noise with the power spectral density of (22). Using the relationship between the power spectral densities of the input and the output of a filter

$$\Phi_{yy}(z) = H(z)H(z^{-1})\Phi_{xx}(z) \qquad (23)$$

we arrive at the expression for the desired filter,

$$H(z) = \frac{A}{\sigma} \frac{\sqrt{1-e^{-2\alpha T}}}{(1-e^{-\alpha T}z^{-1})}. \qquad (24)$$

The equivalent difference equation is identical to (14).

Given a sequence of N values of zero-mean white noise $x_n (1 \leq n \leq N)$, and an initial value of the output y_0, it is possible to generate the next N values of the output $y_n (1 \leq n \leq N)$. For a zero-mean input the output will have zero mean. If a biased output is desired, the bias B must be added to each value of the output, y_n. Equation (14) then becomes

$$y_{n+1} = e^{-\alpha T}y_n + \frac{A}{\sigma}\sqrt{1-e^{-2\alpha T}}\,x_{n+1} + B. \quad (25)$$

This equation yields an exponentially autocorrelated random sequence with standard deviation A, inverse autocorrelation time α, and mean B.

ACKNOWLEDGMENT

The authors wish to acknowledge the assistance of H. Gish, Harvard University, Cambridge, Mass., in the preparation of this note.

A. GELB
P. PALOSKY
Dynamics Research Corp.
Stoneham, Mass.

Discussion of "Comments on the Statistical Design of Linear Sampled-Data Feedback Systems"

F. Csáki

I read with great attention the interesting comments of Steiglitz, Franaszek, and Haddad.[1]

Their statement is that the following well-known formula

$$W_1(z) = \frac{\left\{\dfrac{G(z^{-1})G_d(z)[\phi_{r_sr_s}(z) + \phi_{r_nr_s}(z)]}{[G(z)G(z^{-1})]^-\phi_{rr}^-(z)}\right\}_+}{[G(z)G(z^{-1})]^+\phi_{rr}^+(z)} \qquad (1)$$

is false; therefore a new relation

$$W_2(z) = \frac{z\left\{\dfrac{G(z^{-1})G_d(z)[\phi_{r_sr_s}(z) + \phi_{r_nr_s}(z)]}{z[G(z)G(z^{-1})]^-\phi_{rr}^-(z)}\right\}_+}{[G(z)G(z^{-1})]^+\phi_{rr}^+(z)} \qquad (2)$$

is proposed. Thus, according to their opinion the expression for the sampled-data solution is not quite formally identical to the continuous-data expression.

However, their proof does not convince me of the truth of the second formula. On the contrary, it can readily be shown that (1) is the true solution and (2) is a false one.

First of all, let us concentrate our attention to the meaning of the symbol $\{\ \}_+$. In continuous-data systems this symbol denotes a component of a transfer function belonging to a positive-time function. The same concept is also valid for sampled-data systems. Thus, the component in question can be determined by an inverse two-sided z transform taking for the integration path the unit circle Γ_0, and thereafter by the ordinary one-sided z transform (performing the summation only from $n=0$ to ∞):

$$\{\Psi(z)\} = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi j}\oint_{\Gamma_0}\Psi(\zeta)\zeta^{n'-1}d\zeta\right)z^{-n} \qquad (3)$$

where

$$\Psi(z) = \frac{G(z^{-1})G_d(z)[\phi_{r_sr_s}(z) + \phi_{r_nr_s}(z)]}{[G(z)G(z^{-1})]^-\phi_{rr}^-(z)}.$$

It is fallacious to assume, for example, in case of simple poles that the symbol $\{\ \}_+$ means the sum of the ordinary partial fractions, hence

$$\{\Psi(z)\}_+ \neq \sum_{\mu}\frac{A_\mu}{z - p_\mu^*} \qquad (4)$$

where p_μ^* denotes the simple poles of $\Psi(z)$ inside the unit circle Γ_0 of the z plane. On the contrary, on the basis of (3) it can be shown that

$$\{\Psi(z)\}_+ = \sum_\mu \frac{A_\mu z}{z - p_\mu^*} = \sum_\mu \frac{A_\mu}{1 - p_\mu^* z^{-1}} \qquad (5)$$

where

$$A_\mu = \lim_{z\to p_\mu^*}(z - p_\mu^*)\frac{1}{z}\Psi(z). \qquad (6)$$

Hence, in case of simple poles p_μ^*, (3) and (5) can also be expressed as

$$\{\Psi(z)\}_+ = \sum_\mu \frac{z}{z - p_\mu^*}\lim_{z\to p_\mu^*}(z - p_\mu^*)\frac{1}{z}\Psi(z). \qquad (7)$$

There is then no doubt that the final solution is

$$W_m(z) = \frac{\displaystyle\sum_\mu \frac{A_\mu z}{z - p_\mu^*}}{[G(z)G(z^{-1})]^+\phi_{rr}^+(z)} = \frac{\displaystyle\sum_\mu \frac{A_\mu}{1 - p_\mu^* z^{-1}}}{[G(z)G(z^{-1})]^+\phi_{rr}^+(z)}. \qquad (8)$$

Summarizing, if the proper formula, (3), (5), or (7), is applied, then (1) gives the correct solution. If, on the other hand, the worst relation (4) is used then from (2) the proper solution can also be obtained but only through double mistakes. These mistakes originate from the incorrect application of the symbol $\{\ \}_+$. Thus, (1), (3), (5), and (7) are true and supply, in a correct manner, the final solution (8). On the contrary, (2) and (4) are false and do give the final solution (8) only by making two errors. Equation (1) is true, while (2) is false.

Thus, there is a close analogy between the solution formulas of continuous-data and sampled-data systems, respectively.

To illustrate the proper solution let us see the example given by Steiglitz et al.

$$G(z) = \frac{1}{1 - 0.5z^{-1}}, \qquad G_d(z) = 1.$$

Thus in this case

$$[G(z)G(z^{-1})]^+ = G(z)$$

and

$$[G(z)G(z^{-1})]^- = G(z^{-1})$$

and

$$G_d(z)[\phi_{r_sr_s}(z) + \phi_{r_nr_s}(z)] = \phi_{r_sr_s}(z).$$

Furthermore

$$\phi_{rr}^-(z) = \sqrt{2}\,\frac{1 - 0.1z}{1 - 0.2z};$$

$$\phi_{rr}^+(z) = \sqrt{2}\,\frac{1 - 0.1z^{-1}}{1 - 0.2z^{-1}}.$$

Hence,

$$\Psi(z) = \frac{0.98}{\sqrt{2}}\,\frac{1}{(1 - 0.1z)(1 - 0.2z^{-1})}.$$

Applying (3), (5), or (7)

$$\Psi_+(z) = \frac{1}{\sqrt{2}}\,\frac{1}{1 - 0.2z^{-1}}$$

the latter being, indeed, the part belonging to the positive-time function component of

$$\psi^*(t) = \frac{1}{\sqrt{2}}[\cdots + 0.1^2\delta(t + 2T) + 0.1\delta(t + T) + \delta(t) + 0.2\delta(t - T) + 0.2^2\delta(t - 2T) + \cdots].$$

Finally, applying (1) or (8)

$$W_m(z) = 0.5\,\frac{1 - 0.5z^{-1}}{1 - 0.1z^{-1}}.$$

This does agree, of course, with the proper solution of Steiglitz, Franaszek, and Haddad. As we have seen, however, $W_m(z) = W_1(z)$ and $W_m(z) \neq W_2(z)$.

F. CSÁKI
Dept. of Automation
Polytechnical University
Budapest, Hungary

Manuscript received August 19, 1965.

[1] K. Steiglitz, P. A. Franaszek, and A. H. Haddad, *IEEE Trans. on Automatic Control (Correspondence)*, vol. AC-10, pp. 216–217, April 1965.

Steiglitz, Franaszek and Haddad[2]

We wish to thank Dr. Csáki for his interesting comments. We find his mathematics correct and agree that his solution is consistent. We also find nothing wrong with our original discussion. The difficulty stems only from the fact that the definition of the operator $\{\ \}_+$ adopted by Dr. Csáki dif-

[2] Manuscript received September 7, 1965.

fers from that used in the original[3,4] papers. The following argument shows that the two answers are equivalent.

Let

$$\Psi(z) = \sum_{i=1}^{n} \frac{A_i}{z - \alpha_i} + \sum_{i=1}^{m} \frac{B_i}{z - \beta_i} + \sum_{i=0}^{p} C_i z^i;$$

where

$$|\alpha_i| < 1, \qquad i = 1, \cdots, n;$$
$$|\beta_i| > 1, \qquad i = 1, \cdots, m;$$

assuming for convenience that no poles have magnitude exactly unity and that no poles are multiple. Define the operator $\{\ \}_+$ by

$$\{\Psi(z)\}_+ = \sum_{i=1}^{n} \frac{A_i}{z - \alpha_i}.$$

This is the convention that is used in the continuous case and the one that the authors tacitly adopted in the discussion referred to by Dr. Csáki.

The operator defined by Csáki, which we shall call $\{\ \}_{\text{plus}}$, consists of taking the terms involving z^i for $i=0, 1, 2, \cdots$ in the power series expansion of $\Psi(z)$ that is valid on the unit circle. Thus

$$\{\Psi(z)\}_{\text{plus}} = \sum_{i=1}^{n} \frac{A_i}{z - \alpha_1} - \sum_{i=1}^{m} \frac{B_i}{\beta_i} + C_0.$$

The following relation then follows:

$$z\left\{\frac{\Psi(z)}{z}\right\}_+ = \{\Psi(z)\}_{\text{plus}}.$$

Proof:

$$\frac{\Psi(z)}{z} = \sum_{i=1}^{n} \frac{A_i}{z(z - \alpha_i)} + \sum_{i=0}^{m} \frac{B_i}{z(z - \beta_i)} + \sum_{i=0}^{p} C_i z^{i-1}.$$

By definition of $\{\ \}_+$

$$\left\{\frac{\Psi(z)}{z}\right\}_+ = \sum_{i=1}^{n} \frac{A_i}{z(z - \alpha_i)} - \sum_{i=1}^{m} \frac{B_i}{\beta_i z} + \frac{C_0}{z},$$

and therefore

$$z\left\{\frac{\Psi(z)}{z}\right\}_+ = \sum_{i=1}^{n} \frac{A_i}{z - \alpha_i} - \sum_{i=1}^{m} \frac{B_i}{\beta_i} - C_0$$
$$= \{\Psi(z)\}_{\text{plus}}.$$

Hence the two results should be written

$$W_1(z) = \frac{\{\Psi(z)\}_{\text{plus}}}{[G(z)G(z^{-1})]^+ \phi_{rr}^+(z)} \qquad (1)$$

and

$$W_2(z) = \frac{z\left\{\frac{\Psi(z)}{z}\right\}_+}{[G(z)G(z^{-1})]^+ \phi_{rr}^+(z)}; \qquad (2)$$

and both are correct.

The introduction of the operator $\{\ \}_{\text{plus}}$ thus makes the sampled-data solution formally identical with the continuous case. In any event both formulas will yield the same answer provided that the proper interpretation of the partial-fractioning operator is used. We hope that this finally clarifies the difficulty.

K. STEIGLITZ
P. A. FRANASZEK
A. H. HADDAD
Dept. of Elec. Engrg.
Princeton University
Princeton, N. J.

[3] J. T. Tou, "Statistical design of digital control systems," *IRE Trans. on Automatic Control*, vol. AC-5, pp. 290–297, September 1960.
[4] K. Steiglitz, P. A. Franaszek, and A. H. Haddad, "Comments on the statistical design of linear sampled-data feedback systems," *IEEE Trans. on Automatic Control (Correspondence)*, vol. AC-10, pp. 216–217, April 1965.

On "Error Bounds for Jittered Sampling"

In a recent paper,[1] some interesting results were obtained by the authors. However, the restriction imposed on (15) is not necessary, and hence the results are more general. Indeed the proof is trivial and is seen from

$$R_{ff}(0) - R_{ff}(\sigma_j) = \frac{1}{\pi}\int_0^{\pi\rho} S_{ff}(\omega)[1 - \cos\omega\sigma_j]d\omega$$

where $S_{ff}(\omega)$ denotes the spectral density associated with $f(t)$; in view of

$$1 - \cos\omega\sigma_j = 2\sin^2\frac{\omega\sigma_j}{2}$$

and

$$\frac{\sin^2\frac{\omega\sigma_j}{2}}{\left(\frac{\omega\sigma_j}{2}\right)^2} \le 1$$

it then follows that

$$R_{ff}(0) - R_{ff}(\sigma_j) = \frac{1}{\pi}\int_0^{\pi\rho} S_{ff}(\omega)2\sin^2\frac{\omega\sigma_j}{2}d\omega$$
$$\le \frac{2}{\pi}\int_0^{\pi\rho} S_{ff}(\omega)\frac{\omega^2\sigma_j^2}{4}d\omega$$
$$\le \frac{2}{\pi}\frac{\sigma_j^2}{4}\pi^2\rho^2\int_0^{\pi\rho} S_{ff}(\omega)d\omega$$
$$\le R_{ff}(0)\pi^2\rho^2\frac{\sigma_j^2}{2},$$

as also shown in Papoulis.[2]

OSCAR A. Z. LENEMAN
Lincoln Lab.[3]
Mass. Inst. of Tech.
Lexington, Mass.

Manuscript received September 9, 1965.
[1] B. Liu and T. Stanley, *IEEE Trans. on Automatic Control*, vol. AC-10, pp. 449–454, October 1965.
[2] A. Papoulis, *Probability, Random Variables and Stochastic Processes*. New York: McGraw-Hill, 1965, p. 378.
[3] Operated with support from the U. S. Air Force.

Comment on "A Note on Transfer Function Identification"

The above correspondence[1] was not intended to exhaust the four parameter case but was merely used to demonstrate that the assumption of a four parameter function in either of the proposed forms of 2 was in error. One would obviously attempt identification in the case of an M parameter function with a possible denominator of order M and a possible numerator of order equal to the least integer greater than or equal to $(M-1)/2$.

ARTHUR J. DEEX
Dept. of Elec. Engrg.
University of Texas
Austin, Texas

Manuscript received August 16, 1965.
[1] A. J. Deex and J. M. Mendel, *IEEE Trans. on Automatic Control (Correspondence)*, vol. AC-10, pp. 491–492, October 1965.