Transmission of an Analog Signal Over a Fixed Bit-Rate Channel

K. STEIGLITZ, MEMBER, IEEE

Abstract—The transmission of a nonbandlimited analog signal over a digital channel with a fixed bit-rate is considered. The trade-off between the mean-square error due to quantizing and the mean-square error due to the process of sampling and reconstructing the signal is investigated. Simple approximations to these errors, which are valid in most practical situations, are derived, and simple expressions are obtained from which the optimum sampling interval and number of bits per sample can be calculated. Results for first-, second-, and third-order Butterworth and flat bandlimited spectra, together with the zero-order hold and the linear point connector, are included. The resulting mean-square error goes to zero with large channel bit-rates in a slower manner than the Shannon limit, which assumes a strictly bandlimited signal and perfect reconstruction.

I. Introduction

In order to send a continuous signal over a noisy channel, some overall error must be tolerated. In particular, if a digital link is used and the bit error rate is kept very small by proper coding, the predominant errors will be caused by the quantizing process and by the process of sampling at discrete times and reconstructing the continuous signal. The conflicting requirements imposed by these two errors for a fixed bit-rate channel are evident. On the one hand, if we quantize very finely, we need to send long words and hence cannot afford to sample fast. On the other hand, if we sample very fast, we can afford to quantize only coarsely. We are therefore presented with the problem of optimizing our choice of sampling interval and quantizing fineness.

In this paper, an approximate solution to the problem is derived with reasonable assumptions about the nature of the signals and reconstruction devices. This solution can be used as a general guide in the choice of quantizing fineness and sampling rate in practical situations.

We will assume that the continuous signal to be transmitted, $f(t)$, is a sample function of a wide-sense stationary, ergodic random process with correlation function $\varphi(t)$ and power spectral density $\Phi(\omega)$. This signal will be quantized uniformly into 2^N levels and then sampled at a uniform rate corresponding to a sampling interval T. Thus, each sample point of f will be coded into an N-bit word. When we impose the restriction that the binary coded version of f is to be sent over a channel at a rate R bps, we arrive at the constraint

$$\frac{N}{T} = R. \tag{1}$$

The fundamental problem of this paper will be to choose N and T subject to the constraint (1) so as to minimize the mean-square error between the reconstructed signal and the original signal $f(t)$.

II. The Total Mean-Square Error

Figure 1 is a block diagram of the system to be considered, with the quantizing errors represented by additive noise before the sampler. Quantizing noise can always be so represented, but will be, strictly speaking, statistically dependent on the signal $f(t)$. However, Katzenelson [1], [2] and Watts [2] have shown that, in the Gaussian case with relatively small quantization grain q, the correlation between the signal and the quantization noise is extremely small: of the order of 10^{-8} compared with the autocorrelation function $\varphi(t)$ of the signal. For our purposes, the quantization noise will be represented by uncorrelated additive noise with correlation function $\varphi_{nn}(t)$.

With reference to Fig. 1, using well-known techniques of sampled-data analysis [3], [4] we can write the mean-square error as

$$\overline{\epsilon^2(T)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\Phi(\omega) + \frac{G(\omega, T)G(-\omega, T)}{T^2} \cdot \sum_{n=-\infty}^{\infty}\left[\Phi\left(\omega - n\frac{2\pi}{T}\right) + \Phi_{nn}\left(\omega - n\frac{2\pi}{T}\right)\right] - \frac{1}{T}\left[G(\omega, T) + G(-\omega, T)\right]\Phi(\omega)\right\} d\omega \tag{2}$$

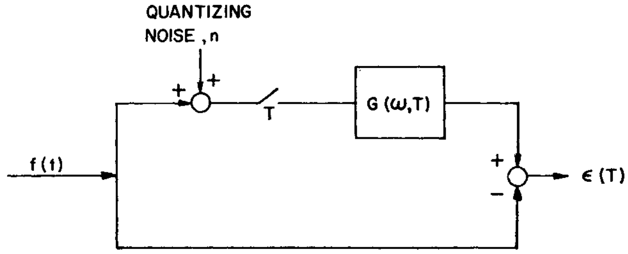

Fig. 1. The quantizing, sampling, and reconstruction process.

Manuscript received August 9, 1965; revised February 11, 1966. This work was supported by Army Research Office Contract DA-31-124-ARO-D-292, and made use of computer facilities sponsored in part by the National Science Foundation Grant NSF-GP579.
The author is with the Department of Electrical Engineering, Princeton University, Princeton, N. J.

where $G(\omega, T)$ is the transfer function of the linear reconstruction filter. On introducing in the summation the change of variable

$$\omega - n\frac{2\pi}{T} \to \omega$$

and rearranging terms, we find that this becomes

$$\begin{aligned}\overline{\epsilon^2(T)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega) \\ \cdot\left\{1 + \sum_{n=-\infty}^{\infty} \frac{G\left(\omega + n\frac{2\pi}{T}, T\right)G\left(-\omega - n\frac{2\pi}{T}, T\right)}{T^2}\right. \\ \left. - \frac{1}{T}[G(\omega, T) + G(-\omega, T)]\right\} d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{nn}(\omega) \\ \cdot \sum_{n=-\infty}^{\infty} \frac{G\left(\omega + n\frac{2\pi}{T}, T\right)G\left(-\omega - n\frac{2\pi}{T}, T\right)}{T^2} d\omega. \qquad (3)\end{aligned}$$

We will now make the assumption that the reconstruction filter does not change its shape as T varies, that is, that

$$G(\omega, T) = TH(\omega T).$$

The function $H(u)$ will be referred to as the normalized reconstruction filter. Most of the commonly used reconstruction filters can be represented in this way. With this assumption, the mean-square error can be written

$$\begin{aligned}\overline{\epsilon^2(T)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega)\left\{1 + \sum_{n=-\infty}^{\infty} H(\omega T + n2\pi)\right. \\ \left.\cdot H(-\omega T - n2\pi) - [H(\omega T) + H(-\omega T)]\right\} d\omega \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{nn}(\omega)\left[\sum_{n=-\infty}^{\infty} H(\omega T + n2\pi)H(-\omega T - n2\pi)\right] d\omega, \qquad (4)\end{aligned}$$

or, letting $u = \omega T$,

$$\begin{aligned}\overline{\epsilon^2(T)} = \overline{\epsilon_R^2(T)} + \overline{\epsilon_Q^2(T)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{T}\Phi\left(\frac{u}{T}\right)\Psi(u)\, du \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{T}\Phi_{nn}\left(\frac{u}{T}\right)\Gamma(u)\, du, \qquad (5)\end{aligned}$$

where

$$\Gamma(u) = \sum_{n=-\infty}^{\infty} |H(u + n2\pi)|^2$$

and

$$\begin{aligned}\Psi(u) = 1 + \Gamma(u) - [H(u) + H(-u)] \\ = |1 - H(u)|^2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |H(u + n2\pi)|^2.\end{aligned}$$

The first integral in (5) $\overline{\epsilon_R^2(T)}$ represents only the effect of sampling and reconstruction; the second integral $\overline{\epsilon_Q^2(T)}$ represents only the effect of quantizing.

It is interesting to examine the mean-square error due to the sampling-reconstruction process as $T \to 0$. When the variance of the signal $f(t)$ is kept fixed at unity, the contracted spectral density approaches a delta function

$$\frac{1}{T}\Phi\left(\frac{u}{T}\right) \to 2\pi\delta(u)$$

so that

$$\lim_{T\to 0} \overline{\epsilon_R^2(T)} = \Psi(0) = |1 - H(0)|^2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |H(n2\pi)|^2. \qquad (6)$$

Let us call any normalized reconstruction filter $H(u)$ *asymptotically perfect* if

$$\lim_{T\to 0} \overline{\epsilon_R^2(T)} = 0.$$

Then, from (6), we can conclude the following.

Theorem

The normalized reconstruction filter $H(u)$ will be asymptotically perfect if, and only if, the following two conditions hold.

1) $H(0) = 1$
2) $H(n2\pi) = 0,\ n \neq 0$.

This is a kind of sampling condition in the frequency domain, ensuring that, as $T \to 0$, $H(u)$ selects only the baseband alias of the sampled function.

Using Parseval's relation and the fact that

$$\mathfrak{F}^{-1}\Gamma(u) = \sum_{k=-\infty}^{\infty} \delta(t - k)\int_{-\infty}^{\infty} h(\tau)h(\tau + k)\, d\tau,$$

we can write (5) in the time domain, as done by Katzenelson [1] and Liff [4],

$$\begin{aligned}\overline{\epsilon^2(T)} = \overline{\epsilon_R^2(T)} + \overline{\epsilon_Q^2(T)} \\ = \int_{-\infty}^{\infty} \varphi(xT)\psi(x)\, dx + \int_{-\infty}^{\infty} \varphi_{nn}(xT)\gamma(x)\, dx, \qquad (7)\end{aligned}$$

where

$$\begin{aligned}\psi(x) = \delta(x) + \sum_{k=-\infty}^{\infty} \delta(x - k)\int_{-\infty}^{\infty} h(\tau)h(\tau + k)\, d\tau \\ - [h(x) + h(-x)]\end{aligned}$$

and

$$\gamma(x) = \sum_{k=-\infty}^{\infty} \delta(x - k)\int_{-\infty}^{\infty} h(\tau)h(\tau + k)\, d\tau. \qquad (8)$$

The functions $\psi(x)$ and $\gamma(x)$ determine the weighting function by means of which the mean-square errors of reconstruction and quantizing are obtained from the correlation functions of the signal and quantizing noise near the origin. Note that $\gamma(x)$ consists of the impulse portions of $\psi(x)$ without the unit impulse at the origin. Figure 2 gives the normalized reconstruction filter $H(u)$ and the corresponding $\psi(x)$ for four of the most frequently encountered reconstruction filters. All of these are asymptotically perfect.

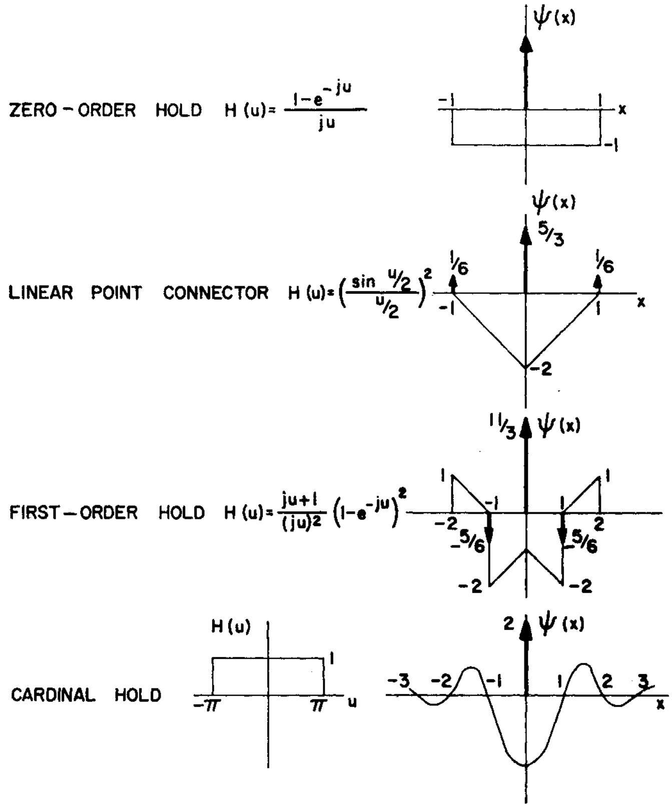

Fig. 2. Asymptotically perfect reconstruction filters and corresponding $\psi(x)$.

III. The Quantizing Noise

As discussed by Katzenelson [1], the autocorrelation function of the quantizing noise can be represented closely by

$$\varphi_{nn}(t) = \frac{q^2}{12} e^{-b|t|}, \tag{9}$$

where q is the quantizing grain size, and b is large enough so that this correlation function decreases with t much faster than the correlation function of the signal $f(t)$. We will make the assumption that the sampling interval T is large enough that the quantizing noise is effectively uncorrelated for lags of T or greater, which is reasonable if successive samples are not redundant. In this case we obtain, from (7) and (8),

$$\overline{\epsilon_Q^2(T)} = \frac{q^2}{12} \int_{-\infty}^{\infty} h^2(x)\, dx. \tag{10}$$

The integral in (10) can be calculated for the four reconstruction filters considered above, with the following results.

Zero-Order Hold

$$\text{(ZOH)}: \quad \int_{-\infty}^{\infty} h^2(x)\, dx = 1$$

Linear Point Connector

$$\text{(LPC)}: \quad \int_{-\infty}^{\infty} h^2(x)\, dx = \tfrac{2}{3}$$

First-Order Hold

$$\text{(FOH)}: \quad \int_{-\infty}^{\infty} h^2(x)\, dx = \tfrac{8}{3}$$

Cardinal Hold

$$\text{(CH)}: \quad \int_{-\infty}^{\infty} h^2(x)\, dx = 1. \tag{11}$$

To evaluate q, we will assume that the signal amplitude can be limited to the range of values between $-A\sqrt{\varphi(0)}$ and $+A\sqrt{\varphi(0)}$, where $\varphi(0)$ is the variance of the signal $f(t)$, and that this range is quantized uniformly. For a Gaussian signal, we can take $A = 3$ with little error involved in truncating the probability distribution of $f(t)$. Hence,

$$q = 2A\sqrt{\varphi(0)}\, 2^{-N} \tag{12}$$

and

$$\frac{\overline{\epsilon_Q^2(T)}}{\varphi(0)} = \frac{A^2}{3}\left(\int h^2\, dx\right) 2^{-2N} \tag{13}$$

using (10). This method of quantizing is realistic so long as the quantizing grain is small compared with the standard deviation of f, which amounts to the requirement

$$2^N \gg 2A. \tag{14}$$

IV. The Reconstruction Error for Small T

In this section the reconstruction error $\overline{\epsilon_R^2(T)}$ will be expanded in a power series in T about $T = 0$. In many practical situations, with asymptotically perfect reconstruction filters, the reconstruction error will be approximated by the lowest-order nonzero term in this expansion. This approximation will enable us to obtain relatively simple expressions for the optimum sampling interval and the resulting mean-square error.

Expanding the even function $\varphi(xT)$ in a power series about the origin, we obtain

$$\varphi(xT) = \sum_{n=0}^{\infty} \varphi^{(n)}(0^+) \frac{T^n}{n!} |x|^n. \tag{15}$$

Substituting in (7) yields

$$\frac{\overline{\epsilon_R^2(T)}}{\varphi(0)} = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0^+)}{\varphi(0)} \frac{T^n}{n!} e_n, \tag{16}$$

where

$$e_n = \int_{-\infty}^{\infty} |x|^n\, \psi(x)\, dx. \tag{17}$$

The power series (16) deserves close attention, first to determine what the lowest order nonzero term may be, and second to determine how well this first term may approximate the reconstruction error for T in the region of interest. We begin by tabulating values of e_n and $\varphi^{(n)}(0^+)$ for different reconstruction devices and power spectra. For the reconstruction filters of Fig. 2, we have the following values for e_0 through e_5.

	e_0	e_1	e_2	e_3	e_4	e_5	
ZOH ···	0	−1	−2/3	−1/2	−2/5	−1/3	
LPC ···	0	−1/3	0	2/15	1/5	5/21	(18)
FOH ···	0	−5/3	0	7/3	93/15	13	
CH ···	0	$-4/\pi^2$	0	$8/\pi^4$	0	$-96/\pi^6$.	

All of these reconstruction filters are asymptotically perfect, and hence, for all of these, $e_0 = 0$.

The power spectra of practical importance to which we will give special attention here are the ideal flat bandlimited spectrum

$$\Phi(\omega) = \begin{cases} \pi/\omega_0 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

and the nth order Butterworth spectra

$$\Phi(\omega) = \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^{2n}}, \qquad n = 1, 2, 3, \cdots .$$

The following tabulation gives some pertinent values of $\varphi^{(n)}(0^+)$.

	$\varphi(0)$	$\varphi'(0^+)$	$\varphi''(0^+)$	$\varphi'''(0^+)$	$\varphi^{\mathrm{IV}}(0^+)$	$\varphi^{\mathrm{V}}(0^+)$	
Flat Bandlimited	1	0	$-\omega_0^2/3$	0	$\omega_0^4/5$	0	
Butterworth, $n = 1$	1/2	$-\omega_0/2$	$\omega_0^2/2$	$-\omega_0^3/2$	$\omega_0^4/2$	$-\omega_0^5/2$	(19)
Butterworth, $n = 2$	$1/2\sqrt{2}$	0	$-\omega_0^2/2\sqrt{2}$	$\omega_0^3/2$	$-\omega_0^4/2\sqrt{2}$	0	
Butterworth, $n = 3$	1/3	0	$-\omega_0^2/6$	0	$\omega_0^4/3$	$-\omega_0^5/2$.	

The first special case that should be investigated is that of a bandlimited spectrum with a cardinal hold, for which $\overline{\epsilon_R^2(T)}$ must be identically zero in a neighborhood of the origin given by $|T| < \pi/\omega_0$. Here, every term of (15) must vanish. This is, indeed, the case, since the $\varphi^{(n)}(0^+)$ vanish for n odd, and the e_n vanish for n even.

Now consider the case of the first-order Butterworth spectrum (the so-called Markov spectrum) for which

$$\varphi(t) = \tfrac{1}{2}e^{-\omega_0|t|}.$$

Equation (15) yields

$$\frac{\overline{\epsilon_R^2(T)}}{\varphi(0)} = -e_1\omega_0 T + e_2\frac{(\omega_0 T)^2}{2} - e_3\frac{(\omega_0 T)^3}{6} + \cdots . \tag{20}$$

For our purposes, we wish to use the approximation

$$\frac{\overline{\epsilon_R^2(T)}}{\varphi(0)} \approx -e_1\omega_0 T. \tag{21}$$

If we use this approximation to obtain an optimum sampling interval $T = T_{\text{opt}}$ (subject to the fixed bit-rate constraint), it is necessary to verify that this is, indeed, a valid approximation for this T_{opt}. One crude indication of the validity of (21) is the ratio of the next term in the series to the first term. This ratio is in magnitude

$$\left|\frac{e_2}{e_1}\right|\frac{\omega_0 T}{2} = \tfrac{1}{3}\omega_0 T$$

for the zero-order hold, and

$$\left|\frac{e_3}{e_1}\right|\frac{(\omega_0 T)^2}{6}$$

for the other reconstruction filters, since, in these cases, $e_2 = 0$. These ratios will become small if $\omega_0 T \ll 1$, and this will be so for sufficiently large bit-rates R. A more exact check of (21) can always be obtained by evaluating more terms in (20), or by an exact calculation using (7).

Similar remarks hold for other spectrum-hold combinations. Consider, for example, the third-order Butterworth spectrum using the linear point connector as a reconstruction filter. Equation (15) yields

$$\frac{\overline{\epsilon_R^2(T)}}{\varphi(0)} \approx \frac{1}{120}(\omega_0 T)^4,$$

where the ratio of the next term to this term is in magnitude

$$\tfrac{5}{14}\omega_0 T.$$

In the following, we will use the approximation

$$\frac{\overline{\epsilon_R^2(T)}}{\varphi(0)} \approx e_k\frac{\varphi^{(k)}(0^+)}{\varphi(0)}\frac{T^k}{k!}, \tag{22}$$

where the kth term is the lowest order nonzero term in (15). The appropriate values of k for various spectrum-hold combinations are listed below.

	ZOH	LPC	FOH	CH	
Flat Bandlimited	2	4	4	∞	
Butterworth, $n = 1$	1	1	1	1	
Butterworth, $n = 2$	2	3	3	3 .	(23)
Butterworth, $n = 3$	2	4	4	5	

V. The Optimum Choice of N and T

The total normalized mean-square error can now be written approximately as

$$\frac{\overline{\epsilon^2}}{\varphi(0)} = \frac{A^2}{3}\left(\int_{-\infty}^{\infty} h^2\,dx\right)2^{-2N} + \frac{e_k}{k!}\frac{\varphi^{(k)}(0^+)}{\varphi(0)}\left(\frac{N}{R}\right)^k \tag{24}$$

which shows clearly the tradeoff between quantizing fineness and sampling rate. We have written this as a function of N, using the constraint $N = RT$. Setting the derivative of this with respect to N equal to zero, we obtain the condition

$$R(N) = \left[\frac{e_k}{(k-1)!}\frac{\varphi^k(0^+)}{\varphi(0)}\frac{N^{k-1}2^{2N}}{(A^2/3)\left(\int h^2\,dx\right)2\ln 2}\right]^{1/k} \tag{25}$$

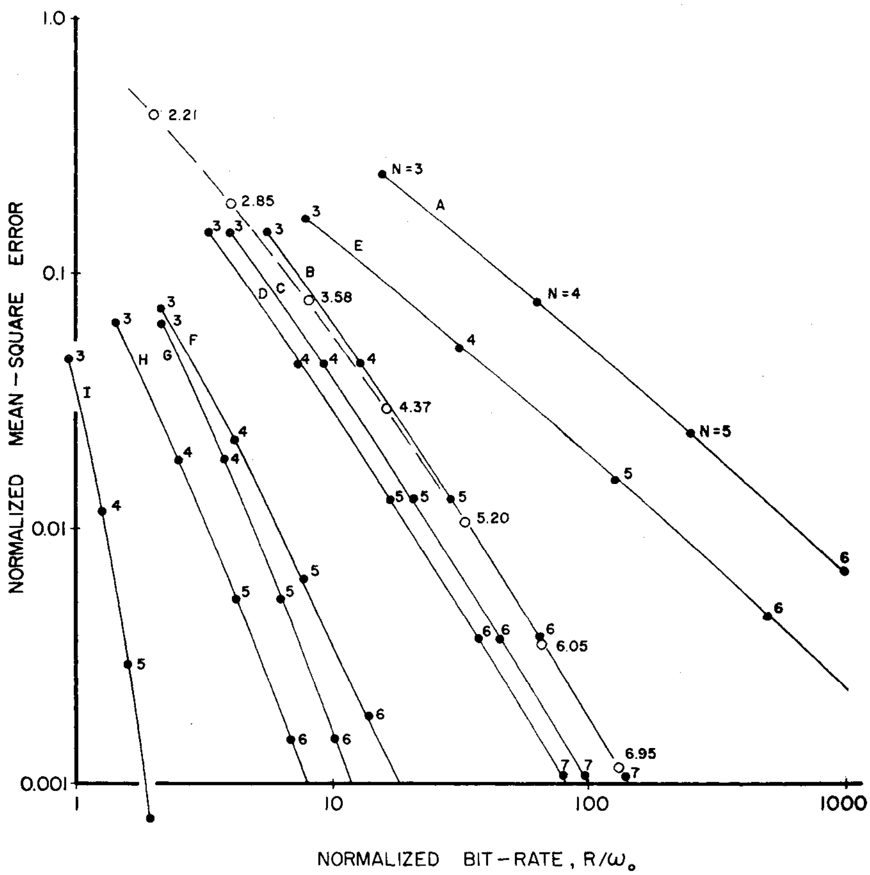

Fig. 3. Curves of optimum mean-square error vs. bit-rate, using the approximation of (22). Curves A through D are for the zero-order hold and first- through third-order Butterworth and flat bandlimited spectra. Curves E through H are for the same spectra but the linear point connector. Curve I is the Shannon limit with a flat band-limited spectrum and a cardinal hold. $A = 3$ for all curves.

which gives the bit-rate $R(N)$ when N is the optimum number of bits per sample. Substituting (25) in (24), we obtain

$$\left.\frac{\overline{\epsilon^2(N)}}{\varphi(0)}\right|_{\text{opt}} = \frac{A^2}{3}\left(\int h^2\,dx\right)\left[1 + \frac{N2\ln 2}{k}\right]2^{-2N}, \qquad (26)$$

which is the mean-square error when N is optimum. It is not possible, in this general case, to eliminate N between (25) and (26) to obtain $\overline{\epsilon^2}/\varphi(0)|_{\text{opt}}$ explicitly as a function of the bit-rate R. It is very simple, however, to evaluate (25) and (26) for integral values of N in the range of interest, and in this way obtain a plot of $\overline{\epsilon^2}/\varphi(0)|_{\text{opt}}$ vs. R parametrically. Such curves are shown in Fig. 3 for the spectra of (19) and for the zero-order hold and the linear point connector. Also shown is the mean-square error vs. normalized bit-rate R/ω_0 in the idealized bandlimited case considered by Shannon [5]. In this case, the sampling rate is fixed at $T = \pi/\omega_0$ by the bandwidth of the signal; the only error is that due to quantization, which, by (13) and (11) for the cardinal hold, is

$$\frac{\overline{\epsilon_Q^2}}{\varphi(0)} = \frac{A^2}{3}\,2^{-2\pi R/\omega_0}. \qquad (27)$$

As can be seen, the mean-square error, in this case, lies below the other curves and goes to zero much faster, reflecting the fact that we have taken into account the reconstruction error. In fact, as can be seen from (26), the mean-square error due to reconstruction is, with an optimum choice of sampling rate, $2N(\ln 2)/k$ times that due to quantization.

To illustrate the accuracy of the approximation made to obtain (25) and (26), consider curve B—a second-order Butterworth spectrum with a zero-order hold. From (7), (8), and (13), the total normalized error can be calculated exactly as

$$\frac{\overline{\epsilon^2}}{\varphi(0)} = 2 - \frac{1}{\varphi(0)}\int_{-1}^{1}\varphi(xT)\,dx + 3\cdot 2^{-2N}$$
$$= 2\,\frac{Q - 1 + e^{-Q}\cos Q}{Q} + 3\cdot 2^{-2N}, \qquad (28)$$

where

$$Q = \frac{N}{\sqrt{2}\,R/\omega_0}.$$

A digital computer program was written to minimize this as a function of N for various fixed R/ω_0, and the results are shown in Fig. 3 as open circles. The agreement is quite good for $R/\omega_0 > 8$, which corresponds to a sampling interval of $\omega_0 T = N/(R/\omega_0) < 0.43$.

VI. The Case when $\varphi'(0^+) \neq 0$

The special case $k = 1$, illustrated by curves A and E, arises only when $\varphi'(0^+) \neq 0$, i.e., when the spectral density falls off as $1/\omega^2$ for large ω. This is the most pessimistic situation, since the aliasing errors caused by sampling are most severe. Of particular interest, however, is the fact that (24)–(26) can, in this case, be solved explicitly for $\overline{\epsilon^2}/\varphi(0)|_{\text{opt}}$ as a function of R. For convenience, define a normalized sampling interval

$$\tau \triangleq T\left(-\frac{\varphi'(0^+)}{\varphi(0)}\right),$$

a normalized bit rate

$$\rho \triangleq 2R \ln 2\left(-\frac{\varphi(0)}{\varphi'(0^+)}\right),$$

and a constant depending only on the reconstruction filter

$$a \triangleq \frac{A^2}{3(-e_1)}\left(\int h^2\, dx\right).$$

Then, the normalized total mean-square error (24) becomes

$$\frac{\overline{\epsilon^2}}{\varphi(0)} = (-e_1)[ae^{-\rho\tau} + \tau]. \tag{29}$$

The optimum sampling interval is

$$\tau_{\text{opt}} = \frac{\ln a\rho}{\rho}. \tag{30}$$

The optimum number of bits per sample is

$$N_{\text{opt}} = \tfrac{1}{2}\log_2 a\rho, \tag{31}$$

and the optimum normalized mean-square error is

$$\left.\frac{\overline{\epsilon^2}}{\varphi(0)}\right|_{\text{opt}} = (-e_1)\left[\frac{1 + \ln a\rho}{\rho}\right]. \tag{32}$$

This goes to zero slower than $1/\rho$, in sharp contrast with the Shannon limit (27).

VII. Conclusion

The purpose of this analysis has been to find the best tradeoff between sampling rate and quantizing fineness for the transmission of an analog signal over a fixed bit-rate digital channel.

An approximate solution to this problem, valid for large bit-rates, was obtained by approximating the reconstruction error by an appropriate power of the sampling interval T. The results, which take into account the error of reconstructing nonbandlimited signals by realizable filters, yield considerably higher total mean-square errors than an analysis which assumes ideal bandlimited signals and perfect reconstruction.

It is hoped that the results will be of some use as a guide to the utilization of data channels. Suppose, for example, that it is desired to transmit voice with a bandwidth of 4 kc/s over a data channel with a bit-rate of 100 kilobits/s. Assuming that the voice signal is very nearly bandlimited, and that a linear point connector is used, curve H applies with $R/\omega_0 = 3.93$. This corresponds to 5 bits per sample, a sampling rate of 20 kc/s (2.5 times the Nyquist frequency), and a mean-square error of 0.0052. These numbers are typical of performance now being achieved, although, in practice, a reconstruction filter with sharper cutoff characteristics than a linear point connector might be used.

Acknowledgment

The author is indebted to S. G. Kamak for many detailed computations justifying the approximations in this work, and to S. A. Tretter for helping simulate the problem on the computer. The author also wishes to thank Professors J. B. Thomas and A. Lo for helpful discussions relating to this problem.

References

[1] J. Katzenelson, "On errors introduced by combined sampling and quantization," *IRE Trans. on Automatic Control*, vol. AC-7, pp. 58–68, April 1962.
[2] D. G. Watts and J. Katzenelson, "Discussion of 'On errors introduced by combined sampling and quantization,'" *IEEE Trans. on Automatic Control* (*Correspondence*), vol. AC-8, pp. 187–188, April 1963.
[3] S. S. L. Chang, *Synthesis of Optimum Control Systems*. New York: McGraw-Hill, 1961, ch. 6.
[4] A. I. Liff, "Mean-square reconstruction error," *IEEE Trans. on Automatic Control* (*Correspondence*), vol. AC-10, pp. 370–371, July 1965.
[5] C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. J.*, vol. 27, pp. 379–423, July 1948; pp. 623–656, October 1948. (See part V: "The rate for a continuous source.")