\frac{\mathbb{E}\, f_G(x')}{\frac{1}{4}\sum_{i,j}\frac{d}{n}\mathbb{E}(x'_i-x'_j)^2} \le \left(\frac{1}{\Delta}\right)\frac{\tilde{\mathbb{E}}_\mu f_G(x)}{\frac{1}{4}\sum_{i,j}\frac{d}{n}\tilde{\mathbb{E}}_\mu (x_i-x_j)^2}$$

Need:
$$\frac{d}{n}\frac{1}{4}\mathbb{E}\sum_{i,j}(x'_i-x'_j)^2 \geq \frac{1}{\Delta}\frac{d}{n}\cdot\frac{1}{4}\tilde{\mathbb{E}}_\mu\sum_{i,j}(x_i-x_j)^2$$

need a "balanced" cut.

NOTHING TO DO WITH THE INPUT GRAPH ANYMORE.

Idea: look at the min-cut rounding a little closely.



how can we guarantee both sides contain $\Omega(n)$ vertices?

Need $\forall j \in B$, $d(j,A)$ is larger than Δ.

Def (Large separated sets)

A, B are large, Δ-separated if

$\forall\, i \in A,\ j \in B \quad D(i,j) \geq \Delta$

(1) $\forall \quad i \in A, \; j \in B,$

& (2) $|A| \cdot |B| = \Omega(n^2)$.

Prop: If we round according to the line embedding $D(j, A)$; & A, B are large, Δ separated, then

$$\sum_{i,j} \mathbb{E}\,(x_i' - x_j')^2 \geq \sum_{\substack{i \in A \\ j \in B}} D(j, A) \geq \Delta |A||B| = \Omega(\Delta \cdot n^2)$$

$$\geq \Omega(\Delta) \sum_{i,j} \tilde{\mathbb{E}}_\mu (x_i - x_j)^2$$

Note: $\sum_{i,j} \tilde{\mathbb{E}}_\mu (x_i - x_j)^2 \leq n^2$. Thus

Using 1D embedding + random threshold rounding with A gives cut size bound & "balancedness"

$\Rightarrow$ rounding with approx ratio $\dfrac{1}{\Theta(\Delta)}$.

Theorem ["Global Structure Theorem"]

G : d-reg graph, μ : deg 4. p.d.

$D(i, j) = \frac{1}{4} \cdot \tilde{\mathbb{E}}_\mu (x_i - x_j)^2$ (D sat Δ inequalities)

Suppose $\frac{1}{n^2}\sum_{i,j} D(i,j) \geq 0.1 \rightarrow$ $\boxed{\text{EXTRA HYP}}$

Then, $\exists$ A, B (can be found in poly time)

s.t. $|A|\cdot|B| \geq \Omega(n^2)$.

& $\forall i\in A, j \in B$, $D(i,j) \geq \underbrace{\Omega\left(\frac{1}{\sqrt{\log(n)}}\right)}_{\Delta}$

$$* \quad \frac{1}{n^2}\sum_{i,j} \tilde{\mathbb{E}}_\mu \underbrace{(x_i - x_j)^2}_{+|\mathbb{1}(x)|(n-\mathbb{1}(x))|} =$$

<u>Extra hypothesis</u>: pseudo-dist^n is on "large" sets

Can reduce to the case where hyp holds WLOG
↪ HW!

Local Analysis gives struct thm with $\Delta = \frac{1}{\Theta(\log(n))}$.

<u>Note</u>: $\frac{1}{n^2}\sum_{i,j} D(i,j) \geq 0.1$

<u>Our goal</u>: Δ - separated large sets.

<u>WLOG</u>, $\tilde{\mathbb{E}}_\mu x = 0$

$D(i,j) = \tilde{\mathbb{E}}_\mu (x_i - x_j)^2$

$\tilde{}$

Let $g \sim N(0, \tilde{\mathbb{E}}_\mu xx')$ $\quad \mathbb{E}(g_i - g_j)^2$

$\forall i$: $\mathbb{E} g_i^2 = 1$, $\mathbb{E} g_i = 0$

Let $A^0 = \{ i \mid g_i \leq -1 \}$, $\Pr[g_i \leq -1] \sim 0.15$

$B^0 = \{ j \mid g_j \geq 1 \}$

Claim: $\mathbb{E} |A^0| \cdot |B^0| \geq \Omega(n^2)$. WHY?

Pf: $\Pr[g_i \leq -1 \;\&\; g_j \geq 1] \geq C \cdot D(i,j)$

$\Downarrow$

$\mathbb{E} |A^0| \cdot |B^0| \geq 0.1 c \cdot n^2$

Is (A^0, B^0) Δ-separated? $\equiv$ is there $i \in A, j \in B$ such that $\mathbb{E}(g_i - g_j)^2 \leq \Delta$? Note: $g_j - g_i \geq 2$.

$$D(i,j) = \tilde{\mathbb{E}}_\mu (x_i - x_j)^2 = \mathbb{E}(g_i - g_j)^2$$

What's the chance that $\exists$ $i \in A$, $j \in B$

s.t. $\mathbb{E}(g_i - g_j)^2 \leq \Delta$

but $g_j - g_i \geq 2$?

GAUSSIAN CONC

$O(1)$

$Pr[\, g_j - g_i \geq 2] \leq e^{-\underbrace{\ }_{}\Delta}$

Union Bound $\Rightarrow$

A^0 & B^0 are Δ separated w.p. $\geq 1 - n^2 \cdot e^{-\frac{O(1)}{\Delta}}$

Thus, if we choose $\Delta \ll \frac{1}{\log(n)}$, then get Δ-separated large sets!

$\rightarrow$ recovered Leighton-Rao approx. for expansion

Next: Improve on Δ by global analysis

Notation: H "short edge graph"

$$E(H) = \{ (i,j) \in [n^2] \mid D(i,j) \leq \Delta \}$$

Goal: is to find $\Omega(n)$ size sets A & B that is a "Vertex Separator": $\nexists\, i \in A, j \in B$ s.t. $(i,j) \in E(H)$

Algorithm:
1. take A^0 & B^0 as before
2. find a maximal matching M greedily in $\underline{E(H) \cap A^0 \times B^0}$
3. Output $A = A^0 \setminus V(M)$

output is clearly

Δ-sep

Is it large?

$$B = B^0 \,|\, V(M)$$

<u>Note:</u> 1) M is a random variable

2) M has directed edges

for any $(i,j) \in M$ then $g_j - g_i \geq 2$

3) <u>Prop:</u> $M(g) = \text{reverse}(M(-g))$

$\Leftrightarrow$ $(i,j) \in M(g)$

$\Leftrightarrow$ $(j,i) \in M(-g)$.

$\Rightarrow$ $\Pr[\text{vertex } i \text{ has an incoming edge in } M]$

$= \Pr[\text{vertex } i \text{ has outgoing edge in } M]$ $\rightarrow$ using symmetry of centered gaussian

4) EASY PROP: If $|A| \cdot |B| = o(n^2)$

then $\mathbb{E}|M| \geq \Omega(n)$.

5) $\forall (i,j) \in M$, $\mathbb{E}(g_j - g_i)^2 \leq \Delta$ "RARE EVENT"

$g_j - g_i \geq 2$

The main lemma bounds the size of the matching M by relating it to max of a certain jointly gaussian vars.

Key $\Omega(1)\left(\frac{\mathbb{E}|M|}{n}\right)^3 \leq \mathbb{E} \max_{i \in S_1 ?} g_j - g_i \quad \Big)$

Lemma 1: $\frac{\cdots}{\Delta} \cdot \left(\frac{\cdots}{n}\right) \le \mathbb{E}_{i,j \in [n]} \cdots$

nontrivial part

$$\le \tilde{O}\left(\sqrt{\log(n)}\right).$$

relies on strong conc^n of Gaussian processes

Fact: $(Z_1, \ldots, Z_t)$ be jointly gaussian.

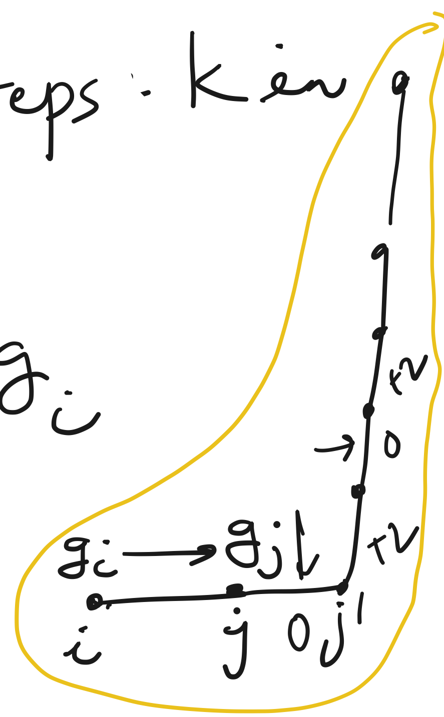
$$\mathrm{Var}\left(\max_{j \le t} Z_j\right) \le O(1) \max\{V(Z_i)\}_{i \le t}.$$

(in fact the r.v. $\max_{j \le t} Z_j$ is concentrated like gaussian of above variance)

Idea of proof of main lemma

"chain" together matching edges.

def: $H^k(i)$ = vertices within steps k in H from i



def: $$\gamma_i^k = \max_{j \in H^k(i)} g_j - g_i$$

Note: $(i,j) \in M \Rightarrow g_j - g_i \geq 2$

Def: $\Phi(k) = \sum_{i=1}^n \mathbb{E}\gamma_i^k$ ("Potential")

Notice: $\dfrac{\Phi(k)}{n} \leq \mathbb{E} \max_{i,j\in[n]} g_j - g_i$

$\Leftrightarrow$ enough to lower bound $\Phi(k)$.

Key Lemma 2: $\Phi(k+1) - \Phi(k)$

$$\geq \mathbb{E}|M| - O(n)\cdot \max_{\substack{i\in[n] \\ j\in H^{k+1}(i)}} \sqrt{\mathbb{E}(g_i-g_j)^2} \quad \text{- - - -} \quad (**)$$

Proof of Key Lemma 1 using Key Lemma 2

Note: $\forall j \in H^{k+1}(i)$: $\mathbb{E}(g_i-g_j)^2 \leq (k+1)\cdot\Delta$.

$\hookrightarrow$ used triangle ineq for D here.

Thus RHS of $(**)$ $\geq \mathbb{E}|M| - O(n)\cdot\sqrt{(k+1)\cdot\Delta}$

Let $k_0 = \left(\dfrac{\mathbb{E}|M|}{n}\right)^2 / \Delta \cdot c \sim O(1/\Delta)$.

Then $\forall\ k \leq k_0$.

$$\Phi(k+1) - \Phi(k) \geq \frac{1}{2}\cdot\mathbb{E}|M|.$$

Main Message

Can chain together a path with $\sim\sqrt{\log(n)}$ matching edges when $|M| = \Omega(n)$.

Thus: $\Phi(k_0) \geq \frac{1}{2}\cdot k_0 \cdot \mathbb{E}|M|$

Thus using that $\max_{i,j} g_j - g_i \geq \dfrac{\Phi(k_0)}{n}$

$$\geq \frac{1}{2} \cdot k_0 \cdot \frac{\mathbb{E}|M|}{n}$$

$$= \Theta(1) \cdot \left(\frac{\mathbb{E}|M|}{n}\right)^3 \cdot \frac{1}{\Delta}$$

Using that $\mathbb{E} \max_{i,j} g_j - g_i \leq O(\sqrt{\log n})$

gives that

$$\left(\frac{\mathbb{E}|M|}{n}\right)^3 \leq \Delta \; O(\sqrt{\log(n)})$$

or $$\left(\frac{\mathbb{E}|M|}{n}\right) \leq \left(\Delta \cdot O(\sqrt{\log(n)})\right)^{\frac{1}{3}}$$

Thus if we choose $\Delta \leq \frac{c}{\sqrt{\log(n)}}$ for c small enough

Then $\mathbb{E}|M| \leq c' n$. for some $c' = c'(c)$.

By choosing c small enough, we can force c' to be small enough.

So $\mathbb{E}|A| \cdot |B| \geq \mathbb{E}|A^0| \cdot |B^0| - n\,\mathbb{E}|M|$

$$= \Omega(n)$$

$\square$

In the next class, we will complete this argument by using the chaining idea to prove Key Lemma 2

$$\Phi(k+1) - \Phi(k) \gtrsim \mathbb{E}|M| - O(n) \cdot \max_{\substack{i \\ j \in H^{k+1}(i)}} \sqrt{\mathbb{E}(g_i - g_j)^2}$$

Sep 29

Recall

$G(V,E)$, d-regular graph

$$\Phi_G = \min_S \Phi_G(S) = \min_{x \in \{\pm 1\}^n} \frac{\frac{1}{4} \sum_{\{i,j\} \in E} (x_i - x_j)^2}{\frac{d}{n} \cdot \frac{1}{4} \cdot \sum_{i,j} (x_i - x_j)^2}$$

$$= \frac{|E(S,\bar{S})|}{\frac{d}{n} \cdot |S| \cdot (n - |S|)}$$

Goal: approximate Φ_G.

Theorem: (ARV Thm) Let μ be a degree 4 pseudo-dist^n over $\{\pm 1\}^n$.

Then, there's a poly time algo to find a set $S \subseteq V$ such that
$$\Phi_G(S) \lesssim O(\sqrt{\log(n)}) \cdot \frac{\tilde{\mathbb{E}}_\mu f_G(x)}{\frac{d}{n} \cdot \frac{1}{4} \sum_{i,j} \tilde{\mathbb{E}}_\mu (x_i - x_j)^2}$$

Proof: 1. For any set $A \subseteq V$,

... "random threshold" rounding on

Can do random threshold ... of
the 1-D embedding of vertices $i \to d(i, A)$.

$d(i,j) = \frac{1}{4}\tilde{\mathbb{E}}_\mu (x_i - x_j)^2$

2. The cut-size in expectation is at most $\tilde{\mathbb{E}}_\mu \cdot f_G(x)$

3. Need: $\sum_{i,j} \frac{1}{4} \mathbb{E}(x_i' - x_j')^2 \geq \Omega\left(\frac{1}{\sqrt{\log(n)}}\right) \frac{1}{4} \sum_{i,j} \tilde{\mathbb{E}}_\mu (x_i - x_j)^2$

enough to find a pair of (A, B) LARGE ($= \Omega(n)$ size) Δ-separated ($\forall\, i \in A, j \in B$ $d(i,j) = \frac{1}{4}\tilde{\mathbb{E}}_\mu (x_i - x_j)^2 \geq \Delta$) sets

"ARV STRUCTURE THEOREM".

Proof of structure theorem: "Global Analysis" of Gaussian Sampling.

Idea: Auxiliary graph $H(V, E(H))$ of "short edges"

Let $A^0 = \{ i \mid g_i \leq -1 \}$ $\quad B^0 = \{ j \mid g_j \geq 1 \}$

Remove a maximal matching of $E(H) \cap A^0 \times B^0$.

Resulting set is Δ-separated. Is it large?

If (A, B) is small, then $\exists$ a large matching in $E(H) \cap A^0 \times B^0$. $\to$ Matching greedy. $M(g) = rev(M(-g))$ each vertex equally likely to be on left or right.

$\to$ Show that we can "chain together" edges in ... is large.

matching so that the $\sup_{i,j} g_j - g_i$ is large

→ Use concentration of suprema of gaussian "process" to derive a contradiction.

Key Chaining Claim: $d(i,j) = \frac{1}{4} \tilde{\mathbb{E}}_\mu (x_i - x_j)^2$

$H^k(i)$: vertices reachable by a path of $\leq k$ edges in H.

$$Y_i^k : \max_{j \in H^k(i)} g_j - g_i$$

$$\Phi(k) = \sum_{i=1}^n \mathbb{E}\, Y_i^k.$$

Then $\frac{\Phi(k)}{n} \leq \mathbb{E} \max_{i,j\in[n]} g_j - g_i$ — (1) EASY

& $\Phi(k+1) - \Phi(k) \geq \mathbb{E}|M| - O(n) \cdot \max_{\substack{i\in[n] \\ j\in H^{k+1}(i)}} \left(\mathbb{E}(g_i - g_j)^2\right)^{1/2}$

Proof: $Y_i^{k+1} \geq Y_j^k + g_j - g_i \quad \forall (i,j) \in E(H)$

(true because $H^{k+1}(i) \supseteq H^k(j)$). $g_j - g_i$

≥ 2

$N \in [n] \times [n]$ arbitrary matching of vertices not in M.

Observe: $(i,j) \in M \Rightarrow g_j - g_i \geq 2$

$\Rightarrow \forall (i,j) \in M: \ r_i^{k+1} \geq r_j^{k} + 2$ (matching M edges)

$\forall (i,j) \in N: \ \frac{1}{2} r_i^{k+1} + \frac{1}{2} \cdot r_j^{k+1} \geq \frac{1}{2} \cdot r_i^{k} + \frac{1}{2} \cdot r_j^{k}$ (edges in matching N)

Summing up:

$$\sum_{i=1}^{n} r_i^{k+1} \cdot L_i \geq \sum_{j=1}^{n} r_j^{k} \cdot R_j + 2 \cdot |M|.$$

$$L_i = \begin{cases} 1 & \text{if } i \text{ has an outgoing edge in } M \\ 0 & \text{if } i \text{ has an incoming edge in } M \\ 1/2 & \text{if } i \text{ is unmatched in } M \end{cases}$$

$$R_i = \begin{cases} 1 & \text{if } \ldots \text{ incoming in } M \\ 0 & \text{if } \ldots \text{ outgoing in } M \\ \frac{1}{2} & \text{if unmatched in } M. \end{cases}$$

$M(g) = \mathrm{rev}(M(-g))$

$$\mathbb{E}\, L_i = \mathbb{E}\, R_j = \frac{1}{2} \quad \forall \ i,j \in [n]$$

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thought experiment

$\Phi(k+1) = \sum_i \mathbb{E}\, r_i^{k+1}$

$$\underbrace{\mathbb{E} \sum_i r_i^{k+1} \cdot L_i}_{\Phi(k+1)\,?} \geq \underbrace{\mathbb{E} \sum_i r_j^{k} \cdot R_j}_{\Phi(k)} + 2\,\mathbb{E}|M|.$$

almost done but γ_i^{k+1} & L_i are not indep

Claim: $\left|\mathbb{E}\gamma_i^{k+1} \cdot L_i - \mathbb{E}\gamma_i^{k+1} \cdot \mathbb{E}L_i\right|$

$$= \left|\mathbb{E}\left(\gamma_i^{k+1} - \mathbb{E}\gamma_i^{k+1}\right)\left(L_i - \mathbb{E}L_i\right)\right|$$

$$\leq \sqrt{\mathbb{E}\left(\gamma_i^{k+1} - \mathbb{E}\gamma_i^{k+1}\right)^2} \cdot \sqrt{\mathbb{E}\left(L_i - \mathbb{E}L_i\right)^2}$$

$\to \mathrm{Var}(\gamma_i^{k+1})$

$\to \mathrm{Var}(L_i) \leq 1$

Maxima of Gaussian Process

$$\mathrm{Var}(\max(z_1, \ldots, z_t)) \lesssim O(1) \cdot \max\{V(z_1), \ldots, V(z_t)\}$$

$$\gamma_i^{k+1} = \max_{j \in H^{k+1}(i)} g_j - g_i$$

$$\mathrm{Var}(\gamma_i^{k+1}) \lesssim \left(\max_{j \in H^{k+1}(i)} \mathrm{Var}(g_j - g_i)\right) O(1)$$

$$\mathrm{Var}(g_j - g_i) = \mathbb{E}(g_i - g_j)^2$$

So: $\left|\mathbb{E}\gamma_i^{k+1} \cdot L_i - \mathbb{E}\gamma_i^{k+1} \cdot \mathbb{E}L_i\right|$

$$\leq O(1) \cdot \sqrt{\max_{j \in H^{k+1}(i)} \mathbb{E}(g_j - g_i)^2}$$

Similar argument gives

$$\left| \mathbb{E}\, \gamma_j^k \cdot R_j - \mathbb{E}\, \gamma_j^k \cdot \mathbb{E} R_j \right|$$

$$\leq O(1) \cdot \sqrt{\max_{j \in H^k(i)} \mathbb{E}(g_j - g_i)^2}$$

Plugging in these bounds finishes the proof.

$$\Phi(k+1) - \Phi(k) \geq 4\, \mathbb{E}|M| - O(n) \cdot \max_{\substack{j \in H^{k+1}(i) \\ i \in [n]}} \sqrt{\mathbb{E}(g_j - g_i)^2}$$