An *RC* Impedance Approximant to $s^{-1/2}$*

Usually, time domain synthesis procedures do not lead easily to realizations. Quite by accident the author came across a method for approximating a $\sqrt{t}$ step response which leads directly to simple *RC* realizations with known component values. It is hoped that such an example will be of interest to readers of these TRANSACTIONS.

The method is based on the observation that

$$(1 - s^{-1/2})^n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \tag{1}$$

if $|1 - s^{-1/2}| < 1$. This region of convergence includes the half-plane $Re\,\{s\} > 1/4$. When (1) is expanded as a binomial and the terms in $s^{-1/2}$ collected, the following rational approximant is obtained:

$$Z_n(s) = \frac{\sum_{k=0,2,\cdots}^{n-1} \binom{n}{k} s^{-k/2}}{\sum_{k=1,3,\cdots}^{n} \binom{n}{k} s^{-(k-1)/2}} = \frac{\sum_{k=1,3,\cdots}^{n} \binom{n}{k} s^{(k-1)/2}}{\sum_{k=0,2,\cdots}^{n-1} \binom{n}{k} s^{k/2}} \rightarrow s^{-1/2}$$

where n is odd. We shall take n to be odd throughout the remainder of this communication. $Z_n(s)$ can also be written

$$Z_n(s) = \left[\frac{(1 + s^{1/2})^n - (1 - s^{1/2})^n}{(1 + s^{1/2})^n + (1 - s^{1/2})^n}\right] s^{-1/2} \tag{2}$$

and the zeros and poles of $Z_n(s)$ are found from this expression to lie on the negative-real axis at the following points:

zeros: $\sigma = -\tan^2 k\pi/n$

poles: $\sigma = -1/\tan^2 k\pi/n, \quad k = 1, 2, \cdots, (n-1)/2.$

Therefore, since $Z_n(\infty) = 1/n$,

$$Z_n(s) = \frac{1}{n} \prod_{k=1}^{(n-1)/2} \frac{s + \tan^2 k\pi/n}{s + 1/\tan^2 k\pi/n}.$$

By using (2) again to evaluate the residues, $Z_n(s)$ can be expanded in a partial fraction expansion as follows:

$$Z_n(s) = \frac{1}{n} + \frac{2}{n} \sum_{k=1}^{(n-1)/2} \frac{1 + 1/\tan^2 k\pi/n}{s + 1/\tan^2 k\pi/n}. \tag{3}$$

Since the poles all lie on the negative-real axis and have positive residues, $Z_n(s)$ is an *RC* driving-point impedance. $Z_n(s)$ can be realized directly from (3) with $(n+1)/2$ resistors and $(n-1)/2$ capacitors, a total of n elements.

The step response of $Z_n(s)$ can be found by expanding $Z_n(s)/s$ in a partial fraction expansion:

$$Z_n(s)/s = \frac{n}{s} - \frac{2}{n} \sum_{k=1}^{(n-1)/2} \frac{1 + \tan^2 k\pi/n}{s + 1/\tan^2 k\pi/n} \rightarrow s^{-3/2}$$

or, in the time domain,

$$\text{step response} = n - \frac{2}{n} \sum_{k=1}^{(n-1)/2} (1 + \tan^2 k\pi/n) e^{-t/\tan^2 k\pi/n}. \tag{4}$$

The step response starts at $1/n$ when $t = 0^+$ and becomes asymptotic to n for large t. It thus can approximate

$$\mathcal{L}^{-1}[s^{-3/2}] = 2(t/\pi)^{1/2} \tag{5}$$

well only for $t < \pi n^2/4$.

Fig. 1 shows the over-all behavior of the first few approximations to a $2\sqrt{t/\pi}$ step response. The accuracy in the range of approximation increases quite rapidly. For example, if the error $e(t)$ is defined as the difference between (5) and (4), we have

$n = 3$:	$\lvert e(t)\rvert \leq .07$	for	$0.1 \leq t \leq 5.0$
$n = 5$:	$\lvert e(t)\rvert \leq .03$	for	$0.1 \leq t \leq 10.0$
$n = 7$:	$\lvert e(t)\rvert \leq .011$	for	$0.2 \leq t \leq 14.5$
$n = 9$:	$\lvert e(t)\rvert \leq .004$	for	$0.2 \leq t \leq 19.5.$

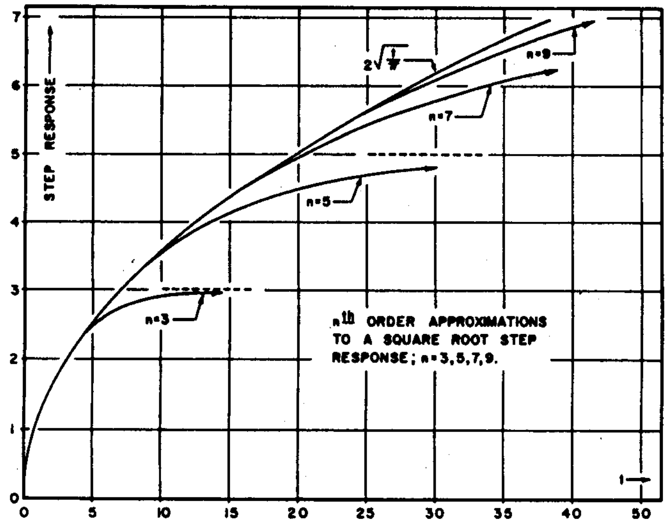

Fig. 1.

We see then that in the special case when a square-root step response is desired, the $Z_n(s)$ give a simple and easily realized solution to the time domain synthesis problem. Of more general interest is the fact that $s^{-1/2}$, an irrational function with a branch point at the origin, can be approximated by rational functions whose poles and zeros alternate on the negative-real axis (a branch cut). Whether or not this is a particular example of a more general situation is an open question.

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