# An Analytical Approach to Root Loci*

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*Summary*—**The general algebraic equations of root loci for real $K$ are found in polar and Cartesian coordinates. A synthesis method is then suggested which leads to linear equations in the coefficients of the open-loop transfer function when closed-loop poles and their corresponding gains are specified. Equations are also found for the gain corresponding to a given point on the root locus.**

**A superposition theorem is presented which shows how the root loci for two open-loop functions place constraints on the locus for their product. With a knowledge of the simple lower-order loci, this theorem can be used in sketching and constructing root loci.**

## I. Introduction

In the usual application of the root locus technique, points are found, by a more or less trial and error procedure, at which the open-loop function is negative real. The 180° locus of the open-loop function is then sketched in the region of interest, and calibrated in terms of gain. While this graphical approach is effective in many practical problems, it is of interest to investigate the actual algebraic equations of the root loci. First of all, these equations can be used to plot, or to help sketch, the loci. Also, the equations can be used to synthesize prescribed closed-loop poles. The development will be considerably expedited by allowing the gain constant $K$ to be both positive and negative. This idea will lead to a kind of superposition theorem for root loci, which can also be used as an aid in sketching the loci. In some cases, this approach provides exact geometrical construction procedures.

We shall be concerned with the locus of the closed-loop poles of the single-loop feedback structure shown in Fig. 1, although the results will be directly applicable

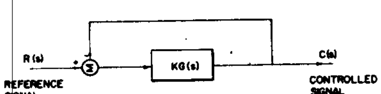


Fig. 1—The basic single-loop system.

to any system where a parameter enters linearly into the characteristic equation. The open-loop transfer function will be assumed to be a real rational function of $s$, such as is encountered in the analysis of linear, lumped, finite systems. We shall assume that $N(s)$, the numerator of $G(s)$, is of degree $n$; that the denominator $D(s)$ is of degree $d$; and that $N(s)$ and $D(s)$ have leading coefficients of unity. Thus, we may write

$$KG(s) = K\frac{N(s)}{D(s)} = K\frac{s^n + a_{n-1}s^{n-1} + \cdots + a_0}{s^d + b_{d-1}s^{d-1} + \cdots + b_0} = K\frac{\sum_{k=0}^{n} a_k s^k}{\sum_{k=0}^{d} b_k s^k}, \qquad (1)$$

where the $a_i$, $b_i$ are real, and $a_n = b_d = 1$.

We shall now define the *root locus* corresponding to the open-loop function $G(s)$ to be the locus of the poles of the closed-loop system as the gain constant $K$ takes on all real values, $-\infty \leq K \leq +\infty$. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)}, \qquad (2)$$

so that the closed-loop poles for a given value of $K$ are given by the solutions of the equation

$$1 + KG(s) = 0, \qquad (3)$$

or

$$G(s) = -\frac{1}{K} \cdot \qquad (4)$$

Since $K$ takes on all real values, any value of $s$ for which $G(s)$ is real will be a solution of (4). Therefore, the root locus is just the image in the $s$ plane of the entire real axis in the $G$ plane, and the equation of the root locus can be expressed as[1,2]

$$\text{Im}\,[G(s)] = 0, \qquad (5)$$

or

$$\arg\,[G(s)] = 0°, 180°, 360°, \cdots. \qquad (6)$$

We shall call those segments of the root locus for which the argument of $G(s)$ is 0°, or an even multiple of 180°, the 0° locus; and similarly, those segments for which the argument of $G(s)$ is an odd multiple of 180°, we shall call the 180° locus. Clearly, the 0° locus and the 180° locus can intersect only at infinity, or at a zero or pole of $G(s)$. Eq. (5) will give us the equation of the entire root locus, and it will remain for us to determine which segments are on the 0° locus and which are on the 180° locus.

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[1] F. M. Reza, "Some mathematical properties of root loci for control systems design," *Trans. AIEE*, vol. 75 (*Commun. and Electronics*), pp. 103–108; March, 1956.

[2] H. Lass, "A note on the root locus method," Proc. IRE (Correspondence), vol. 44, p. 693; May, 1956.

The well-known properties of root loci are susceptible to obvious extensions under the more general definition. For instance, segments of the real axis with an odd total number of poles and zeros to the right are on the 180° locus; and the other segments are on the 0° locus. The asymptotes at infinity are at angles

$$\pm \frac{k360°}{n-d} \qquad k = 0, 1, 2, \cdots \tag{7}$$

for the 0° locus, and

$$\pm \frac{k360° + 180°}{n-d} \qquad k = 0, 1, 2, \cdots \tag{8}$$

for the 180° locus. Furthermore, these asymptotes radiate from the asymptotic center

$$\sigma_\infty = \frac{a_{n-1} - b_{d-1}}{d-n}. \tag{9}$$

We shall use the notation of Yeh,[3] and denote the root locus for an open-loop function with $n$ zeros and $d$ poles by $T(n, d)$.

## II. The Equations of Root Loci in Polar Coordinates

We now turn to the problem of finding the general algebraic equations of root loci. First, we shall find the equations in terms of the polar coordinates shown in Fig. 2. Substituting $s = Re^{j\theta}$ into (1), we have for $G(s)$

$$G(s) = \frac{\sum_{k=0}^{n} a_k R^k e^{jk\theta}}{\sum_{l=0}^{d} b_l R^l e^{jl\theta}}. \tag{10}$$

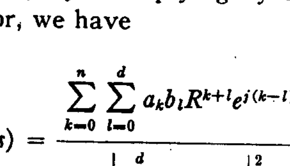


Fig. 2—The polar coordinates, $R$ and $\theta$.

Rationalizing (10) by multiplying by the conjugate of the denominator, we have

$$G(s) = \frac{\sum_{k=0}^{n}\sum_{l=0}^{d} a_k b_l R^{k+l} e^{j(k-l)\theta}}{\left|\sum_{l=0}^{d} b_l R^l e^{jl\theta}\right|^2}. \tag{11}$$

[3] V. C. M. Yeh, "The study of transients in linear feedback systems by conformal mapping and root locus," *Trans. ASME*, vol. 76, pp. 349–361; April, 1954.

We may now set the imaginary part of $G(s)$ to zero as in (5) to obtain the equation of the root locus

$$\sum_{k=0}^{n}\sum_{l=0}^{d} a_k b_l R^{k+l} \sin (k-l)\theta = 0. \tag{12}$$

Since the real axis will always be part of the root locus, $R \sin \theta$ will be a factor of (12). Removing this factor, we have as the equation of the nonreal root locus

$$\sum_{k=0}^{n}\sum_{l=0}^{d} a_k b_l R^{k+l-1} \frac{\sin (k-l)\theta}{\sin \theta} = 0. \tag{13}$$

We may recognize the trigonometric functions in (13) as Tchebycheff polynomials[4] of the second kind in $\cos \theta$, defined by

$$U_{n-1}(\cos \theta) = \frac{\sin n\theta}{\sin \theta}. \tag{14}$$

Thus, the nonreal locus may be written in terms of these polynomials as

$$\sum_{k=0}^{n}\sum_{l=0}^{d} a_k b_l R^{k+l-1} U_{k-l-1}(\cos \theta) = 0. \tag{15}$$

If $\theta$ is prescribed, tables will facilitate the numerical evaluation of the $U_{k-l-1}(\cos \theta)$, and the resultant polynomial in $R$ will have roots at the intersections of the line $\theta = \text{const.}$ with the nonreal root locus.

## III. The Equations of Root Loci in Cartesian Coordinates

To find the root locus equation in terms of $\sigma$ and $\omega$, we shall use an idea that was suggested by Bendrikov and Teodorchik.[5] The idea is that of expanding $N(s)$ and $D(s)$ in a power series in $j\omega$.

$N(s)$ is analytic everywhere in the $s$ plane, and can indeed be expanded in a Taylor series about any $\sigma$ with an infinite radius of convergence to obtain the identity

$$N(\sigma + j\omega) = N(\sigma) + j\omega \frac{N'(\sigma)}{1!} + (j\omega)^2 \frac{N''(\sigma)}{2!} + \cdots + (j\omega)^n \frac{N^{(n)}(\sigma)}{n!}. \tag{16}$$

Grouping the real and imaginary components of this expression, we have

$$N(\sigma + j\omega) = \left[N(\sigma) - \omega^2 \frac{N''(\sigma)}{2!} + \cdots\right] + j\omega\left[\frac{N'(\sigma)}{1!} - \omega^2 \frac{N'''(\sigma)}{3!} + \cdots\right]. \tag{17}$$

[4] See, for example, "Tables of Chebyshev Polynomials $S_n(x)$ and $C_n(x)$," Natl. Bureau of Standards Appl. Math. Ser., no. 9, Washington, D. C.; 1952.

[5] G. A. Bendrikov and K. F. Teodorchik, "The analytical theory of constructing root loci," *Avtomat. i Telemekh.* (*Automation and Remote Control*), vol. 20; March, 1959. (English Translation.)

When this is also done for $D(s)$, $G(s)$ may be written

$$G(s) = \frac{\left[N(\sigma) - \omega^2 \frac{N''(\sigma)}{2!} + \cdots\right] + j\omega\left[\frac{N'(\sigma)}{1!} - \omega^2\frac{N'''(\sigma)}{3!} + \cdots\right]}{\left[D(\sigma) - \omega^2 \frac{D''(\sigma)}{2!} + \cdots\right] + j\omega\left[\frac{D'(\sigma)}{1!} - \omega^2\frac{D'''(\sigma)}{3!} + \cdots\right]}. \tag{18}$$

Multiplying by the conjugate of the denominator, and setting the imaginary part equal to zero as before, we have as the equation of the nonreal root locus

$$\left[\frac{N(\sigma)D'(\sigma)}{0!1!} - \frac{N'(\sigma)D(\sigma)}{1!0!}\right] - \omega^2\left[\frac{N(\sigma)D'''(\sigma)}{0!3!} - \frac{N'(\sigma)D''(\sigma)}{1!2!} + \frac{N''(\sigma)D'(\sigma)}{2!1!} - \frac{N'''(\sigma)D(\sigma)}{3!0!}\right] + \omega^4\left[\frac{N(\sigma)}{0!}\frac{D^{V}(\sigma)}{5!} - \cdots\right] - \cdots = 0. \tag{19}$$

This may be written

$$Q_1(\sigma) - \omega^2 Q_3(\sigma) + \omega^4 Q_5(\sigma) - \cdots = 0, \tag{20}$$

where $Q_R(\sigma)$ is a polynomial in $\sigma$ defined by

$$Q_R(\sigma) = \sum_{r=0}^{R} (-1)^r \frac{N^{(r)}(\sigma)}{r!} \frac{D^{(R-r)}(\sigma)}{(R-r)!}. \tag{21}$$

In general, $Q_R(\sigma)$ will be of degree $d+n-R$. The author has not been able to find an interpretation for all the polynomials $Q_R(\sigma)$. However, it is evident from (21) that

$$Q_{n+d}(\sigma) = (-1)^n. \tag{22}$$

It can also be shown that

$$Q_{n+d-1}(\sigma) = (-1)^n[(d-n)\sigma - (a_{n-1} - b_{d-1})], \tag{23}$$

so that if $n \neq d$,

$$Q_{n+d-1}(\sigma) = (-1)^n(d-n)(\sigma - \sigma_\infty). \tag{24}$$

Also, $Q_1(\sigma)$ can be factored by observing that the zeros of $Q_1(\sigma)$ are the roots of the equation

$$Q_1(\sigma) = N(\sigma)D'(\sigma) - N'(\sigma)D(\sigma) = 0, \tag{25}$$

or

$$N(\sigma)D(\sigma)\left[\frac{d}{d\sigma}\log\frac{N(\sigma)}{D(\sigma)}\right] = 0. \tag{26}$$

Therefore, $Q_1(\sigma)$ has zeros of appropriate orders at all the zeros of the logarithmic derivative of $G(s)$, whether they are on the root locus or not, and at the multiple poles and zeros of $G(s)$. The zeros of the logarithmic derivative represent multiple points on the phase loci of $G(s)$, and will here be called *critical points*.[6,7] By writing out the first few terms in (25), it can be seen that $Q_1(\sigma)$ has a leading coefficient of $(d-n)$, if $n \neq d$. Hence, we may write

$$Q_1(\sigma) = (d-n)(\sigma - s_{k1})(\sigma - s_{k2}) \cdots (\sigma - s_{k(n+d-1)}), \tag{27}$$

where the $s_k$ may be complex and are solutions of (25). Ur[8] has shown that the asymptotes of the root locus for a system with no poles or zeros at infinity can be obtained by considering the asymptotes of the system $(N-D)/D$. Indeed, the root locus for $(N-D)/D$ is the same as that for $N/D$, since the root locus is defined by

$$\text{Im}\left\{\frac{N}{D}\right\} = \text{Im}\left\{\frac{N-D}{D}\right\} = 0. \tag{28}$$

The lower-order loci may now be written in terms of the solutions of (26) and the asymptotic center with a little more effort.

The locus for $G$ is the same as that for $1/G$, so that $T(n, d) = T(d, n)$, and we need only consider loci for which $n \leq d$. The loci $T(1, 2)$ and $T(2, 2)$ reduce to

$$\omega^2 + (\sigma - s_{k1})(\sigma - s_{k2}) = 0, \tag{29}$$

which is in general the equation of a circle that intersects the real axis at $s_{k1}$ and $s_{k2}$. The locus $T(0, 3)$ is

$$3(\sigma - s_{k1})(\sigma - s_{k2}) - \omega^2 = 0, \tag{30}$$

which is the equation of a hyperbola. The loci $T(1, 3)$ and $T(0, 4)$ are the only cubic loci, and represent the

[6] J. L. Walsh, "The Location of Critical Points of Analytic and Harmonic Functions," American Mathematical Society, New York, N. Y.; 1950.

[7] M. Marden, "The Geometry of the Zeros of a Polynomial in a Complex Variable," American Mathematical Society, New York, N. Y.; 1949.

[8] H. Ur, "Root locus properties and sensitivity relations in control systems," IRE TRANS. ON AUTOMATIC CONTROL, vol. AC-5, pp. 57–65; January, 1960.

next order of complexity after the quadratic loci. $T(1,3)$ may be written

$$\omega^2(\sigma - \sigma_\infty) + (\sigma - s_{k1})(\sigma - s_{k2})(\sigma - s_{k3}) = 0, \quad (31)$$

and $T(0, 4)$ may be written

$$\omega^2(\sigma - \sigma_\infty) - (\sigma - s_{k1})(\sigma - s_{k2})(\sigma - s_{k3}) = 0. \quad (32)$$

## IV. Equations Involving the Gain Constant, $K$

The basic equation for the root locus, (3), has a real and imaginary part, and involves $K$ as well as $s$. The equations for the root loci given above are the result of eliminating $K$ between the two equations. If we break up $N(s)$ and $D(s)$ into real and imaginary parts as follows,

$$N(s) = N_R(\sigma, \omega) + j\omega N_I(\sigma, \omega)$$
$$D(s) = D_R(\sigma, \omega) + j\omega D_I(\sigma, \omega), \quad (33)$$

(3) may be written

$$1 + K\frac{N_R(\sigma, \omega) + j\omega N_I(\sigma, \omega)}{D_R(\sigma, \omega) + j\omega D_I(\sigma, \omega)} = 0, \quad (34)$$

or

$$D_R(\sigma, \omega) + KN_R(\sigma, \omega) = 0$$
$$j\omega[D_I(\sigma, \omega) + KN_I(\sigma, \omega)] = 0. \quad (35)$$

These last two simultaneous equations fully represent the calibrated locus. In terms of polar coordinates, these equations are

$$\sum_{k=0}^{d} b_k R^k T_k(\cos\theta) + K\sum_{k=0}^{n} a_k R^k T_k(\cos\theta) = 0$$
$$j\omega\left[\sum_{k=1}^{d} b_k R^{k-1} U_{k-1}(\cos\theta) + K\sum_{k=1}^{n} a_k R^{k-1} U_{k-1}(\cos\theta)\right] = 0. \quad (36)$$

Here, $T_n(\cos\theta)$ is a Tchebycheff polynomial of the first kind, defined by

$$T_n(\cos\theta) = \cos n\theta. \quad (37)$$

Again, it might be mentioned that tables of these polynomials can facilitate computation. In terms of Cartesian coordinates, these equations are

$$\left[\frac{D(\sigma)}{0!} - \omega^2\frac{D''(\sigma)}{2!} + \cdots\right] + K\left[\frac{N(\sigma)}{0!} - \omega^2\frac{N''(\sigma)}{2!} + \cdots\right] = 0$$
$$j\omega\left\{\left[\frac{D'(\sigma)}{1!} - \omega^2\frac{D'''(\sigma)}{3!} + \cdots\right] + K\left[\frac{D'(\sigma)}{1!} - \omega^2\frac{N'''(\sigma)}{3!} + \cdots\right]\right\} = 0. \quad (38)$$

These equations may be used to synthesize desired closed-loop characteristics. Suppose, for example, that we require the pole-pair $s = R_1e^{\pm j\theta_1} = \sigma_1 + j\omega_1$ to be closed-loop poles at a value of gain $K = K_1$. Then these values may be substituted into (36) or (38) to give two linear equations in the coefficients. Thus, we may specify all but two coefficients and have a closed-loop pole-pair determine these remaining two. If we specify a real closed-loop pole and its corresponding value of $K$, we need only substitute in the first of (36) or (38). It can be seen in fact that by specifying points in the space $(s, K)$, we can determine any number of the coefficients by linear equations. The resultant equations will be linear in the coefficients, $a_i$ and $b_i$, but will not be linear in the root positions. Thus, while the use of synthesis procedures based on (36) and (38) will lead to linear algebra in the solution for the coefficients, it will remain for the designer to bridge the gap between the unknown pole-zero positions and the coefficients in the polynomials $N(s)$ and $D(s)$.

Eqs. (35) can be solved for $K$ as follows:

$$K = -\frac{D_R}{N_R} = -\frac{D_I}{N_I}. \quad (39)$$

Since $\omega$ is a factor of the second of (35), the second of (39) is necessarily valid only off the real axis, where $\omega \neq 0$. This restriction also applies to the second of (40), (41), and (43), which follow. We may now write $K$ on the root locus in terms of the coordinates in the $s$ plane and the coefficients in the open-loop transfer function. First in terms of the polar coordinates, $R$ and $\theta$:

$$K = -\frac{\sum_{k=0}^{d} b_k R^k T_k(\cos\theta)}{\sum_{k=0}^{n} a_k R^k T_k(\cos\theta)} = -\frac{\sum_{k=1}^{d} b_k R^{k-1} U_{k-1}(\cos\theta)}{\sum_{k=1}^{n} a_k R^{k-1} U_{k-1}(\cos\theta)}. \quad (40)$$

Or, in terms of $\sigma$ and $\omega$:

$$K = -\frac{\frac{D(\sigma)}{0!} - \omega^2\frac{D''(\sigma)}{2!} + \omega^4\frac{D^{IV}(\sigma)}{4!} - \cdots}{\frac{N(\sigma)}{0!} - \omega^2\frac{N''(\sigma)}{2!} + \omega^4\frac{N^{IV}(\sigma)}{4!} - \cdots} = -\frac{\frac{D'(\sigma)}{1!} - \omega^2\frac{D'''(\sigma)}{3!} + \omega^4\frac{D^{V}(\sigma)}{5!} - \cdots}{\frac{N'(\sigma)}{1!} - \omega^2\frac{N'''(\sigma)}{3!} + \omega^4\frac{N^{V}(\sigma)}{5!} - \cdots}. \quad (41)$$

$(R, \theta)$ and $(\sigma, \omega)$ in these equations are points on the root locus.

It is often of interest to find the intersections of the root locus with the $j\omega$ axis. To find these crossover points, we set $\cos\theta=0$ in (15) or $\sigma=0$ in (19). After some simplification, we have

$$(a_0b_1 - a_1b_0) - \omega^2(a_0b_3 - a_1b_2 + a_2b_1 - a_3b_0) + \omega^4(a_0b_5 - a_1b_4 + \cdots) - \cdots = 0. \quad (42)$$

The real solutions of this equation will give the values of $\omega$ at which the locus crosses the $j\omega$ axis. To find the values of $K$ corresponding to these crossover points, we set $\cos\theta=0$ in (40) or $\sigma=0$ (41) to obtain

$$K = -\frac{b_0 - b_2\omega^2 + b_4\omega^4 - \cdots}{a_0 - a_2\omega^2 + a_4\omega^4 - \cdots} = -\frac{b_1 - b_3\omega^2 + b_5\omega^4 - \cdots}{a_1 - a_3\omega^2 + a_5\omega^4 - \cdots}, \quad (43)$$

where $\omega$ in this equation is an appropriate crossover point, a solution of (42). Note that if either $N(s)$ or $D(s)$ is a constant, both the numerator and denominator of the right-most fraction in (43) vanish, and the first expression must be used for calculations.

## V. A Superposition Theorem for Root Loci

We thus have investigated the general algebraic equations of root loci. We now turn to a kind of superposition theorem for root loci; in particular, we shall show how the root loci for two open-loop functions place constraints on the locus for their product.

*Theorem:* Let $T_1$ be the root locus associated with $G_1$, and let $T_2$ be the locus associated with $G_2$. Then intersections of $T_1$ and $T_2$ are on the root locus associated with $G_1 \cdot G_2$. Furthermore, the locus for $G_1 \cdot G_2$ cannot cross the remaining parts of $T_1$ and $T_2$.

*Proof:* At any point which is on both $T_1$ and $T_2$, $G_1$ and $G_2$ are both real, and hence, so is $G_1 \cdot G_2$. At a point on $T_1$ and not on $T_2$, $G_1$ is real and $G_2$ is not; so that $G_1 \cdot G_2$ is not real and this point is not on the root locus for $G_1 \cdot G_2$.

When a point on the root locus is found by this theorem, the angle of $G=G_1 \cdot G_2$ at this point can be determined by adding the angles of $G_1$ and $G_2$. Thus, if a point is on the 0° locus of $T_1$ and the 180° locus of $T_2$, for instance, the point must be on the 180° locus of $G_1 \cdot G_2$. On the other hand, if the point is on the 180° locus of both $T_1$ and $T_2$, it is on the 0° locus of $G_1 \cdot G_2$, and so on.

This theorem is most useful when the total open-loop function can be broken up into the product of two other functions whose loci can be drawn immediately. It is important, therefore, that the user of this theorem be able to draw immediately as many loci as possible. For $T(0, 1)$, and $T(1, 1)$, or for any function which has simple poles and zeros alternating on the real axis, there is no nonreal locus. $T(0, 2)$ is a line $\sigma=$const. through the center of gravity of the poles. As pointed out by Yeh,[8] if the open-loop function consists just of an $N$th order pole, the root locus coincides with the asymptotes. More generally, Lorens and Titsworth[9] state that the locus coincides with an asymptote if the pole-zero pattern is symmetric about the asymptote line extended through the asymptotic center. The Loci $T(1, 2)$ and $T(2, 2)$ are in general circles, and Fig. 3 shows these cases with enough information so each locus can be traced with a compass.

As an example of how this theorem can be used in sketching a locus, consider the open-loop function shown in Fig. 4. The asymptotes are drawn first; then the zero and poles can be divided into various groups for which simple loci can be drawn. The zero at $-3$ can be associated with the double pole at $-5$ and a circle drawn. The locus for the remaining two poles is a straight line through $-3.5$, and the intersections of these two loci give two points on the final locus. Moreover, this circle and line represent barriers for the final locus. Another circle-line combination is possible, and this gives two more points on the locus. When the zero is associated with other pairs of poles, it lies between them and produces no locus off the real axis, and the lines for the remaining two poles represent barriers to the locus. Thus, it is seen that the 180° locus cannot come back to meet the real axis again between the origin and the zero, and the general shape of the final locus can be sketched.

## VI. Construction Procedures for Root Loci

With the device of introducing coincident pole-zero pairs, the preceding ideas give rise to construction procedures for certain loci. The simplest example of this is the hyperbolic locus $T(0, 3)$.

Consider the three-pole open-loop function shown in Fig. 5(a). Now introduce a pole and a zero which coincide, as in Fig. 5(b), so that the open-loop function and the locus is unchanged. We may now take the two real poles as one factor of $G(s)$, and the zero together with the complex pair of poles as the other. As shown in Fig. 5(c), these produce a straight line and a circle, whose intersections give a pair of points on the final locus $T(0, 3)$. By introducing another pole-zero pair along the real axis, another pair of points may be found. In this way, the locus $T(0, 3)$ can be quickly sketched as shown in Fig. 5(d). These loci may in turn be used to sketch higher-order loci.

$T(1, 3)$ can be constructed in a similar manner, as shown in Fig. 6. Here, points are located by the intersections of two circles. To construct a $T(0, 4)$ locus, such as the one shown in Fig. 7, one might introduce a real

[9] C. S. Lorens and R. C. Titsworth, "Properties of root locus asymptotes," IRE Trans. on Automatic Control, vol. AC-5, pp. 71–72; January, 1960.

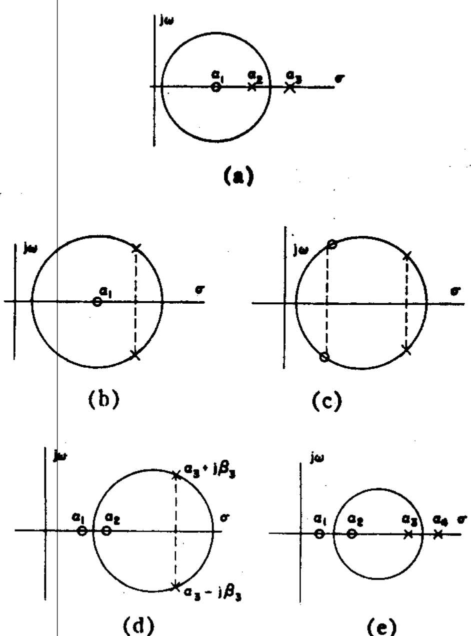


Fig. 3—The circular loci:

$(\sigma - \sigma_0)^2 + \omega^2 = R^2$

(a) $\sigma_0 = \alpha_1$, $R^2 = (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)$

(b) $\sigma_0 = \alpha_1$

(d) $\sigma_0 = \dfrac{\alpha_3^2 + \beta_3^2 - \alpha_1\alpha_2}{2\alpha_3 - \alpha_1 - \alpha_2}$

(e) $\sigma_0 = \dfrac{\alpha_3\alpha_4 - \alpha_1\alpha_2}{\alpha_4 + \alpha_3 - \alpha_2 - \alpha_1}$, $R^2 = \sigma_0^2 + \dfrac{(\alpha_3 + \alpha_4)\alpha_1\alpha_2 - (\alpha_1 + \alpha_2)\alpha_3\alpha_4}{\alpha_4 + \alpha_3 - \alpha_2 - \alpha_1}$

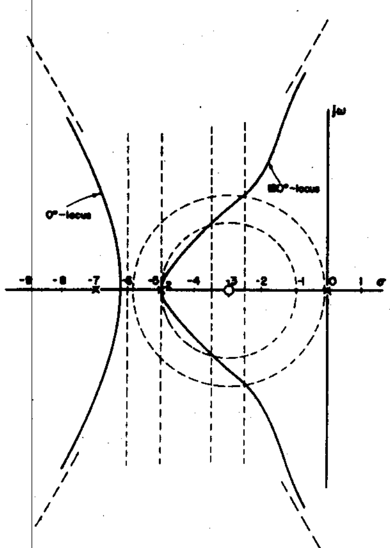


Fig. 4—The locus $T(1, 4)$ for the open-loop system

$$G(s) = \frac{(s+3)}{s(s+5)^2(s+7)}$$

sketched with the aid of the theorem of Section V.

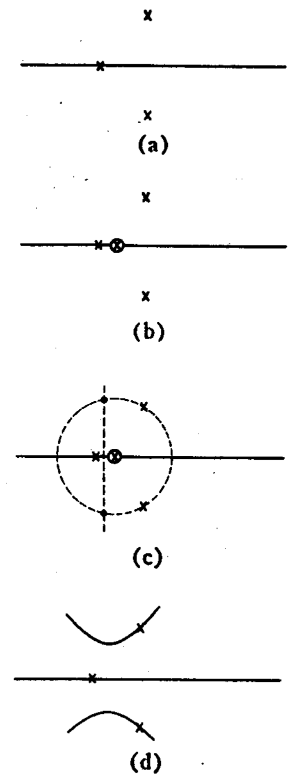


Fig. 5—A graphical procedure for constructing $T(0, 3)$. (a) The open-loop function. (b) The addition of a coincident pole and zero does not change the locus. (c) The composite loci, $T(0, 2)$ and $T(1, 2)$. (d) The final locus constructed as above.

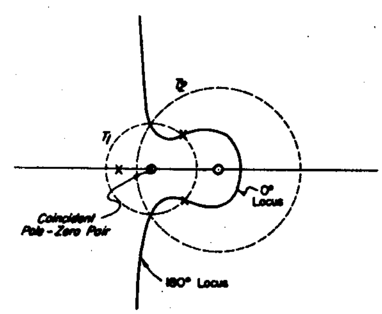


Fig. 6—Construction of the locus $T(1, 3)$ by introduction of coincident pole-zero pairs on the real axis.

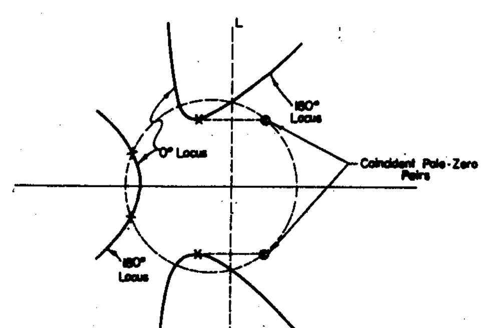


Fig. 7—Construction of the locus $T(0, 4)$ by introduction of complex pole-zero pairs.

pole-zero pair and find the intersections of hyperbolas and circles. The construction of hyperbolas can be avoided, however, in the following way. Introduce a coincident complex pair of zeros and poles with the same imaginary component as the poles in the original function. The four complex poles are now symmetrical in the line $L$, $G$ is real on $L$, and so the line $L$ must be part of the locus $T_1(0, 4)$. The locus $T_2(2, 2)$ is a circle, and this locates two points on the final locus.

## VII. Conclusions

The general algebraic equations of root loci have been presented here in terms of the coefficients of the polynomials $N(s)$ and $D(s)$. We have seen that the specification of closed-loop poles, with their associated gains, leads to linear equations in these coefficients. The necessity of dealing with the coefficients rather than the pole-zero locations is evidently the price to be paid for linear algebra in the synthesis of closed-loop poles.

It has also been shown how the root locus for a higher-order system is restrained by the loci of its lower-order factors. A familiarity with the simple circular loci will enable the designer to use this idea as an aid in sketching loci. In certain cases, this idea leads to exact construction procedures for root loci.

## Appendix

To illustrate the use of the equations for the root locus, we shall consider as an example the $T(1, 3)$ locus constructed in Fig. 6. If the zero is taken at the origin, the function is

$$G(s) = \frac{s}{[(s+1)^2+1](s+3)} = \frac{s}{s^3+5s^2+8s+6}\cdot \tag{44}$$

Thus,

$$\begin{aligned} a_0 &= 0 \qquad & b_0 &= 6 \\ a_1 &= 1 & b_1 &= 8 \\ & & b_2 &= 5 \\ & & b_3 &= 1. \end{aligned} \tag{45}$$

Eq. (13) then becomes

$$\sum_{l=0}^{d} b_l R^l \frac{\sin(1-l)\theta}{\sin\theta} = 6 + 5R^2\frac{\sin(-\theta)}{\sin\theta} + R^3\frac{\sin(-2\theta)}{\sin\theta} = 0. \tag{46}$$

or

$$2R^3\cos\theta + 5R^2 - 6 = 0. \tag{47}$$

Since $R\cos\theta = \sigma$, this may be written simply as

$$R^2 = \frac{6}{5+2\sigma}, \tag{48}$$

which can be plotted quickly on polar graph paper.

Alternatively, we may find the equation of the root locus in Cartesian coordinates. Then,

$$\begin{aligned} \frac{N(\sigma)}{0!} &= \sigma \qquad & \frac{D(\sigma)}{0!} &= \sigma^3+5\sigma^2+8\sigma+6 \\ \frac{N'(\sigma)}{1!} &= 1 & \frac{D'(\sigma)}{1!} &= 3\sigma^2+10\sigma+8 \\ & & \frac{D''(\sigma)}{2!} &= 3\sigma+5 \\ & & \frac{D'''(\sigma)}{3!} &= 1; \end{aligned} \tag{49}$$

and (19) becomes

$$[-(\sigma^3+5\sigma^2+8\sigma+6)+\sigma(3\sigma^2+10\sigma+8)] - \omega^2[\sigma(1)-1(3\sigma+5)] = 0, \tag{50}$$

or

$$2\sigma^3+5\sigma^2-6+\omega^2[2\sigma+5] = 0, \tag{51}$$

which is, of course, the same as (47). To plot this, we may wish to solve for $\omega^2$

$$\omega^2 = -\frac{2\sigma^3+5\sigma^2-6}{2\sigma+5}\cdot \tag{52}$$

The gain constant on the nonreal root locus may be found by the second of (40) or (41) to be

$$K = \omega^2 - 3\sigma^2 - 10\sigma - 8. \tag{53}$$

To find the crossings of the $j\omega$ axis, for example, we have, by (48) or (52), and (53), with $\sigma = 0$

$$\omega^2 = \frac{6}{5}$$

and

$$K = -\frac{34}{5}\cdot \tag{54}$$

Thus, these two crossover points are on the 0° locus. This may be verified by substituting directly in the system function

$$G\left(j\sqrt{\frac{6}{5}}\right) = \frac{j\sqrt{\frac{6}{5}}}{-j\frac{6}{5}\sqrt{\frac{6}{5}} - 5\frac{6}{5} + 8j\sqrt{\frac{6}{5}} + 6} = \frac{1}{34/5} = -\frac{1}{K}\cdot \tag{55}$$

## Acknowledgment

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