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A SEMIRING ON CONVEX POLYGONS AND ZERO-SUM CYCLE PROBLEMS*

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Abstract. Two natural operations on the set of convex polygons are shown to form a closed semiring; the two operations are vector summation and convex hull of the union. Various properties of these operations are investigated. Kleene's algorithm applied to this closed semiring solves the problem of determining whether a directed graph with two-dimensional labels has a zero-sum cycle or not. This algorithm is shown to run in polynomial time in the special cases of graphs with one-dimensional labels, BTTSP (Backedged Two-Terminal Series-Parallel) graphs, and graphs with bounded labels. The undirected zero-sum cycle problem and the zero-sum *simple* cycle problem are also investigated.

Key words. semiring, convex polygon, dynamic graph, algorithm, complexity

AMS(MOS) subject classifications. 05, 13, 16, 52, 68

1. Introduction. In this paper, we show that two natural operations on the set of convex polygons form a closed semiring; the two operations are vector summation and convex hull of the union. We then investigate the time complexity of each operation and its effect on the number of edges of the polygons.

Kleene's algorithm applied to various closed semirings leads to efficient algorithms for a variety of problems; for example, finding the shortest paths for all pairs of nodes [3], converting a finite automaton into a regular expression, and finding the most reliable or largest-capacity path [5]. In this paper we use the above closed semiring to solve the *zero-sum cycle problem* in doubly weighted directed graphs.

Doubly weighted graphs, which have a two-dimensional weight on each edge, have been studied by Lawler [19], Dantzig, Blattner, and Kao [7], and Reiter [24]. The cost of a path is defined as the sum of weights of edges on the path. The *doubly weighted zero-sum cycle problem* is to find a cycle whose cost in each dimension is 0. In [12], [13], [14], [15], [17], we saw that certain problems in VLSI applications involving a regular structure can be transformed to problems in two-dimensional infinite graphs consisting of repeated finite graphs. Repeated use of a doubly weighted digraph, called the *static graph* G^0, forms a *dynamic graph* G^2. As shown in Fig. 1, each label of the static graph G^0 indicates the differences between the x- and y-coordinates of two connected vertices in G^2. The absence of a zero-sum cycle in the specified static graph is then necessary and sufficient for the acyclicity of the associated dynamic graph. If a two-dimensional regular electrical circuit is associated with a dynamic graph, acyclicity of the dynamic graph implies that the associated electrical circuit is free of an electrical "short circuit" [12].

Since the cost of each path between any two vertices can be regarded as a point in the two-dimensional Euclidean plane, we can associate a pair of vertices v and w with a convex polygon α_{vw} as follows: α_{vw} is the convex hull of all points associated

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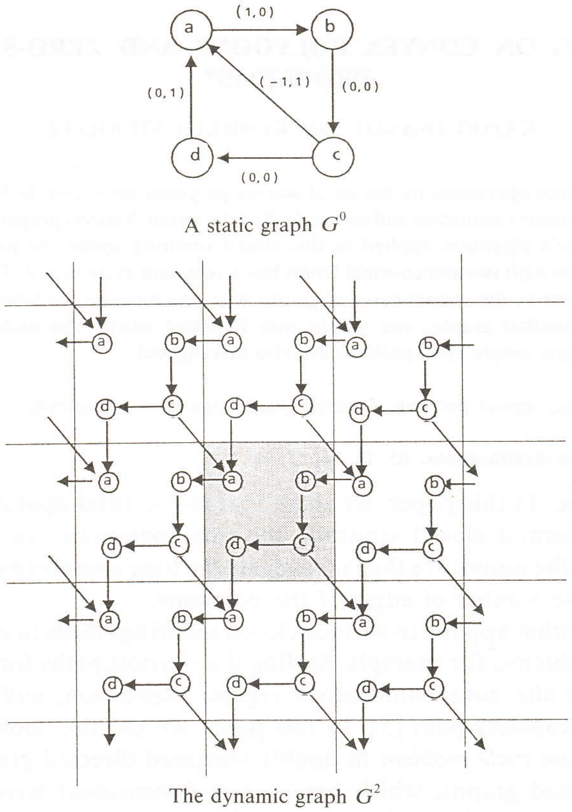


FIG. 1. *A static graph G^0 shows how to connect nodes in G^2. The shaded area shows a basic cell.*

with costs of paths from v to w. We apply the two operations above to the set of these convex hulls, and use the closed semiring defined by these two operations to solve the doubly weighted zero-sum cycle problem. We show that this algorithm runs in polynomial time in the special cases of bounded label graphs, BTTSP graphs (the Backedged Two-Terminal Series-Parallel graphs), and graphs with one-dimensional labels. The 1-*bounded graphs*, whose labels are 0, 1, or -1, arise in VLSI applications where the interconnections between regular basic cells are made locally. The BTTSP graphs are an extension of the class of *Two-Terminal Series-Parallel* [1], [8], [26], [27]. When the extended abstract of the present paper appeared in [16], the question of whether the zero-sum cycle problem for general graphs is in P remained open. Kosaraju and Sullivan [18] subsequently showed that the zero-sum cycle problem for any dimension can be formulated in terms of linear programming and is thus solvable in polynomial-time. Recently Cohen and Megiddo [6] proved that the zero-sum cycle problem for any fixed dimension belongs to the class NC, and can be solved in the two-dimensional case in serial time $O(nm)$, the best result to date. We hope the present paper retains independent interest as a new connection between convex polygons and semirings, and as a novel application of Kleene's closure algorithm.

Finally, we discuss variations of the zero-sum cycle problem, the undirected case, and the zero-sum *simple* cycle problem.

2. Two operations and a semiring. We define our closed semiring [21] as follows: Let S be the set of all convex polygons whose vertices have integer coordinates. That is, $S=\{\alpha^{\uplus} \mid \alpha \in 2^{Z\times Z}\}$, where $\alpha^{\uplus}$ indicates the convex hull of α. Notice that this definition allows polytopes with an infinite number of edges, unbounded area, or zero area, but does not allow curves. Thus our usage of the term *convex polygon* is more general than the conventional one. We conventionally denote an element in S by a lowercase Greek letter. We regard a point or a line segment as an element of S.

For any two sets $\alpha, \beta \in S$, we define the new set called a *vector sum of* α and β as follows: $\alpha * \beta = \{(x, y) \mid$ there exist elements $(a_x, a_y) \in \alpha$ and $(b_x, b_y) \in \beta$ such that $x = a_x + b_x$, $y = a_y + b_y\}$. See [11] for details of this operation. Let $\mathbf{0} = \{(0,0)\} \in S$ and $\varnothing$ be the empty set.

We define the $\uplus$ operation as the convex hull of the union of two convex polygons in S; that is, $\alpha \uplus \beta = (\alpha \cup \beta)^{\uplus}$ for any $\alpha, \beta \in S$. In this paper, we call the $\uplus$ operation *union-sum*. We can naturally define a union-sum of a countable number of convex polygons as follows: Let I be a countable (finite or infinite) index set and $\alpha_i \in S$ for all $i \in I$. Then we define *union-sum* $\biguplus_{i\in I} \alpha_i$ by $\biguplus_{i\in I} \alpha_i = (\bigcup_{i\in I} \alpha_i)^{\uplus}$. Since $\bigcup_{i\in I} \alpha_i$ exists and is unique, its convex hull $\biguplus_{i\in I} \alpha_i$ exists and is unique. Note that α_i is the convex hull of some set in $2^{Z\times Z}$, and thus every vertex of $\bigcup_{i\in I} \alpha_i$ is in $2^{Z\times Z}$. Hence $\biguplus_{i\in I} \alpha_i \in S$, and thus the union-sum above is well defined.

We now define the $+$ operation as the convex hull of the vector summation of two convex polygons in S; that is, $\alpha + \beta = (\alpha * \beta)^{\uplus}$. By convention, we define $\alpha + \varnothing = \varnothing + \alpha = \varnothing$. Note as we show later (Corollary 3.4, § 3), that $\alpha * \beta$ is itself a convex polygon when α and β are convex polygons. Therefore $\alpha + \beta = (\alpha * \beta)^{\uplus} = \alpha * \beta$ for any $\alpha, \beta \in S$. Therefore, we identify $+$ with $*$, and call the $+$ operation *vector-sum*. From the definitions, the vector-sum operation is commutative. Fig. 2 shows an example of the vector-sum of two convex polygons.

We now have the following theorem.

THEOREM 2.1. *The system* $(S, \uplus, +, \varnothing, \mathbf{0})$ *is a closed semiring.*

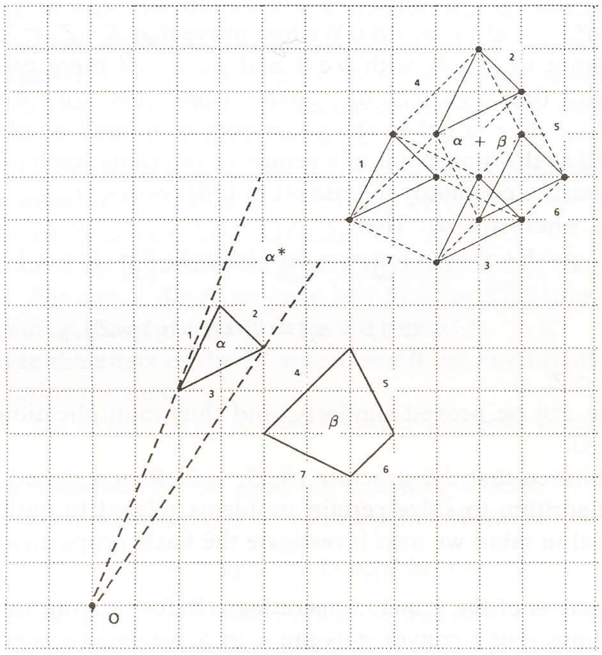


FIG. 2. *$\alpha+\beta$ is bounded by edges that are aligned with the edges in α or β. Aligned edges have the same numbers.*

Before proving this theorem, we need the following lemma.

LEMMA 2.2. *Let I be a countable index set. Let $\alpha_i \in 2^{Z\times Z}$ for all $i \in I$. Then we have* $(\bigcup_{i\in I} \alpha_i^{\uplus})^{\uplus} = (\bigcup_{i\in I} \alpha_i)^{\uplus}$.

Proof. Let A be the left-hand side of the above equation and B the right-hand side. Since $\alpha_i \subset A$ for all $i \in I$, we have $\bigcup_{i\in I} \alpha_i \subset A$. Since A is a convex hull, we have $(\bigcup_{i\in I} \alpha_i)^{\uplus} = B \subset A$.

Note that $\alpha_i \subset B$ and thus $\alpha_i^{\uplus} \subset B$ for all $i \in I$. Therefore $\bigcup_{i\in I} (\alpha_i^{\uplus}) \subset B$, and moreover, since B is a convex polygon, $A = (\bigcup_{i\in I} \alpha_i^{\uplus})^{\uplus} \subset B$. □

Now we prove Theorem 2.1.

Proof of Theorem 2.1. We show that the system $(S, \uplus, +, \varnothing, \mathbf{0})$ satisfies the six properties of a closed semiring.

(1) $(S, \uplus, \varnothing)$ *is a commutative monoid.* This is immediate from the definition and Lemma 2.2.

(2) $(S, +, \mathbf{0})$ *is a monoid.* From the definition, this is trivial.

(3) $+$ *distributes over* $\uplus$. Let α, β, $\gamma \in S$ be convex polygons. Since $\alpha + (\beta \uplus \gamma)$ is convex and contains $(\alpha+\beta)$ and $(\alpha+\gamma)$, we have $(\alpha+\beta) \uplus (\alpha+\gamma) \subset \alpha + (\beta \uplus \gamma)$.

For the opposite direction, let x be a point in $\alpha + (\beta \uplus \gamma)$. Then x can be expressed as $a + \lambda b + (1-\lambda)c$, where $\lambda \in [0, 1]$. Then we have $x = a + \lambda b + (1-\lambda)c = \lambda(a+b) + (1-\lambda)(a+c) \in (\alpha+\beta) \uplus (\alpha+\gamma)$. Therefore, $+$ distributes over $\uplus$. Note that we can also prove that $+$ distributes over finite union-sums by induction.

Since $\alpha \uplus \alpha = (\alpha \cup \alpha)^{\uplus} = \alpha^{\uplus} = \alpha$, $+$ is idempotent.

(4) Let $I = \{i_1, i_2, \cdots, i_k\}$ be a finite nonempty index set. Let $\alpha_i \in S$ for all $i \in I$. Then we can prove $\biguplus_{i\in I} \alpha_i = \alpha_{i_1} \uplus \alpha_{i_2} \uplus \cdots \uplus \alpha_{i_k}$ by induction on k and Lemma 2.2. For the empty index set $I = \varnothing$, we have $\biguplus_{i\in\varnothing} \alpha_i = \varnothing$.

(5) *The result of union-sum does not depend on the ordering of the factors.* The proof is straightforward from the definition of $\uplus$ and Lemma 2.2.

(6) In addition to (3), $+$ *distributes over countably infinite union-sums* $\uplus$. Let $\beta \in S$ and $\alpha_i \in S$ for $I = \{1, 2, \cdots\}$. Then we prove that $\beta + \biguplus_{i\in I} \alpha_i = \biguplus_{i\in I} (\alpha_i + \beta)$. Let $Z_\alpha = \biguplus_{i\in I} \alpha_i$ and $Z_{\alpha+\beta} = \biguplus_{i\in I} (\alpha_i+\beta)$. We first prove that $\beta + Z_\alpha \subset Z_{\alpha+\beta}$. Let $p = b + x$ be an arbitrary point in $\beta + Z_\alpha$ with $b \in \beta$ and $x \in Z_\alpha$. If there exists a finite set of indexes J such that $x \in \biguplus_{j\in J} \alpha_j$, then from (3), $p = b + x \in \beta + \biguplus_{j\in J} \alpha_j \in \biguplus_{j\in J} (\beta + \alpha_j) \subset Z_{\alpha+\beta}$. If x is not in the union-sum of a finite number of α_i's, then x can be represented as the limit point of a sequence of points, each of which is in some α_i; that is, there exists a countably infinite set of indexes $J = \{j_1, j_2, \cdots, j_i, \cdots,\}$ such that $x = \lim_{i\to\infty} x_{j_i}$ where $x_{j_i} \in \alpha_{j_i}$. Then

$$p = b + x = b + \lim_{i\to\infty} x_{j_i} = \lim_{i\to\infty} (b + x_{j_i}) \in \lim_{i\to\infty} (\beta + \alpha_{j_i})$$

$$\subset \biguplus_{j\in J} (\beta + \alpha_j) \subset \biguplus_{i\in I} (\beta + \alpha_i) = Z_{\alpha+\beta}.$$

Therefore $\beta + Z_\alpha \subset Z_{\alpha+\beta}$.

The converse can be proved similarly, and thus multiplication distributes over infinite sums. □

Having established that the structure $(S, \uplus, +, \varnothing, \mathbf{0})$ is a closed semiring, we can apply Kleene's algorithm to solve certain problems related to paths in a graph [2], [21]. With this goal in mind we next investigate the basic properties of the operations $+$ and $\uplus$.

3. Some properties of the + and $\uplus$ operations. Before stating some properties, we need some definitions. For a convex polygon α in S, we denote its edge set by α_E and its vertex set by α_V. Let l be an edge of α or a line that does not intersect α. Then we regard l as an oriented line with respect to α and define its direction, denoted by $\theta_l(\alpha)$, in the range $0 \leqq \theta_e < 2\pi$ such that α lies on the right-hand side of l when we traverse l in its positive direction. Unless specified, θ_e means $\theta_e(\alpha)$ for an edge $e \in \alpha_E$. We regard $e \in \alpha_E$ as a vector $\mathbf{e}$ with the direction of $\theta_e(\alpha)$. Let $\alpha_{vector} = \{\mathbf{e} \mid e \in \alpha_E\}$. By convention, we define the following special cases: When α is either a point or the entire plane, we regard α as a special symbol and define $\alpha_{vector} = \{\alpha\}$. When α is a line segment e, we define $\alpha_{vector} = \{\mathbf{e}, -\mathbf{e}\}$.

Let $A = \{\alpha_i\}_{i \in I}$ be a set of convex polygons. We define $|A| = |\bigcup_{i \in I} (\alpha_i)_{vector}|$, that is, $|A|$ is the number of distinct vectors in $\bigcup_{\alpha_i \in A} (\alpha_i)_{vector}$. We also write $|A| = |\alpha|$ when A has the single element α. We say that two edges $e \in \alpha_E$ and $f \in \beta_E$ are *aligned* when $\theta_e(\alpha) = \theta_f(\beta)$.

Now we have the following lemma about the relationship between two consecutive edges of a convex polygon and their directions.

LEMMA 3.1. *Let e and f be two consecutive edges of a convex polygon α in clockwise order. Then $\theta_f < \theta_e$ or $\theta_e + \pi < \theta_f$.*

Proof. From the definition, f lies in the right-half plane of e. □

COROLLARY 3.2. *Let $\alpha_E = \{e_1, e_2, \cdots, e_m\}$ be the edges of a convex polygon α in clockwise order. Let θ_{e_1} be the maximum of $\{\theta_{e_i}\}$. Then $\theta_{e_1} > \theta_{e_2} > \cdots > \theta_{e_m}$. The set α_E is called the edge sequence when the elements of α_E are ordered as above.*

Proof. The proof is clear from Lemma 3.1. □

To analyze how the + operation affects the number of distinct vectors, we will use the following well-known theorem.

THEOREM 3.3 ([11]). *Let α, β be two convex polygons in S. Then for every $e \in \alpha_E \cup \beta_E$, there exists an edge $f \in (\alpha + \beta)_E$ that is aligned with e; that is, $\theta_f = \theta_e$. This enables us to define a function $f = \varphi(e)$ from $\alpha_E \cup \beta_E$ to $(\alpha + \beta)_E$. Moreover, the function φ is onto. Figure 2 illustrates this theorem.* □

COROLLARY 3.4. *Let α, β be convex polygons. Then $\alpha + \beta = \alpha * \beta = \beta * \alpha = \beta + \alpha$.*

Proof. For the proof see [11], [20], [28]. □

COROLLARY 3.5. *Let α and β be convex polygons in S such that both have a finite number of edges and $n = |\alpha + \beta|$. Then the edge sequence of $\alpha + \beta$ can be computed in $O(n)$ steps from two edge sequences α_E and β_E.*

Proof. From Theorem 3.3, every edge e in $\alpha + \beta$ has an associated edge f in $\alpha_E \cup \beta_E$ such that $\theta_f = \theta_e$. Thus the edge sequence of $(\alpha + \beta)_E$ can be obtained by merging the two sets $\{\theta_e \mid e \in \alpha_E\}$ and $\{\theta_e \mid e \in \beta_E\}$. □

COROLLARY 3.6. *Let $\alpha_1, \alpha_2, \cdots,$ and α_n be convex polygons in S. Then we have an onto function φ from $(\alpha_1)_E \cup (\alpha_2)_E \cup \cdots \cup (\alpha_n)_E$ to $(\alpha_1 + \alpha_2 + \cdots + \alpha_n)_E$ such that $\theta_{\varphi(e)} = \theta_e$ for any $e \in (\alpha_1)_E \cup (\alpha_2)_E \cup \cdots \cup (\alpha_n)_E$.*

Proof. The proof is by induction on n and Theorem 3.3. □

THEOREM 3.7. *For any $\alpha, \beta \in S$, we have $|\alpha + \beta| \leqq |\{\alpha, \beta\}| \leqq |\alpha| + |\beta|$.*

Proof. The proof is straightforward from Theorem 3.3. □

Next we analyze the effect of the $\uplus$ operation on the number of distinct vectors. First we have the following theorem.

THEOREM 3.8. *Let α and β be bounded convex polygons in S. Then $|\alpha \uplus \beta| \leqq |\alpha| + |\beta|$.*

Before proving Theorem 3.8, we need the following lemma.

LEMMA 3.9. *Let $\alpha \in S$ and $p_1, p_2, \cdots, p_n$ be points. Then*

$$|\alpha \uplus p_1 \uplus p_2 \uplus \cdots \uplus p_n| \leqq |\alpha| + n.$$

Proof. This can be proved by induction on n. Suppose $n = 1$. If α contains p_1, then $|\alpha \uplus p_1| = |\alpha|$. Otherwise, $\alpha \uplus p_1$ contains at least one edge of α, and thus $|\alpha \uplus p_1| \leqq |\alpha| + 1$. Suppose the lemma holds for numbers less then n. Let $\beta_k = \alpha \uplus p_1 \uplus p_2 \uplus \cdots \uplus p_k$. Then from the induction hypothesis, $|\beta_{n-1}| \leqq |\alpha| + (n-1)$, and thus $|\beta_n| \leqq |\beta_{n-1}| + 1 \leqq |\alpha| + n$. □

Proof of Theorem 3.8. Note that $\alpha \uplus \beta$ is $\alpha \uplus p_1 \uplus p_2 \uplus \cdots \uplus p_n$ where $\beta_V = \{p_1, p_2, \cdots, p_n\}$. Thus from Lemma 3.9, $|\alpha \uplus \beta| \leqq |\alpha| + n = |\alpha| + |\beta|$. □

Theorem 3.11 covers the case when α or β is an unbounded convex polygon α^*, which is defined as follows: For a convex polygon α and a nonnegative integer n, we define a convex polygon α^n as follows: (1) $\alpha^0 = \mathbf{0}$ and (2) $\alpha^n = \alpha + \alpha^{n-1}$ for $n > 1$. Since a system $(S, \uplus, +, \emptyset, \mathbf{0})$ is a closed semiring, we can define the convex polygon α^* by $\alpha^0 \uplus \alpha^1 \uplus \cdots = \uplus_{i=0}^{\infty} \alpha^i$. As shown in Fig. 2, α^* is $\cup_{p \in \alpha} (\cup_{\lambda > 0} \lambda p)$. Thus α^* is essentially a cone emanating from the origin. As a special case, α^* may be the entire plane, a half plane, a line, a half line, or the origin itself. Now we analyze the effect of the * operation on the number of distinct vectors.

LEMMA 3.10. *For two convex polygons α and γ, we have* $|\alpha + \gamma^*| \leqq |\alpha| + 1$.

Proof. If γ^* is either the entire plane, a half plane, a line, a half line, or the origin itself, the proof is straightforward. Otherwise γ^* is a cone emanating from the origin and has two edges g_1' and g_2'. Let g_1 (respectively, g_2) be the support lines at v (respectively, w) of the convex polygon α such that $\theta_{g_1}(\alpha) = \theta_{g_1'}(\gamma^*)$ and $\theta_{g_2}(\alpha) = \theta_{g_2'}(\gamma^*)$. If $v = w$, then $|\alpha + \gamma^*| = |\gamma^*| \leqq 2$. If $v \neq w$, then as shown in Fig. 3, there must be at least one edge of α which is inside $\alpha + \gamma^*$. Thus the lemma is proved. □

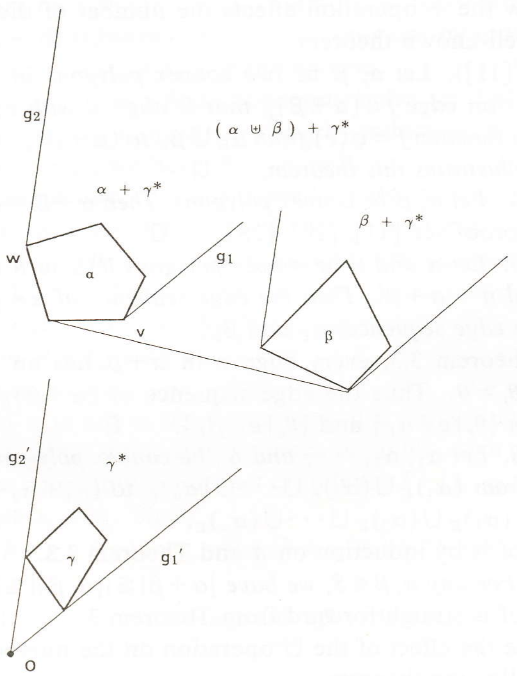


FIG. 3. $|\alpha + \gamma^*| \leqq |\alpha| + 1$ *and* $\alpha \uplus (\beta + \gamma^*) = (\alpha \uplus \beta) + \gamma^*$.

The above lemma shows that replacement of α by $\alpha + \gamma^*$ does not increase the number of edges by more than one. Moreover, we have a stronger result in the following theorem, which shows the same result for any number of such replacements in a series of $\uplus$ operations. We will use this theorem in §§ 5 and 6.

THEOREM 3.11. *Let β_i, $\gamma_i \in S$ for $i = 1, 2, \cdots, n$. Then we have*

$$|(\beta_1 + \gamma_1^*) \uplus (\beta_2 + \gamma_2^*) \uplus \cdots \uplus (\beta_n + \gamma_n^*)| \leqq |\beta_1 \uplus \beta_2 \uplus \cdots \uplus \beta_n| + 1.$$

Before proving Theorem 3.11, we need some lemmas.

LEMMA 3.12. *Let* $\gamma_i \in S$ *for* $i = 1, 2, \cdots, n$. *Then*

$$\gamma_1^* + \gamma_2^* + \cdots + \gamma_n^* = (\gamma_1 \uplus \gamma_2 \uplus \cdots \uplus \gamma_n)^*.$$

Proof. We prove this by induction on n, using k for the index of induction. The lemma trivially holds for $k = 1$. When $k = 2$, we prove that $\gamma_1^* + \gamma_2^* = (\gamma_1 \uplus \gamma_2)^*$. Since $\gamma_1^*, \gamma_2^* \subset (\gamma_1 \uplus \gamma_2)^*$, we have $\gamma_1^* + \gamma_2^* \subset (\gamma_1 \uplus \gamma_2)^*$. We next prove the opposite direction. Since $\mathbf{0} \in \gamma_1^* + \gamma_2^*$ we have from the distributive law $\gamma_1 \uplus \gamma_2 \subset (\gamma_1 \uplus \gamma_2) + (\gamma_1^* + \gamma_2^*) = (\gamma_1 + \gamma_1^* + \gamma_2^*) \uplus (\gamma_1 + \gamma_2 + \gamma_2^*) \subset \gamma_1^* + \gamma_2^*$. Thus the lemma holds for $k = 2$.

Assume that the lemma holds for $k < n$, then $\gamma_1^* + \gamma_2^* + \cdots + \gamma_n^* = (\gamma_1 \uplus \gamma_2 \uplus \cdots \uplus \gamma_{n-1})^* + \gamma_n^* = (\gamma_1 \uplus \gamma_2 \uplus \cdots \uplus \gamma_n)^*$. Note that we used the case $k = n - 1$ for the first transformation and $k = 2$ for the latter. □

LEMMA 3.13. *Let* α, β, *and* γ *be convex polygons. Then*

$$\alpha \uplus (\beta + \gamma^*) = (\alpha \uplus \beta) + \gamma^*.$$

Proof. For the proof see Fig. 3. Since $\alpha \subset \alpha + \gamma^*$, we have

$$\alpha \uplus (\beta + \gamma^*) \subset (\alpha + \gamma^*) \uplus (\beta + \gamma^*) = (\alpha \uplus \beta) + \gamma^*.$$

We now prove the opposite direction; that is, $(\alpha \uplus \beta) + \gamma^* \subset \alpha \uplus (\beta + \gamma^*)$. Since $\beta + \gamma^* \subset \alpha \uplus (\beta + \gamma^*)$, we only have to prove that $\alpha + \gamma^* \subset \alpha \uplus (\beta + \gamma^*)$. Let $p = a + g$ be a point in $\alpha + \gamma^*$ with $a \in \alpha$ and $g \in \gamma^*$. Let b be an arbitrary point in β. Let p_n be a point obtained by the following equation when we regard p_n, a, b, and g as points in the $x-y$ plane: $p_n = (1 - 1/n)a + (1/n)(b + ng)$. Then p_n is on the line segment $a, (b + g^n)$, and thus $p_n \in \alpha \uplus (\beta + \gamma^*)$. Note that $p_\infty = \lim_{n\to\infty} p_n$ is also in $\alpha \uplus (\beta + \gamma^*)$ and $p_\infty = a + g = p$. Therefore $\alpha + \gamma^* \subset \alpha \uplus (\beta + \gamma^*)$. □

LEMMA 3.14. *Let* $\alpha_i, \gamma_i \in S$ *for* $i = 1, 2, \cdots, n$. *Then*

$$\begin{aligned}&(\alpha_1 + \gamma_1^*) \uplus (\alpha_2 + \gamma_2^*) \uplus \cdots \uplus (\alpha_n + \gamma_n^*)\\ &\quad = (\alpha_1 \uplus \alpha_2 \uplus \cdots \uplus \alpha_n) + \gamma_1^* + \gamma_2^* + \cdots + \gamma_n^*.\end{aligned}$$

Proof. Denote the left-hand side of the above equation by A_n, and the right-hand side by B_n. We prove this by induction on n and use k for the index of induction. The lemma holds trivially for $k = 1$. When $k = 2$, from Lemma 3.13, $A_2 = ((\alpha_1 + \gamma_1^*) \uplus \alpha_2) + \gamma_2^* = (\alpha_1 \uplus \alpha_2) + \gamma_1^* + \gamma_2^* = B_2$. Assume that the lemma holds for $k < n$. From the induction hypothesis for $k = n - 1$, $A_n = B_{n-1} \uplus (\alpha_n + \gamma_n^*)$. We then obtain $A_n = ((\alpha_1 \uplus \alpha_2 \uplus \cdots \uplus \alpha_{n-1}) + (\gamma_1 \uplus \gamma_2 \uplus \cdots \uplus \gamma_{n-1})^*) \uplus (\alpha_n + \gamma_n^*)$ by applying Lemma 3.12 to B_{n-1}. From the basis of the induction ($k = 2$), $A_n = (\alpha_1 \uplus \alpha_2 \uplus \cdots \uplus \alpha_n) + (\gamma_1 \uplus \gamma_2 \uplus \cdots \uplus \gamma_{n-1})^* + \gamma_n^*$. From Lemma 3.12, we get $A_n = B_n$. □

We can now prove Theorem 3.11.

Proof of Theorem 3.11. From Lemma 3.12 and 3.14, we have

$$\begin{aligned}&(\beta_1 + \gamma_1^*) \uplus (\beta_2 + \gamma_2^*) \uplus \cdots \uplus (\beta_n + \gamma_n^*)\\ &\quad = (\beta_1 \uplus \beta_2 \uplus \cdots \uplus \beta_n) + \gamma_1^* + \gamma_2^* + \cdots + \gamma_n^*\\ &\quad = (\beta_1 \uplus \beta_2 \uplus \cdots \uplus \beta_n) + (\gamma_1 \uplus \gamma_2 \uplus \cdots \uplus \gamma_n)^*.\end{aligned}$$

Let A (respectively B) be the left- (respectively right-)hand side of the equation in the theorem. From Lemma 3.10, $|A| \leqq |\beta_1 \uplus \beta_2 \uplus \cdots \uplus \beta_n)| + 1 = |B| + 1$. □

THEOREM 3.15. *Let* $|\alpha| + |\beta| = n$. *The operations* $+$, $\uplus$, *and* $*$ *can all be done in* $O(n)$ *steps.*

Proof. We assume that two edge sequences $(\alpha)_E$ and $(\beta)_E$ are available. From Corollary 3.5, we know that the + operation takes $O(n)$ time. Given the edge-sequences of two convex polygons, the convex hull can be found in $O(n)$ time. There is also an algorithm in [4] that computes $l \cap \alpha$ in $O(\log(|\alpha|))$ steps where l is a line segment and α is a convex hull. Therefore the * operation takes $O(\log(|\alpha|))$ time. □

4. Application of the closed semiring. In this section, we define the doubly weighted zero-sum cycle problem and solve it by using the closed semiring defined in § 2.

Our instance is a doubly weighted digraph $G=(V, E, T)$ where V is its vertex set, E is its edge set, and T is a two-dimensional labeling such that $T(e)=(e_x, e_y)\in Z\times Z$ for every $e\in E$. We use n (respectively, m) to denote the number of vertices (respectively, edges) in a graph. We also use $\mathbf{0}$ to denote $(0,0)$. A path P in G is a sequence of vertices $P=v_0, v_1, \cdots, v_k$ where $e_i=(v_{i-1}, v_i)\in E$ and $v_i\in V$. If all vertices $v_0, v_1, \cdots, v_{k-1}$ are distinct, a path P is *simple*. A path P such that $v_0=v_k$ is called a *cycle*. Given a path $P=v_0, v_1, \cdots, v_k$, we define the *cost of the path* $T(P)$ by the component-wise summation of edge-labels on that path; that is, $T(P)=\sum_{i=1}^k T(e_i)=(\sum_{i=1}^k e_{i_x}, \sum_{i=1}^k e_{i_y})$. A cycle W with $T(W)=\mathbf{0}$ is called a *zero-sum cycle*. We can now define the *doubly weighted zero-sum cycle problem* as follows:

Problem ZSC. Doubly Weighted Zero-sum Cycle Problem.

Instance: A doubly weighted digraph $G=(V, E, T)$ where T is a two-dimensional labeling such that $T(e)=(e_x, e_y)\in Z\times Z$ for every $e\in E$.

Question: Does G have a zero-sum cycle? In other words, is there a cycle W such that $T(W)=\mathbf{0}$?

By using the fact that the two operations defined on convex polygons form a closed semiring, we can answer this question with the Floyd–Warshall algorithm [2], [3], [10], [23].

Algorithm ZSC.

Input: A doubly weighted graph G with $V=\{v_1, v_2, \cdots, v_n\}$.

Output: This algorithm answers "Yes" if the digraph G has a zero-sum cycle; otherwise the algorithm answers "No."

Method: Let $PATH(v_i, v_j, k)$ denote the set of all paths from v_i to v_j such that all vertices on the path, except possibly the endpoints, are in the set $\{v_1, v_2, \cdots, v_k\}$. We compute the convex hull α_{ij}^k for $1\leqq i, j\leqq n$ and $0\leqq k\leqq n$, which is the convex hull of costs of all paths in $PATH(v_i, v_j, k)$.

procedure zero-sum cycle
  begin
1.   **for** $1\leqq i, j\leqq n$ **do** $\alpha_{ij}^0=\begin{cases}\{T((v,w))\} & \textbf{if } (v,w)\in E\\ \varnothing & \textbf{otherwise.}\end{cases}$
2.   **for** $k=1$ **to** n
    do
3.   **for** $1\leqq i, j\leqq n$ **do**
4. $\alpha_{ij}^k=\alpha_{ij}^{k-1}\uplus(\alpha_{ik}^{k-1}+(\alpha_{kk}^{k-1})^*+\alpha_{kj}^{k-1})$;
5.   **if** $\exists i\in\{1,2,\cdots,n\}$ s.t. $\mathbf{0}\in\alpha_{ii}^k$
        then exit ("Yes");
    od
6.   **exit** ("No");
  end

THEOREM 4.1. *Algorithm ZSC works correctly.*

Before proving Theorem 4.1, we need the following lemmas.

LEMMA 4.2. *If there is a zero-sum cycle W, there must be a vertex v_i such that $\mathbf{0} \in \alpha_{ii}^n$.*

Proof. Let v_i be a vertex that is on the cycle W. Since the convex hull α_{ii}^n includes all costs of paths from v_i to v_i, we have $\mathbf{0} \in \alpha_{ii}^n$. □

LEMMA 4.3. *If $\mathbf{0} \in \alpha_{ii}^n$, there must be a zero-sum cycle W, and the vertex v_i is on the cycle W.*

Proof. Suppose that $\mathbf{0} \in \alpha_{ii}^n$. Let $(\alpha_{ii}^n)_V = \{s_1, s_2, \cdots, s_l\}$ such that $s_j \in Z \times Z$. Since α_{ii}^n is a convex polygon, any point z in α_{ii}^n can be represented as $z = \sum_{s_j \in (\alpha_{ii}^n)_V} k_j s_j$ such that $k_j \geqq 0$. Let C_j be a cycle corresponding to s_j such that $T(C_j) = s_j$ and v_i is on the cycle C_j. Note that since every s_j has integral coordinates, k_j can be chosen rational, if $z \in Z \times Z$. Thus there are rational numbers k_j' such that $\mathbf{0} = \sum_{s_j \in (\alpha_{ii}^n)_V} k_j' s_j$. There is an integer K such that all $K \cdot k_j'$ are integers. Thus $K \cdot \mathbf{0} = \mathbf{0} = \sum_{s_j \in (\alpha_{ii}^n)_V} (K \cdot k_j) s_j$. Then the desired cycle W consists of $K \cdot k_j$ copies of C_j for $s_j \in (\alpha_{ii}^n)_V$. □

Now we prove Theorem 4.1.

Proof of Theorem 4.1. From Lemma 4.2 and 4.3, in order to find a zero-sum cycle, we only have to check whether or not there exists some i such that $\mathbf{0} \in \alpha_{ii}^n$. We can prove that α_{ij}^k is correctly computed by the algorithm by induction on k (as in [2]). □

THEOREM 4.4. *Algorithm ZSC uses $O(n^3)$ $\uplus$, $+$, and $*$ operations from the closed semiring defined above, where n is the number of vertices in G.*

Proof. Line 4 is executed n^3 times in total. □

5. Special cases of the zero-sum cycle problem. In this section, we discuss the special cases of the zero-sum cycle problem where (1) the graphs have one-dimensional labels, (2) the graphs are undirected, (3) the graphs have labels with magnitude at most M, and (4) we are looking for a *simple* cycle with zero-sum. The first three cases have low order polynomial algorithms, whereas the fourth is NP-complete.

(1) The one-dimensional zero-sum cycle problem. We can solve the problem efficiently in the one-dimensional case as follows.

THEOREM 5.1. *The-dimensional zero-sum cycle problem can be solved in $O(n^3)$ time, where n is the number of vertices. (This result is implicit in Orlin [22].)*

Proof. We can apply our algorithm ZSC by ignoring the second labels. Note that in the one-dimensional case, every α_{ij}^k has at most two vertices, since it is either a point, a line segment, or a line on the x-axis. Thus $|\alpha_{ij}^k| \leqq 2$. From Theorem 3.15, each operation $\uplus$, $+$, or $*$ takes constant time. Hence from Theorem 4.4, the algorithm ZSC takes $O(n^3)$ time. □

(2) The two-dimensional undirected zero-sum cycle problem. We assume that G is connected. We will show that the undirected version of the zero-sum cycle problem can be solved in $O(m \log m)$ time, where m is the number of edges. In the undirected case, a path can traverse an edge in either direction.

An instance of the undirected problem is as follows:

Instance: A connected undirected graph $G = (V, E)$ with $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{e_1, e_2, \cdots, e_m\}$. A two-dimensional labeling T from E to $Z \times Z$ with $T(e) = (e_x, e_y)$ for every $e \in E$.

Now we have the following lemma:

LEMMA 5.2. *Let G and T be defined above. Let H_G be the convex hull of $\{T(e) \mid e \in E\}$. A necessary and sufficient condition for the existence of a zero-sum cycle is that exactly one of the following two conditions holds:*

(1) *The convex polygon H_G properly contains the origin.*

(2) *The origin is on an edge h of the convex polygon H_G. Let $Y = \{e \in E \mid T(e)$ is on $h\}$. Then there exists an edge $e \in Y$ such that $T(e) = \mathbf{0}$, or there are two edges $e_1, e_2 \in Y$ such that e_1 and e_2 are adjacent in G and the origin is on the line segment $\overline{T(e_1), T(e_2)}$.*

Before proving Lemma 5.2, we need some definitions. Let $X = \{C_e \mid e \in E\}$ such that C_e is the cycle $v \to w \to v$ where $e = (v, w)$. Then $T(C_e) = 2T(e)$.

We call a set of cycles $A = \{W_i \mid i \in I\}$ *nullable* if there exists a set of nonnegative integers $A_Z = \{n_i \in Z^+ \cup \{0\} \mid i \in I\}$ such that the n_i are not all 0 and $\sum_{i \in I} n_i T(W_i) = \mathbf{0}$. If $(\bigcup_{i \in I} W_i)$ is connected, we say that A is *connected.*

Note that we can construct a zero-sum cycle from a connected nullable set. Now we have the following lemmas.

LEMMA 5.3. *Let G, T, and X be defined as above. Let $A = \{W_i \mid i \in I\}$ be a nullable set of cycles. Then we can find a connected nullable set B.*

Proof. Since A is nullable, there exists a set of nonnegative integers $A_Z = \{n_i \in Z^+ \cup \{0\} \mid i \in I\}$ such that the n_i are not all 0 and $\sum_{i \in I} n_i T(W_i) = \mathbf{0}$. If A is connected, A is the desired set. Suppose A is not connected. Let v_i be an arbitrary point on W_i for every $i \in I$. Since G is connected, there is a cycle P_i that passes through v_1 and v_i for every $i \in I - \{1\}$. Let k be a large positive integer. Let Q_i be a cycle consisting of k copies of W_i and one copy of P_i for every $i \in I - \{1\}$. Let $Q_1 = W_1$. Then

$$T(Q_i) = \begin{cases} kT(W_1) & \text{for } i = 1, \\ kT(W_i) + T(P_i) & \text{for } i \in I - \{1\}. \end{cases}$$

Since the convex hull of $\{T(W_i) \mid i \in I\}$ contains $\mathbf{0}$, the convex hull of $\{T(Q_i) \mid i \in I\}$ contains $\mathbf{0}$ for some large k. Therefore $B = \{Q_i \mid i \in I\}$ is nullable for large k. Since $v_1 \in \bigcap_{i \in I} Q_i$, B is connected. Thus B is the desired set. □

Now we prove Lemma 5.2.

Proof of Lemma 5.2. Suppose (1) holds. Note that $T(C_e) = 2T(e)$, where C_e is the cycle for every $e \in E$ defined as above. Since $A = \{2T(e) \mid e \in E\}$ is a nullable set, we can find a connected nullable set, by Lemma 5.3. Thus there is a zero-sum cycle in G. When (2) holds, it is obvious that there is a zero-sum cycle in G.

Conversely, suppose there exists a zero-sum cycle W. From the definition, there exist positive integers n_e for $e \in W$ such that $\sum_{e \in W} n_e T(e) = \mathbf{0}$. This means that the convex hull of $\{T(e) \mid e \in E\}$, denoted by H_G, contains the origin. If H_G contains the origin properly, (1) holds. Otherwise, there must be an edge $e \in E$ such that $T(e) = \mathbf{0}$, or the origin must be on an edge h of H_G. Now we assume that $T(e) \neq \mathbf{0}$ for every $e \in E$. Let $Y = \{e \in E \mid T(e)$ is on the edge $h\}$. Since W is nullable, every edge in W is in Y. Let $\tilde{e}$ be an edge in Y. Then for every edge $e \in Y$, there exists k_e and $T(e) = k_e T(\tilde{e})$. Let $W_+ = \{e \in W \mid T(e) = k_e T(\tilde{e}),\ k_e > 0\}$, and let $W_- = \{e \in W \mid T(e) = -k_e T(\tilde{e}),\ k_e > 0\}$. From the definition of $\{n_e\}$, we have $\sum_{e \in W} n_e T(e) = (\sum_{e \in W_+} n_e k_e - \sum_{e \in W_-} n_e k_e) T(\tilde{e}) = \mathbf{0}$. Note that $W_+ \neq \emptyset$ and $W_- \neq \emptyset$. Since $W = W_+ \cup W_-$ is connected, there must be connected edges $e_1 \in W_+$ and $e_2 \in W_-$. Thus (2) holds. □

THEOREM 5.4. *The two-dimensional undirected zero-sum cycle problem can be solved in $O(m \log m)$ time, where m is the number of edges.*

Proof. We only have to check condition (1) and (2) in Lemma 5.2, which can be done in $O(m \log m)$ time. □

(3) Graphs with bounded labels. A doubly weighted digraph $G = (V, E, T)$ is called an *M-bounded graph* if each dimension of every label is an integer in $[-M, M]$.

In many VLSI applications, the communication between regular cells is made locally: that is, interconnections are made only to neighbors. For example, $n \times n$ multipliers can be constructed from arrays of one-bit full adders with carry and sum signal connections to the neighbors of each cell [12], [13], [14], [15]. Parallel adders can also be constructed from one-bit full adders with carry connections to the neighbor of each cell [13]. Many systolic arrays are also implemented with interconnections to neighbors. In such VLSI applications, the associated static digraphs of the regular structures are all 1-bounded graphs [12].

We have the following lemma about the number of edges of a convex polygon included in a bounded region.

LEMMA 5.5. *Let R be a rectangle of width w and height h. Let H be an arbitrary convex polygon included in R. Then $|H| \leqq 2 \max(w, h) + 2$.*

Proof. Without loss of generality, we can assume that $\max(w, h) = w$. Let H_u be the set of edges in H from its highest leftmost vertex to its highest rightmost vertex in clockwise order. When we traverse an edge in H_u, we move at least one unit in the x-direction. Thus the number of edges in H_u is at most w. There are at most two vertical edges in H. Thus $|H| \leqq 2 \max(w, h) + 2$. □

LEMMA 5.6. *Let G be an M-bounded graph with n vertices, then we have $|\alpha_{ij}^k| \leqq 4nM + 3$.*

Proof. Let β_{ij}^k be the convex hull of the costs of all simple paths in $PATH(v_i, v_j, k)$ (see the previous section for the definition). Note that the length of a simple path is at most nM in each dimension. Thus β_{ij}^k is bounded by the rectangle $[-nM, nM] \times [-nM, nM]$. Therefore, from Lemma 5.5, $|\beta_{ij}^k| \leqq 2 \cdot (2nM) + 2 = 4nM + 2$. From Theorem 3.11, $|\alpha_{ij}^k| \leqq |\beta_{ij}^k| + 1 \leqq 4nM + 3$. □

THEOREM 5.7. *The algorithm ZSC takes $O(n^4 M)$ time for M-bounded graphs with n vertices.*

Proof. From Theorem 4.4 and Lemma 5.6, the algorithm ZSC takes $O(n^3 \cdot nM) = O(n^4 M)$ time. □

(4) The zero-sum simple cycle problem.

THEOREM 5.8. *The zero-sum simple cycle problem* (ZSSC) *is* NP-*complete.*

Proof. Here we use a variant of the reduction from the subset sum to the directed path problem in the one-dimensional dynamic graphs discussed in [22]. It is obvious that ZSSC is in NP. We use reduction from the subset sum problem SS to ZSSC, where the problem SS is defined as follows:

Input: $\{a_i \in Z^+ \mid i \in I\}$ where $I = \{1, 2, \cdots, n\}$ and $B \in Z^+$.

Question: Is there a subset J of I such that $\sum_{j \in J} a_j = B$?

Given an instance I_{SS} of SS, we construct an instance I_{ZSSC} of the zero-sum simple cycle problem as follows: A directed graph $G = (V, E)$ is shown in Fig. 4 where

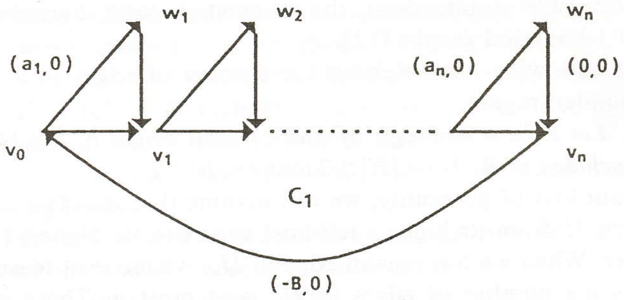
$$V = \{v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n\},$$

$$\begin{aligned} E = &\{e_i = (v_{i-1}, v_i) \mid i = 1, 2, \cdots, n\} \\ &\cup \{f_i = (v_{i-1}, w_i) \mid i = 1, 2, \cdots, n\} \\ &\cup \{g_i = (w_i, v_i) \mid i = 1, 2, \cdots, n\} \\ &\cup \{e_0 = (v_n, v_1)\}. \end{aligned}$$

Let T be a two-dimensional labeling from E to $Z \times Z$ as follows:

$$\begin{cases} T(e_0) = (-B, 0), & \\ T(e_i) = T(g_i) = (0, 0) & \text{for } i = 1, 2, \cdots, n, \\ T(f_i) = (a_i, 0) & \text{for } i = 1, 2, \cdots, n. \end{cases}$$

Suppose I_{SS} has a solution J such that $\sum_{j \in J} a_j = B$. Then I_{ZSSC} has a solution of a simple cycle consisting of e_0, f_j and g_j for $j \in J$, and e_i for $i \notin J$.



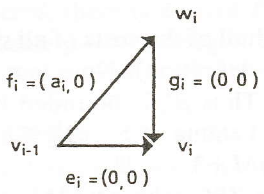


FIG. 4. *The graph above has a zero-sum cycle if and only if there exists a set* $J \subset I = \{1, 2, \cdots, n\}$ *such that* $\sum_{j \in J} a_j = B$.

Conversely, suppose that I_{ZSSC} has a solution; that is, there exists a simple cycle W such that $T(W) = \mathbf{0}$. Note that W must use e_1. Let $J = \{j \mid f_j \in W\}$. Then $\sum_{j \in J} a_j = B$. Thus I_{SS} has the solution J. □

6. Backedged two-terminal series-parallel multidigraphs. Two-Terminal Series-Parallel (TTSP) graphs have been well studied: the undirected version in [1], [8], [25], [27] because of its relationship to electrical networks and the directed version in [26] because it provides an algorithm to recognize general series-parallel digraphs.

A digraph is called a *multidigraph* if we allow multiple edges between the same two vertices. The definition of the class of TTSP multidigraphs appears in [26] as follows:

(1) A digraph consisting of two vertices joined by a single edge is in TTSP.

(2) If G_1 and G_2 are TTSP multidigraphs, so too is the multidigraph obtained by either of the following operations:

(a) *Two terminal parallel composition*: identify the source of G_1 with the source of G_2 and the sink of G_1 with the sink of G_2.

(b) *Two terminal series composition*: identify the sink of G_1 with the source of G_2.

Let TTSP (m) be the class of TTSP multidigraphs that have m edges.

From this definition, a TTSP multidigraph has a single source, denoted by s, and a single sink, denoted by t. Let G be a TTSP graph. A multidigraph, obtained by adding any number of *backedges* to a TTSP graph G, is called a BTTSP (*Backedged Two-Terminal Series-Parallel*) multidigraph. An edge (x, y) is called a *backedge* if there is a path from y to x in G. The graph G is called *the underlying* TTSP *graph of* G_B. Let BTTSP (m) be the class of BTTSP multidigraphs that have m edges. Fig. 5(a) shows an example of a BTTSP graph G_B that consists of a backedge indicated by dotted lines and the underlying TTSP graph G.

Let $G = (V, E, T)$ be a doubly weighted multidigraph with $V = \{v_1, v_2, \cdots, v_n\}$. Then for all v_i, v_j in V and $k \in \{1, 2, \cdots, n\}$, we define the convex polygon $\alpha_{ij}^k(T)$ in

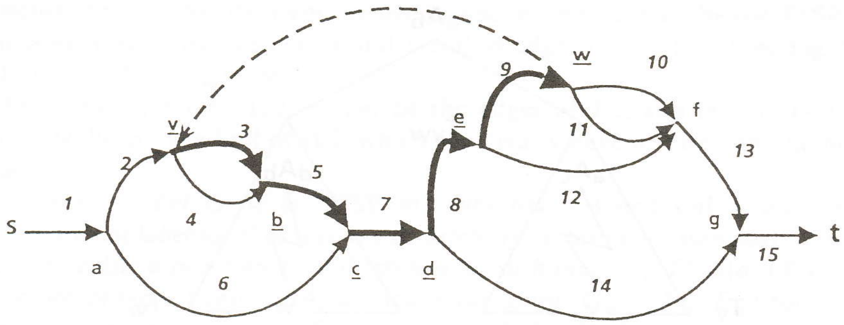


FIG. 5(a). *A* BTTSP *multidigraph* G_B; *a backedge is indicated by the dotted line.*

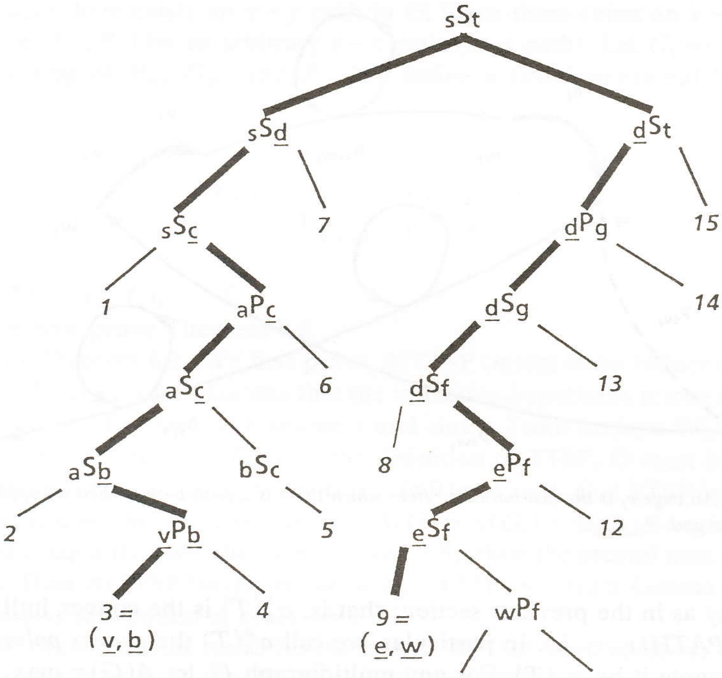


FIG. 5(b). *A binary decomposition tree* BDT (G). *The wide solid line corresponds to the path from* v *to* w *in Fig.* 5(a).

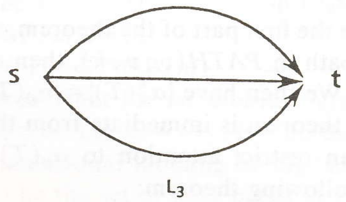


FIG. 5(c). $|\alpha_{st}(L_3)| \leqq 3$.

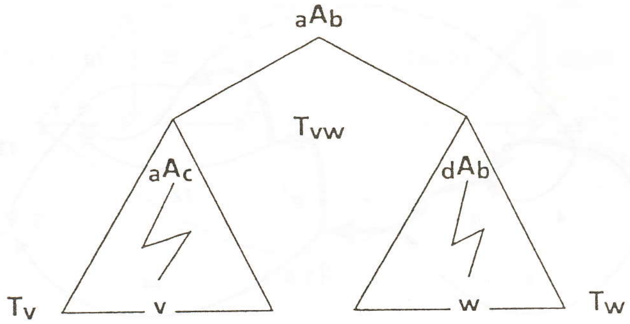


FIG. 5(d). ${}_aA_b = {}_aS_b$ *and* $c = d$. *Every path from* v *to* $x \notin T_v$ *passes through* c. *Every path from* $x \notin T_w$ *to* w *passes through* $d = c$.

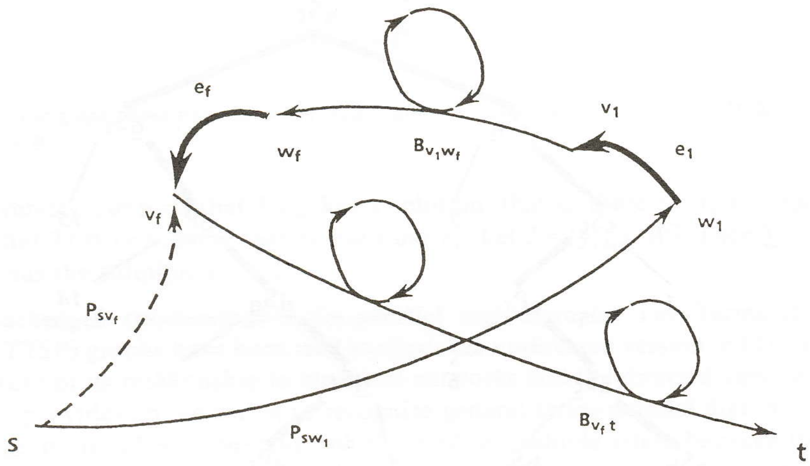


FIG. 5(e). *An edge* e_f *is the last backedge from which there is a path to* w_1. *Then we apply the induction hypothesis to the path* $P_{sv_f}B_{v_ft}$.

the same way as in the previous section: that is, $\alpha_{ij}^k(T)$ is the convex hull of all costs of paths in $PATH(v_i, v_j, k)$. In particular, we call $\alpha_{ij}^n(T)$ the *convex polygon* of $v_i - v_j$ *paths* and denote it by $\alpha_{ij}(T)$. For any multidigraph G, let $A(G) = \max_{i,j,k,T} |\alpha_{ij}^k(T)|$ and similarly for a class of graphs we write $A(\{G\})$. That is, $A(G)$ is the maximum number of edges in $\alpha_{ij}^k(T)$ when i, j, k, and T are arbitrary and G is fixed. We then have the following theorem.

THEOREM 6.1. *Let G be a doubly weighted multidigraph defined as above. For any* i, j, k, *and* T, *there exists a two-dimensional labeling* T' *such that* $\alpha_{ij}^k(T) = \alpha_{ij}(T')$. *Therefore,* $A(G) = \max_{i,j,T} |\alpha_{ij}(T)|$.

Proof. In order to prove the first part of the theorem, we only have to define $T'(e)$ as follows: (1) if e is on a path in $PATH(v_i, v_j, k)$, then define $T'(e) = T(e)$, and (2) otherwise define $T'(e) = \emptyset$. We then have $|\alpha_{ij}^k(T)| = |\alpha_{ij}(T')|$.

The second part of the theorem is immediate from the first part. $\square$

From this result we can restrict attention to $\alpha_{ij}(T)$ instead of $\alpha_{ij}^k(T)$ in what follows. We now have the following theorem:

THEOREM 6.2. $A(\mathrm{TTSP}\,(m)) = m$.

Before proving the theorem, we need some lemmas. Let L_m be the TTSP multidigraph consisting of two vertices s and t, and m edges from s to t. (See Fig. 5c.)

LEMMA 6.3. $A(L_m) = m$.

Proof. Let e_i for $i = 1, 2, \cdots, m$ be the edges of L_m, and let $T(e_i) = v_i \in Z \times Z$. Then α_{st} is the convex hull of $\{v_i\}$, which can clearly have m sides, and no more than m sides. □

LEMMA 6.4. *Let G be in* TTSP(m) *with source s and sink t, and let T be a two-dimensional labeling of G. Let x, y be arbitrary vertices in G such that $(x, y) \neq (s, t)$. Then there exists a two-dimensional labeling T' such that $|\alpha_{xy}(T)| \leqq |\alpha_{st}(T')|$.*

Before proving Lemma 6.4, we define the graph $G_{xy} = (V_{xy}, E_{xy})$ for $x, y \in V$ by the following operations on a TTSP graph $G = (V, E)$: (1) First, we delete all incoming edges to x and all outgoing edges from y. (2) We then delete all *useless* vertices and their adjacent edges. A vertex v is called *useless* when there is no $v - x$ path or $y - v$ path.

Proof of Lemma 6.4. If there is no $x - y$ path in G, we have $\alpha_{xy}(T) = \varnothing$. Thus $|\alpha_{xy}(T)| = 0 \leqq |\alpha_{st}(T)|$. Choose T as T'.

Otherwise there exists an $x - y$ path in G. Since there exists an $s - x$ path and a $y - t$ path, let $P_{sx}(P_{yt})$ be an arbitrary $s - x$ path ($y - t$ path). Let $G_1 = (V_1, E_1)$ be the graph consisting of P_{sx}, G_{xy}, and P_{yt}. We define a two-dimensional labeling T' as follows:

$$T'(e) = \begin{cases} \varnothing & \text{if } e \in E - E_1 \\ \mathbf{0} & \text{if } e \in P_{sx} \cup P_{yt} \\ T(e) & \text{if } e \in E_{xy}. \end{cases}$$

Then $|\alpha_{xy}(T)| = |\alpha_{st}(T')|$. □

We can now prove Theorem 6.2.

Proof of Theorem 6.2. We first prove $A(\text{TTSP}(m)) \leqq m$ by induction on m. It is clear that $A(\text{TTSP}(1)) = 1$. Assume that the induction hypothesis is true for $k < m$. Let $G = (V, E)$ be in TTSP(m) with source s and sink t. From Lemma 6.4, we only have to show $|\alpha_{st}(T)| \leqq m$ for any T. From the definition of TTSP, G must be constructed either in series or in parallel from $G_1 \in \text{TTSP}(m_1)$ and $G_2 \in \text{TTSP}(m_2)$ such that $m = m_1 + m_2$ and $m_1, m_2 > 0$. Then we have $A(G) \leqq A(G_1) + A(G_2) \leqq m_1 + m_2 = m$. Note that the first inequality uses Theorems 3.7 and 3.8, while the second uses the induction hypothesis. Thus $A(\text{TTSP}(m)) \leqq m$. Since $L_m \in \text{TTSP}(m)$, from Lemma 6.3, $A(L_m) = m$, which shows this bound is achievable. □

We will show the same result for the class of BTTSP multidigraphs. The following lemma says that every backedge in an $s - t$ path in a BTTSP graph lies on a cycle that lies on the $s - t$ path.

LEMMA 6.5. *Let G_B be a* BTTSP *graph with source s and sink t, and let P be a path from s to t possibly using some backedges in G_B. Then P can be represented as follows: $P = P_1 C_1^{r_1} P_2 C_2^{r_2} \cdots P_k C_k^{r_k}$ where $P_1 P_2 \cdots P_k$ is a path from source to sink in the underlying* TTSP *graph G, the C_i's are cycles in G_B, and $r_i \geqq 0$ for $1 \leqq i \leqq k$.*

Proof. For the proof see § 7. □

THEOREM 6.6. $A(\text{BTTSP}(m)) = m$.

Proof. Since TTSP$(m) \subset$ BTTSP(m), we have $m = A(\text{TTSP}(m)) \leqq A(\text{BTTSP}(m))$. We now prove that for an arbitrary graph $G_B \in \text{BTTSP}(m)$ with at least one backedge, $A(G_B) \leqq m$. Let $G = (V_1, E_1)$ be the underlying TTSP graph of G_B, and let T be a two-dimensional labeling of G_B. Let $P_B(s, t)$ be the set of $s - t$ paths in G_B, and let $P(s, t)$ be the set of $s - t$ paths in G. Let P be an arbitrary path in $P_B(s, t)$. Then from Lemma 6.5, P can be expressed as $P = P_1 C_1^{r_1} P_2 C_2^{r_2} \cdots P_k C_k^{r_k}$, where $P_1P_2 \cdots P_k$ is a path from source to sink in the underlying TTSP graph G, the C_i's are cycles in G_B, and $r_i \geqq 0$ for $1 \leqq i \leqq k$. Let $\beta_P = T(P_1P_2 \cdots P_k)$ and $\gamma_{P_i} = T(C_i)$ for $1 \leqq i \leqq k$. Then $T(P) = \beta_P + \gamma_{P_1}^{r_1} + \gamma_{P_2}^{r_2} + \cdots + \gamma_{P_k}^{r_k}$. Let $P^* = \{P_1 C_1^{n_1 r_1} P_2 C_2^{n_2 r_2} \cdots P_k C_k^{n_k r_k} \mid P = P_1 C_1^{r_1} P_2 C_2^{r_2} \cdots P_k C_k^{r_k} \in P_B(s, t)$, and $n_i \in Z^+ \cup \{0\}$ for $1 \leqq i \leqq k\}$. Let $T(P^*)$ be defined as $T(P^*) = \uplus_{Q \in P^*} T(Q)$. Since $P^* \subset P_B(s, t)$, we have

$$T(P^*) = \beta_P + (\gamma_{P_1} \uplus \gamma_{P_2} \uplus \cdots \uplus \gamma_{P_k})^* \subset {}^+ \bigcup_{P \in P_B(s,t)} T(P).$$

Note that $T(P) \subset T(P^*)$. Therefore $\uplus_{P \in P_B(s,t)} T(P) = \uplus_{P \in P_B(s,t)} T(P^*)$. Thus we now have

$$\begin{aligned}
|\alpha_{st}(T)| &= \left| \biguplus_{P \in P_B(s,t)} T(P) \right| = \left| \biguplus_{P \in P_B(s,t)} T(P^*) \right| \\
&= \left| \biguplus_{P \in P_B(s,t)} \beta_P + (\gamma_{P_1} \uplus \gamma_{P_2} \uplus \cdots \uplus \gamma_{P_k})^* \right| \\
&\leqq \left| \biguplus_{P \in P_B(s,t)} \beta_P \right| + 1 \quad \text{(using Theorem 3.11)} \\
&\leqq \left| \biguplus_{P \in P_B(s,t)} T(P) \right| + 1 \\
&\leqq |\alpha_{st}(G)| + 1 \quad \text{(from the definition)} \\
&\leqq A(\text{TTSP}(|E_1|)) + 1 \\
&= |E_1| + 1 \quad \text{(using Theorem 6.2)} \\
&\leqq m
\end{aligned}$$

because $|E_1| \leqq m - 1$ by the assumption that G has a backedge. Thus $A(\text{BTTSP}(m)) \leqq m$. □

COROLLARY 6.7. *For* BTTSP, *the algorithm* ZSC *runs in* $O(n^3 m)$ *time where n is the number of vertices and m is the number of edges.*

Proof. The proof is clear from Theorems 4.4 and 6.6. □

7. Proof of Lemma 6.5. Let $G = (V, E)$ be a TTSP multidigraph with source s and sink t. A *binary decomposition tree for* G, denoted by BDT (G), which was discussed in [26], represents the construction process of G by a binary tree. A binary tree BDT (G) can be created by following the sequence of series and parallel compositions that construct G. Initially we have a set of singletons $\{e \mid e \in E\}$. Suppose we apply a two terminal parallel (respectively, series) composition to two TTSP graphs G_{xy} and G_{uv} and obtain the new TTSP graph G_{ab} where x, u, a are sources and y, v, b are sinks. Then we create BDT (G_{ab}) by creating the root ${}_aP_b$ (respectively, ${}_aS_b$) and make BDT (G_{xy}) a left subtree and BDT (G_{uv}) a right subtree. Thus in BDT (G), every leaf represents an edge in G and each internal node ${}_aP_b$ (respectively, ${}_aS_b$) represents a parallel (respectively, series) composition. Fig. 5(b) shows an example. Note that every path in G has a corresponding route in BDT (G). For example, the path

$$P = v - 3 - b - 5 - c - 7 - d - 8 - e - 9 - w,$$

shown in bold lines in Fig. 5(a), has the following corresponding route in BDT(G):

$$P_{\mathrm{BDT}}: \quad (\mathbf{v}, \mathbf{b}) = 3 - {}_vP_{\mathbf{b}} - {}_aS_{\mathbf{b}} - {}_aS_{\mathbf{c}} - {}_aP_{\mathbf{c}} - {}_sS_{\mathbf{c}} - {}_sS_{\mathbf{d}} - {}_sS_t - {}_{\mathbf{d}}S_t$$
$$- {}_{\mathbf{d}}P_g - {}_{\mathbf{d}}S_g - {}_{\mathbf{d}}S_f - {}_{\mathbf{e}}P_f - {}_{\mathbf{e}}S_f - 9$$
$$= (\mathbf{e}, \mathbf{w}).$$

Note that the vertices shown in bold face in the path P_{BDT}, (b, c, d, and e), appear in P in this order.

Let T_{vw} be the smallest subtree in BDT(G) that includes vertices v and w. (Find the nearest common ancestor and include the appropriate subtree.) Let T_v (respectively, T_w) be the subtree of T_{vw} in which v (respectively, w) exists as shown in Fig. 5(d). We use ${}_aA_b$ for representing either ${}_aS_b$ or ${}_aP_b$. Let ${}_aA_b$, ${}_aA_c$, and ${}_dA_b$ be the root of the subtrees T_{vw}, T_v, and T_w, respectively. Then we have the following lemma:

LEMMA 7.1. *Suppose* ${}_xA_y$ *appears in* P_{BDT}. *If* ${}_xA_y$ *appears in* T_v, *then* y *is in* P. *If* ${}_xA_y$ *appears in* T_w, *then* x *is in* P.

Proof. Suppose ${}_xA_y$ appears in T_v. The vertex v is in the TTSP graph with source x and sink y. Thus every path from v to a vertex that is not in T_v must pass through y. We can prove the other case in the same way. □

COROLLARY 7.2. *Suppose there is a* $v-w$ *path in* G *and* T_v, T_w, *and* T_{vw} *are defined as above. Let* ${}_aA_b$, ${}_aA_c$, *and* ${}_dA_b$ *be the roots of the subtrees* T_{vw}, T_v, *and* T_w, *respectively. Then we have the following*:

(1) ${}_aA_b = {}_aS_b$; *that is, the root of* T_{vw} *corresponds to a series composition, and* $c = d$.

(2) *Every path from* v *to* t *passes through the vertex* c.

(3) *Every path from* s *to* w *passes through the vertex* c.

(4) *Any* $v-t$ *path and any* $s-w$ *path intersect at some vertex.*

Proof. (1) If the root of T_{vw} corresponds to a parallel composition, there is no path from v to w. Thus ${}_aA_b = {}_aS_b$. And the series composition identifies the sink of ${}_aC_c$ and the source of ${}_dA_b$, thus $c = d$.

(2) Since there is a $w-t$ path, $t \notin T_v$. Therefore, from the proof of the above lemma, every path from v to t passes through the vertex c.

(3) We can prove this in the same way as (2).

(4) This is obvious from (2) and (3). □

Proof of Lemma 6.5. Let k be the number of backedges in P. Let B_{xy} (P_{xy}) denote an $x-y$ path in G_B (G). We prove the lemma by induction on k.

Suppose $k=1$ and let $e=(w, v)$ be the backedge in P. Note that there must be a $v-w$ path in the underlying TTSP graph G. P can be represented as $P = P_{sw}eP_{vt}$. P_{sw} and P_{vt} are paths in G, since $k=1$, so that from Corollary 7.2, they pass through the same vertex c. Therefore, we can express P as $P = P_{sc}P_{cw}eP_{vc}P_{ct}$. Thus we obtain the cycle $C_1 = P_{cw}eP_{vc}$.

Suppose the lemma holds for numbers less than k. Let $E_B = \{e_1, e_2, \cdots, e_l\}$ be the backedges that appear in P in this order. Let $e_i = (w_i, v_i)$ for $1 \leqq i \leqq l$. Let $e_f = (w_f, v_f)$ be the last backedge in E_B such that there is a path from v_f to w_1 in G. Assume that $e_f \neq e_1$. (When $e_f = e_1$, we can easily modify the following proof.) Then as shown in Fig. 5(e), P can be represented as $P = P_{sw_1}e_1B_{v_1w_f}e_fB_{v_ft}$. Let P_{sv_f} be an arbitrary $s-v_f$ path in G, and let $P_1 = P_{sv_f}B_{v_ft}$. Then P_1 has l backedges where $l \leqq k-1$, because e_1 is not on P_1. From the induction hypothesis, P_1 can be formed from an $s-t$ path P_{st} and cycles $\{C_j \mid j \in J\}$. Note that P_{sv_f} is part of P_{st}; that is, there exists a $v_f - t$ path P_{v_ft} such that $P_{st} = P_{sv_f}P_{v_ft}$. Suppose not. Let $e=(x, y)$ be the first edge in P_{sv_f} such that $y \notin P_{st}$. Then there exists a backedge (z, x) in B_{v_ft} and a cycle C_j such that $(z, x) \in C_j$. Since there exists an $x - v_f$ path and a $v_f - w_1$ path in G, there exists an $x - w_1$ path in G. This contradicts the definition of v_f, since the backedge (z, x) is in $B_{v_f t}$, and thus appears after e_f in E_B.

Thus $P_{st} = P_{sv_f} P_{v_f t}$, and the path P consists of $P_{sw_1} e_1 B_{v_1 w_f} e_f$, $P_{v_f t}$, and cycles $\{C_j \mid j \in J\}$. Let $P_2 = P_{sw_1} e_1 B_{v_1 w_f} e_f P_{v_f t}$. Since there is a $v_f - w_1$ path in G, from Corollary 7.2, P_{sw_1} and $P_{v_f t}$ intersect at some vertex c. Thus P_2 can be expressed as $P_2 = P_{sc} \tilde{C} P_{ct}$ where $\tilde{C} = P_{cw_1} e_1 B_{v_1 w_f} e_f P_{v_f c}$. Therefore the path P consists of the path $P_{sc} P_{ct}$, cycle $\tilde{C}$, and cycles $\{C_j \mid j \in J\}$. □

8. Conclusion. We showed that the two operations of vector summation (+) and convex hull of union (⊎) defined on the set of convex polygons form a closed semiring. We then investigated some properties of these operations. For example, the + operation can be done in $O(m)$ time, where m is the number of edges involved in the operation.

We then obtained the algorithm ZSC by using Kleene's closure algorithm on the above closed semiring. The algorithm ZSC solves the two-dimensional zero-sum cycle problem, which has a close relationship to the problem of acyclicity in two-dimensional regular electrical circuits. The complexities of our algorithm ZSC in some special cases are $O(n^3)$ time for the one-dimensional labeling case, $O(n^4 M)$ time for M-bounded graphs, and $O(n^3 m)$ time for BTTSP graphs, where n is the number of vertices and m is the number of edges. We also showed that the undirected version of the zero-sum cycle problem can be solved in $O(m \log m)$ time and that the zero-sum *simple* cycle problem is NP-complete.

We make the following conjecture about the number of edges of the convex polygons that appear in the algorithm ZSC:

CONJECTURE. *Let G, T, and $\alpha_{ij}(T)$ be defined in the same way as in the text. Then*

$$A(G) = \max_{i,j,T} |\alpha_{ij}(T)| \leqq m,$$

where m is the number of edges in G.

If this conjecture is true, then algorithm ZSC runs in $O(n^3 m)$ time on general graphs.

After the extended abstract of this paper appeared in [16], Kosaraju and Sullivan [18] showed that the zero-sum cycle problem for any dimension can be formulated in terms of linear programming, and thus is solvable in polynomial-time; Cohen and Megiddo [6] proved that the zero-sum cycle problem for any fixed dimension belongs to the class NC and can be solved in the two-dimensional case in serial time $O(nm)$. As mentioned in the Introduction, we hope the results in the present paper are of interest as a new connection between convex polygons and semirings, and as a novel application of Kleene's closure algorithm, even though faster algorithms are now available for the zero-sum cycle problem.

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