Zero-Temperature Point of the Blackbody Chromaticity Locus

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BECAUSE of the extended use of visual color coordinates, in connection with devices other than the human eye, the chromaticity coordinates of objects too dim to be visually perceived has gained interest. In particular, the coordinates of blackbody radiators of temperatures below 1000°K was of concern to the authors recently. In the course of related work the zero-temperature point of the blackbody locus was determined and found to be the long-wavelength limit point of the spectrum locus, in agreement with intuition. The determination is given below, as being of possible interest to other workers in the field. CIE coordinates X, Y, and Z are used.

The procedure is not as simple as in the case of the infinite-temperature point. In the latter case, it is possible to use the Rayleigh–Jeans approximation to the Planck function, with consequent separation of temperature and wavelength variables. This approximation is the first term in the Taylor expansion of the complete function about zero exponent; a similar procedure is not possible for zero temperature (infinite exponent) because of the essential singularity of the exponential at that point.

However, noting that

$$\lim_{T\to 0}\left[\frac{1}{\exp(c_2/\lambda T)-1}\right]=\exp(-c_2/\lambda T),$$

we have

$$\lim_{T\to 0}(x)=\lim_{T\to 0}\left[\int_0^\infty \frac{\bar{x}(\lambda)}{\lambda^5}\exp(-c_2/\lambda T)d\lambda \Big/ \int_0^\infty \frac{\bar{s}(\lambda)}{\lambda^5}\exp(-c_2/\lambda T)d\lambda\right],$$

where the sum of $\bar{x}$, $\bar{y}$, and $\bar{z}$ has been called $\bar{s}$.

The form of the above integrals suggests a Laplace transform. Indeed, with the change of variables $t=1/\lambda$ and $c_2/T=s$,

$$\lim_{T\to 0}(x)=\lim_{\substack{s\to\infty\\ \mathrm{Im}(s)\to 0}}\left[\int_0^\infty t^3\bar{x}\left(\frac{1}{t}\right)e^{-st}dt \Big/ \int_0^\infty t^3\bar{s}\left(\frac{1}{t}\right)e^{-st}dt\right]$$

$$=\lim_{s\to\infty} L[t^3\bar{p}(t)]/L[t^3\bar{\varphi}(t)],$$

where $\left\{\begin{matrix}\bar{p}(t)=\bar{x}(1/t)\\ \text{and}\\ \bar{\varphi}(t)=\bar{s}(1/t)\end{matrix}\right\}$ and L is the Laplace transform operator.

According to the initial-value theorem of Laplace transform theory[1]

$$\lim_{\substack{s\to\infty\\ \mathrm{Im}(s)\to 0}} sL[f(t)]=f(0^+).$$

In the present case the ratio of the limits at $t=0$ is indeterminate because both functions are zero in a finite neighborhood of the origin. We therefore examine the equivalent expression

$$\lim_{s\to\infty}\left[\int_{t_0}^\infty t^3\bar{p}(t)e^{-st}dt \Big/ \int_{t_0}^\infty t^3\bar{\varphi}(t)e^{-st}dt\right],$$

where t_0 is the least upper bound of t for which $\bar{\varphi}$ vanishes everywhere to the left of t. (And $\lambda_0=1/t_0$ is then the corresponding wavelength, at which $\bar{s}$ goes to zero.) Again, changing variables,

$$\int_{t_0}^\infty t^3\bar{\varphi}(t)e^{-st}dt=\int_0^\infty (u+t_0)^3\bar{\varphi}(u+t_0)e^{-su}e^{-st_0}du,$$

$$\therefore \lim_{s\to\infty}(x)=\lim_{s\to\infty} e^{-st_0}\int_0^\infty (u+t_0)^3\bar{p}(u+t_0)e^{-su}du \Big/ e^{-st_0}\int_0^\infty (u+t_0)^3\bar{\varphi}(u+t_0)e^{-su}du$$

$$=\lim_{s\to\infty}\frac{sL[(u+t_0)^3\bar{p}(u+t_0)]}{sL[(u+t_0)^3\bar{\varphi}(u+t_0)]}=\frac{\bar{p}(t_0)}{\bar{\varphi}(t_0)}=\frac{\bar{x}(\lambda_0)}{\bar{s}(\lambda_0)}.$$

The result for y is identical in form. The wavelength λ_0 at which the last of the tristimulus curves goes to zero is at the red end of the spectrum locus; consequently it is this which is the zero-temperature point.

[1] M. F. Gardner and J. L. Barnes, *Transients in Linear Systems* (John Wiley & Sons, Inc., New York, 1958), Vol. I, p. 267.

Wavefront Reconstruction for Incoherent Objects

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ALTHOUGH it is possible to record the amplitude or a nonlinear function of the amplitude of a lightwave on photographic film, it has not been possible to record the phase of a lightwave. Gabor,[1] however, found an indirect way to record both the amplitude and the phase. Such a "total recording" or "hologram" enabled him to reconstruct the original wavefront after it had been temporarily frozen into a photographic plate.

Most optical objects are incoherently illuminated or self-luminous. These objects have been excluded from wavefront reconstruction because of the general misconception that in recording a hologram the original object must be illuminated coherently. In other words, it was believed to be necessary for light coming from any two object points to be mutually coherent.

Mertz and Young,[2] making use of Rogers's[3] way to explain holography, realized that coherence is not necessary. It is only essential that each object point somehow generate its own Fresnel zone-plate pattern on the hologram. Later, each Fresnel zone plate (FZP) acts as a lens in the reconstruction process and, hence, produces an "image." On this general basis, Mertz and Young proposed a new way of generating a hologram, which could be used as a camera for x-ray astronomy. The camera lens is replaced by an FZP mask. Each star produces its own FZP shadow on the film. This process of hologram generation relies on geometrical optics, which is often justified, for example, for x rays.

Here we want to propose some methods of hologram generation which apply for incoherent or self-luminous objects, and which are not limited to the realm of geometrical shadow casting. The basic idea is to split up the spherical wave coming from each object point into two portions. Later, these two portions are recombined to generate interference fringes. These two portions are mutually coherent because they are originated by the same object point. Hence, it does not matter whether two different object points are mutually coherent or not. If both portions are still spherical waves but with different curvatures, the interference pattern has an FZP shape. The center point of the FZP pattern is collinear with the centers of the two spherical waves.

In order to achieve interference fringes of good contrast, it is not enough to have spatial coherence between the two interfering portions as discussed before, the light has to be monochromatic also. Otherwise, different wavelengths would generate different concentric overlapping FZP patterns of different fineness. The net result would be an FZP pattern with good contrast only in the innermost rings. For objects which are not self-luminous, the problem of monochromaticity is a matter of choosing the proper light source. If that is not possible or if the object is self-luminous and not monochromatic, a spectral filter has to be inserted between object and hologram. The larger the aperture of the hologram, the finer the outermost fringes of the FZP patterns. Hence, for large