Design of FIR Digital Phase Networks

KENNETH STEIGLITZ, FELLOW, IEEE

Abstract—The problem of the minimax design of FIR digital filters with prescribed phase characteristics and unit magnitude is a nonlinear optimization problem. In this paper it is approximated by a linear programming problem, and it is shown that the solution of this linear program is optimal to first order. That is, if δ_0 and ϵ_0 are optimal deviations of magnitude and phase characteristics, then the actual deviations obtained from the linear program solution satisfy $\delta < \delta_0 + \epsilon_0^2$ and $\epsilon < \epsilon_0(1 + \delta_0)/(1 - \delta_0)$. Numerical examples are given, including design results for full-band M-term chirp filters which (like linear phase filters) can be implemented with $(M + 1)/2$ multiplications per point.

I. Introduction

We consider here the design of FIR digital filters with prescribed phase characteristics and approximately unit magnitude response, using a weighted Chebyshev error criterion. Such filters are the FIR counterpart to IIR all-pass filters [1], [2] and have the same applications—nonlinear phase equalization, pulse shaping for chirp radar, and so on. These FIR filters can be implemented readily with the FFT or CCD devices, advantages over IIR filters. On the other hand, all-pass IIR digital filters can have magnitude response precisely one, but FIR filters can only approximate an all-pass characteristic. Thus, the design problem we consider here is more complicated than IIR all-pass design in that we must approximate magnitude and phase simultaneously. However, the IIR problem has parameters which enter nonlinearly into the transfer function, and the phase optimization problem is inherently more difficult. It requires an algorithm like that of Davidon-Fletcher-Powell, and is limited to relatively few parameters. We will apply linear programming to the FIR phase design problem, and show that this approach is easily practical for filters of length of at least 61. Also, when the phase characteristic has certain symmetries, the FIR approximating filter can be implemented with one-half the usual number of multiplications, in a manner analogous to the linear phase case.

Previous work by Cuthbert [3], Holt *et al.* [4], and Razzak and Cuthbert [5] takes the following approach. Let the unknown transfer function be

$$H(\omega) = \sum_{k=0}^{M-1} h_k e^{-jk\omega}. \tag{1}$$

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The author is with the School of Engineering and Applied Science, Princeton University, Princeton, NJ 08540.

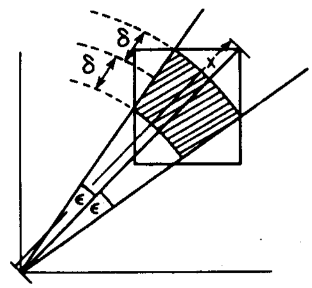

Fig. 1. Optimal approximation of a point at an angle of $\pi/4$; fitting real and imaginary parts can result in a radius deviation of x.

Let $M = 2L + 1$ be an odd integer (the even case is similar), and write

$$e^{jL\omega}H(\omega) = h_L + \sum_{k=1}^{L} (h_{L-k} + h_{L+k}) \cos k\omega + j \sum_{k=1}^{L} (h_{L-k} - h_{L+k}) \sin k\omega = \sum_{k=0}^{L} a_k \cos k\omega + j \sum_{k=1}^{L} b_k \sin k\omega. \tag{2}$$

If the desired complex-frequency response is $D(\omega)$, we can separately approximate Re $[e^{jL\omega}D(\omega)]$ by the cosine polynomial in (2) and Im $[e^{jL\omega}D(\omega)]$ by the sine polynomial, using the Remez algorithm [4], for example. This approach is quite practical, but suffers from two important difficulties.

1) Suppose for illustration that we wish to approximate unit amplitude at an angle of $\pi/4$ at some particular frequency, and that the optimal approximation results in an amplitude deviation of δ and angle deviation of ϵ. Fig. 1 shows the sector within which the optimal solution lies at that frequency. Also shown is the smallest box which can, in general, result from the method above, corresponding to a radius of x. An application of the law of sines shows that

$$x = (1 + \delta)(\cos \epsilon + \sin \epsilon). \tag{3}$$

Take $\epsilon = \delta$ for simplicity; then to first order

$$x \approx 1 + 2\delta. \tag{4}$$

That is, the deviation in magnitude resulting from the minimax approximation of real and imaginary parts can be twice the optimal value.

2) The above approach does not allow the independent weighting of radius and phase angle deviations.

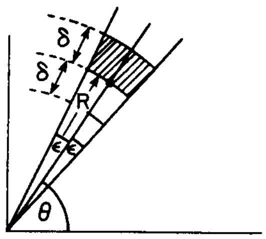

Fig. 2. The constraints corresponding to optimal approximation.

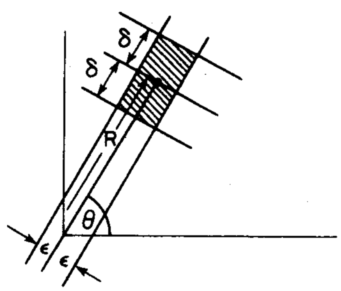

Fig. 3. Linear constraints approximating those of Fig. 2.

We will now show that when the desired radius is 1, the use of linear programming can, to first order in ϵ and δ, eliminate both of these problems.

II. Linear Programming Formulation

The general problem of simultaneous magnitude-phase minimax approximation can be stated as follows. We are given desired values of magnitude R_k and phase θ_k at a set of frequencies ω_k, $k = 1, \cdots, K$ on a grid, $0 \leqslant \omega_k \leqslant \pi$, and a corresponding set of tolerances TM_k and TP_k. We wish to find M real coefficients $h_i, i = 0, \cdots, M-1$ in the transfer function

$$H(\omega) = \sum_{i=0}^{M-1} h_i e^{-ij\omega} \tag{5}$$

so as to minimize the "squeezing" parameter λ in

$$\begin{aligned} \big|\, |H(\omega_k)| - R_k \big| &\leqslant \lambda TM_k \\ \big|\operatorname{Arg} H(\omega_k) - \theta_k\big| &\leqslant \lambda TP_k. \end{aligned} \tag{6}$$

These constraints are nonlinear in the parameters h_i. Fig. 2 shows these constraints at one particular frequency and value of R.

The idea of our approach is to replace this nonlinear problem with a linear one, so that we can use the highly reliable simplex algorithm for linear programming [6]-[12]. Fig. 3 shows linear constraints which approximate those in Fig. 2—we will study the question of how closely in the next section. These constraints in algebraic form are

$$\begin{aligned} \pm(y_k \sin\theta_k + x_k \cos\theta_k - R_k) &\leqslant \lambda TM_k \\ \pm(y_k \cos\theta_k - x_k \sin\theta_k) &\leqslant \lambda TP_k \end{aligned} \tag{7}$$

where

$$y_k = \operatorname{Im}[H(\omega_k)] = -\sum_{i=0}^{M-1} h_i \sin i\omega_k$$

$$x_k = \operatorname{Re}[H(\omega_k)] = \sum_{i=0}^{M-1} h_i \cos i\omega_k. \tag{8}$$

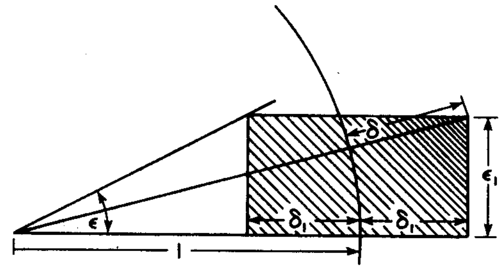

Fig. 4. The linear program results in deviations δ_1 and ϵ_1; the actual deviations can be δ and ϵ.

The linear programming problem is then to find the h_i and λ so as to minimize λ subject to the linear constraints (7).

A program was written especially to implement the simplex algorithm for this problem. This program is similar to that given in [11], except that the tableau elements now correspond to the more complicated inequalities in (7). The two-phase simplex method was used to solve the dual problem [6] with particular care in the resolution of ties in the pivot row selection.

III. Bounds on Quality of the Linear Programming Solution

Suppose for simplicity that the desired values R_k are all one; this is the phase approximation problem on which we want to concentrate. Also, assume that the phase tolerances are all equal, and so are the magnitude to tolerances, and let

$$\begin{aligned} \delta_1 &= \lambda_0 TM \\ \epsilon_1 &= \lambda_0 TP \end{aligned} \tag{9}$$

where λ_0 is the optimal value of λ in the solution of the linear program (7). Also, as shown in Fig. 4, let δ and ϵ be the *maximum possible actual* magnitude and phase deviations. From the figure it is clear that

$$\begin{aligned} \tan\epsilon &= \epsilon_1/(1-\delta_1) \\ (1+\delta)^2 &= \epsilon_1^2 + (1+\delta_1)^2. \end{aligned} \tag{10}$$

Thus, we can state the following.

Result 1: If in the $R = 1$ (all-pass) case the linear program yields optimal values of magnitude and phase deviations δ_1 and ϵ_1, then the actual deviations satisfy

$$\delta \leqslant \sqrt{(1+\delta_1)^2 + \epsilon_1^2} - 1 \leqslant \delta_1 + \frac{\delta_1^2 + \epsilon_1^2}{2} \tag{11}$$

$$\epsilon \leqslant \tan^{-1}[\epsilon_1/(1-\delta_1)] \leqslant \epsilon_1 + \epsilon_1\delta_1 + \epsilon_1\delta_1^2/(1-\delta_1). \tag{12}$$

This result tells how good the actual approximation is from the results of the linear programming problem. It reveals that the actual deviations are equal to those of the linear program to first order in the (small) parameters ϵ_1 and δ_1.

Next let

$$\begin{aligned} \delta_0 &= \lambda_0 TM \\ \epsilon_0 &= \lambda_0 TP \end{aligned} \tag{13}$$

be values of phase and magnitude deviations which are *truly optimal* for the ideal problem of (6). The corresponding

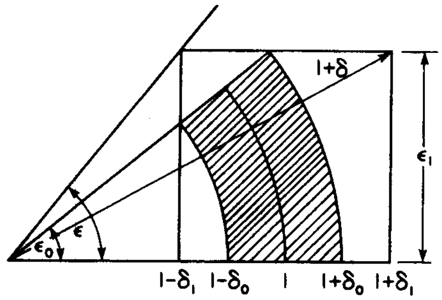

Fig. 5. The optimal deviations are δ_0 and ϵ_0; the linear program deviations are δ_1 and ϵ_1; the actual deviations can be δ and ϵ.

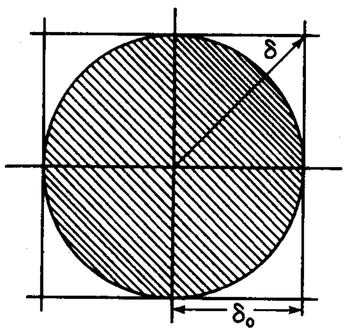

Fig. 6. Approximation to zero. The optimal deviation is δ_0 and the actual deviation can be δ.

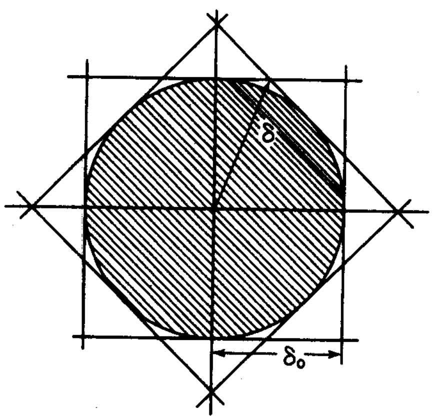

Fig. 7. The addition of linear constraints decreases the possible actual deviation.

geometry is shown in Fig. 5, and implies that

$$\sin \epsilon_0 = \epsilon_1/(1 + \delta_0)$$
$$\cos \epsilon_0 = (1 - \delta_1)/(1 - \delta_0)$$
$$\tan \epsilon = \epsilon_1/(1 - \delta_1)$$
$$(1 + \delta)^2 = \epsilon_1^2 + (1 + \delta_1)^2. \qquad (14)$$

Some algebra yields the following.

Result 2: If in the $R = 1$ (all-pass) case δ_0 and ϵ_0 are the ideally optimal magnitude and phase deviations (the solution to the nonlinear problem), then the actual deviations satisfy

$$\delta \leqslant \delta_0 + \epsilon_0^2 \qquad (15)$$

$$\epsilon \leqslant \tan^{-1}\left(\frac{1 + \delta_0}{1 - \delta_0} \tan \epsilon_0\right) \leqslant \epsilon_0 \frac{1 + \delta_0}{1 - \delta_0}. \qquad (16)$$

Thus, we are assured that the actual deviations are within second-order terms of optimal—we say, therefore, that the linear program yields results which are "optimal to first order."

When the desired magnitude is zero in some band, this particular approximation of the nonlinear problem by linear programming does not work as well, and in fact is no improvement over approximating real and imaginary parts separately. Fig. 6 shows that the actual deviation δ can be $\sqrt{2}$ times the optimal δ_0, a possible sacrifice in rejection in a stopband of about 3 dB. The situation can be improved by adding linear constraints as shown in Fig. 7 to approximate the circle with an octagon instead of a square. This guarantees that $\delta \leqslant \delta_0/\cos(\pi/8) = 1.08239\delta_0$, or only about 0.69 dB. Since we were mainly interested in phase approximation, these extra constraints were not included in the program. (This idea can clearly be extended to an *n-gon* approximation to the circle with a guarantee of $\delta \leqslant \delta_0/\cos(\pi/n)$).

IV. Symmetry Properties of Certain All-Pass Phase Specifications

We now remind the reader of some simple but useful symmetry properties of the Fourier coefficients when the desired function $e^{jL\omega}H(\omega)$ is all-pass with an (odd) phase function $\phi(\omega)$ which is symmetric or antisymmetric about $\omega = \pi/2$. Suppose that we expand the desired response in a Fourier series:

$$e^{jL\omega}H(\omega) = e^{j\phi(\omega)}$$
$$= \sum_{k=-\infty}^{\infty} g_k e^{-jk\omega}. \qquad (17)$$

Then

$$g_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j[\phi(\omega) + k\omega]} \, d\omega$$
$$= \frac{1}{\pi} \int_0^{\pi} \cos(\phi(\omega) + k\omega) \, d\omega$$
$$= \frac{1}{\pi} \int_0^{\pi} \cos \phi(\omega) \cos k\omega \, d\omega$$
$$- \frac{1}{\pi} \int_0^{\pi} \sin \phi(\omega) \sin k\omega \, d\omega. \qquad (18)$$

If $\phi(\omega)$ is even about $\pi/2$ and k is even, the second integral vanishes, so $g_k = g_{-k}$. If, on the other hand, $\phi(\omega)$ is even about $\pi/2$ but k is odd, the first integral vanishes and $g_k = -g_{-k}$. This translates into the following.

Result 3: If the desired transfer function $e^{jL\omega}H(\omega)$ is all-pass with a phase function $\phi(\omega)$ even about $\pi/2$,

$$h_{L-k} = h_{L+k} \qquad k \text{ even}$$
$$h_{L-k} = -h_{L+k} \qquad k \text{ odd}. \qquad (19)$$

We can insist on these conditions when $\phi(\omega)$ is even, thus reducing the number of unknown parameters in (2) to $(M + 1)/2$. It turns out that the linear programming formulation is symmetric and the optimal solutions satisfy the conclusions of Result 3 when $\phi(\omega)$ is even, without additional constraints. (A specialized and faster program could be written for this case, as in [12], but was not done here.)

The conclusions of Result 3 hold for the important case of a full-band chirp filter, and show that such filters can be implemented with $(M+1)/2$ multiplications per point, just as linear phase filters can.

When $\phi(\omega)$ is an odd function about $\pi/2$, we have a similar property.

Result 4: If the desired transfer function $e^{jL\omega}H(\omega)$ is all-pass with a phase function $\phi(\omega)$ odd about $\pi/2$,

$$h_{L-k} = h_{L+k} = 0 \qquad k \text{ odd.} \tag{20}$$

V. Examples

Example 1 (Chirp All-Pass)

The first example deals with the important class of digital chirp filters, with phase specification

$$\operatorname{Arg} H(\omega) = -\left(\frac{M-1}{2}\right)\omega - \beta(\omega - \pi/2)^2 \tag{21}$$

and group delay

$$\tau = -\frac{d}{d\omega}\operatorname{Arg} H(\omega) = \left(\frac{M-1}{2}\right) + 2\beta(\omega - \pi/2). \tag{22}$$

The parameter β must be chosen with some presence of mind, since $\operatorname{Arg} H(\pi)$ must be a multiple of π. We ran two sets of designs, for $\beta = 8/2\pi$ and $16/2\pi$, with the results shown in Table I.

Figs. 8, 9, and 10 show the magnitude (in decibels), group delay, and impulse response, for the last case, $M = 61$ and $\beta = 16/2\pi$. The impulse response satisfies the symmetry relations (19) to nine significant figures, and the upper bound (11) checks.

This particular magnitude characteristic is very nearly equiripple. The reason for this can be seen by examining Fig. 4. When the solution lies on the rightmost vertical constraint boundary, for example, the actual magnitude (assuming $\epsilon_1 = \delta_1$)

$$1 + \delta_1 \leq 1 + \delta \leq \sqrt{(1+\delta_1)^2 + \delta_1^2} \leq 1 + \delta_1 + \delta_1^2 \tag{23}$$

so when δ_1 is small (as it is when $M = 61$), the magnitude response at a positive ripple is always within δ_1^2 of $1 + \delta_1$. When $M = 15$ there is more departure from equiripple behavior. In general, the solution points wander around inside of rectangles, since we are simultaneously approximating two functions, and we should expect more complicated looking minimax solutions than we are used to in the one-function case.

Example 2 (Twin-Delay All-Pass)

This is a specification of a filter of length 61 with unit magnitude and a delay of 34 in [0, 0.25] and 26 in [0.25, 0.5], with equal tolerances on magnitude and phase. The resultant

TABLE I

M	Number of Grid Points	LP Deviation	Actual Magnitude Deviation
		$\beta = 8/2\pi$	
15	250	1.922 *D*-2	1.941 *D*-2
31	500	1.667 *D*-3	1.669 *D*-3
61	610	3.793 *D*-4	3.793 *D*-4
		$\beta = 16/2\pi$	
15	250	2.299 *D*-1	2.512 *D*-1
31	500	7.745 *D*-3	7.774 *D*-3
61	610	8.765 *D*-4	8.769 *D*-4

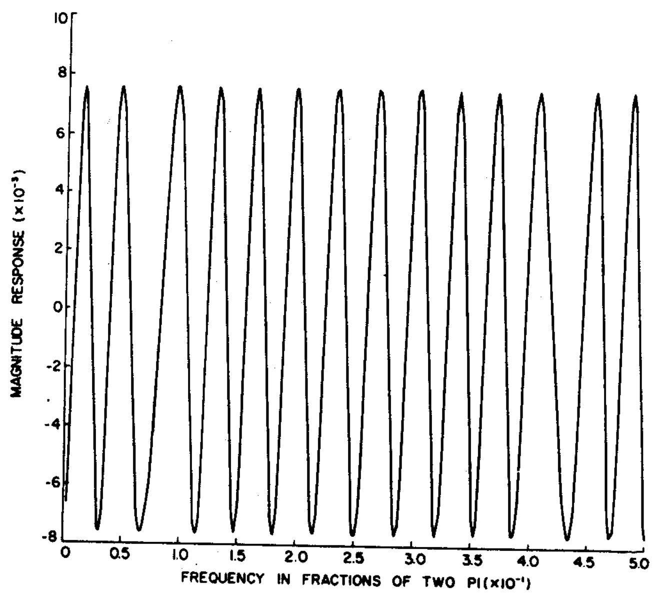

Fig. 8. Magnitude in decibels versus frequency for a full-band chirp filter with $M = 61$, Example 1.

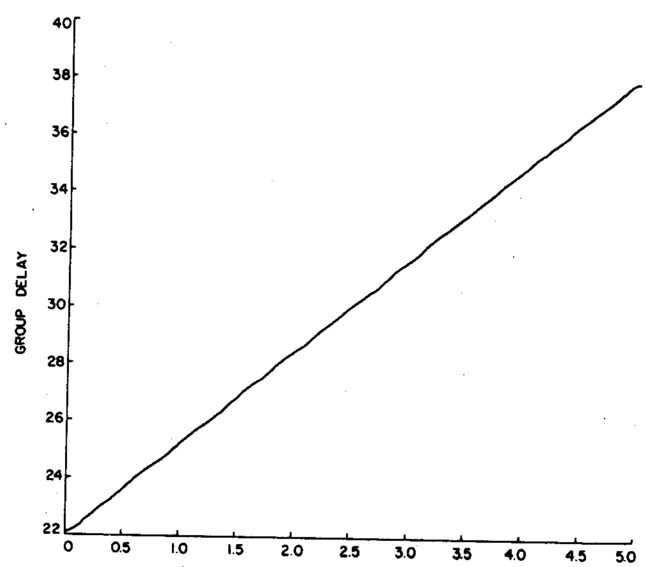

Fig. 9. Group delay versus frequency for Example 1.

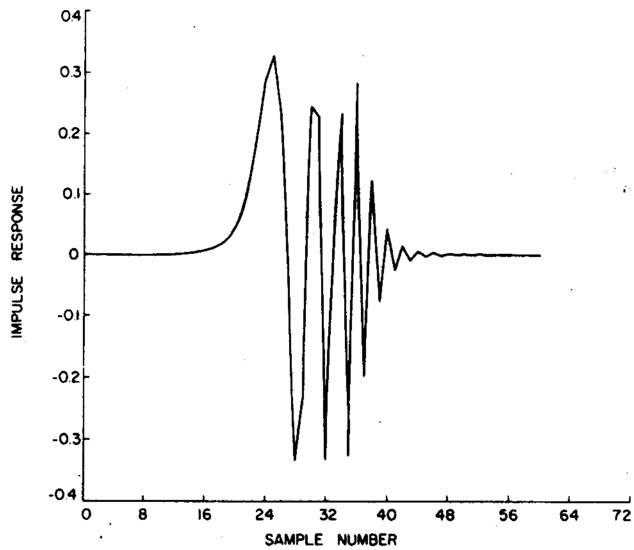

Fig. 10 Impulse response for Example 1.

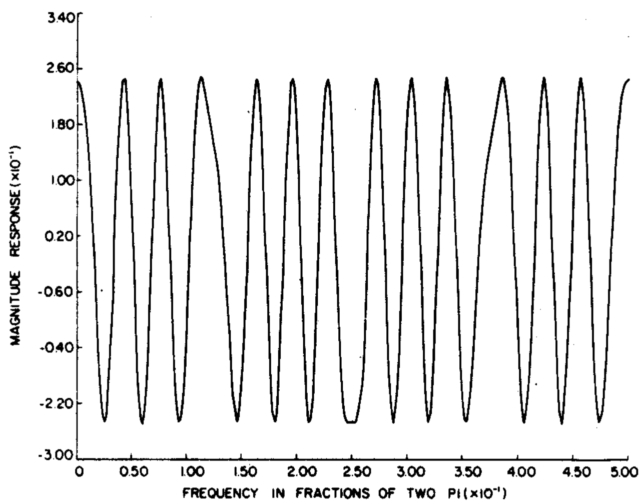

Fig. 11. Magnitude in decibels versus frequency for a twin-delay all-pass filter, $M = 61$, Example 2.

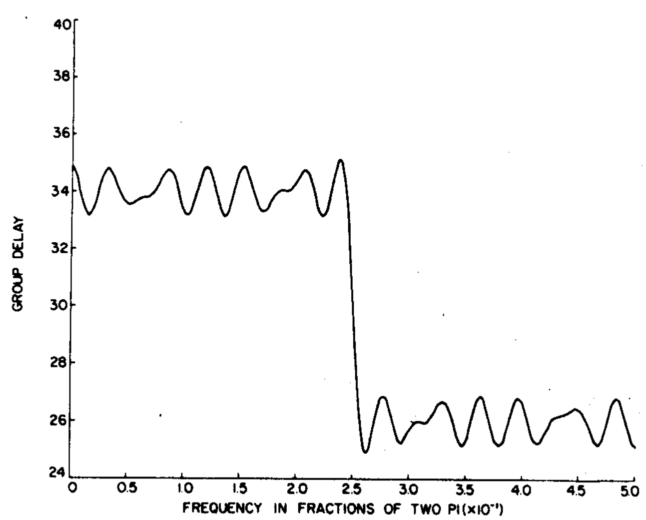

Fig. 12. Group delay versus frequency for Example 2.

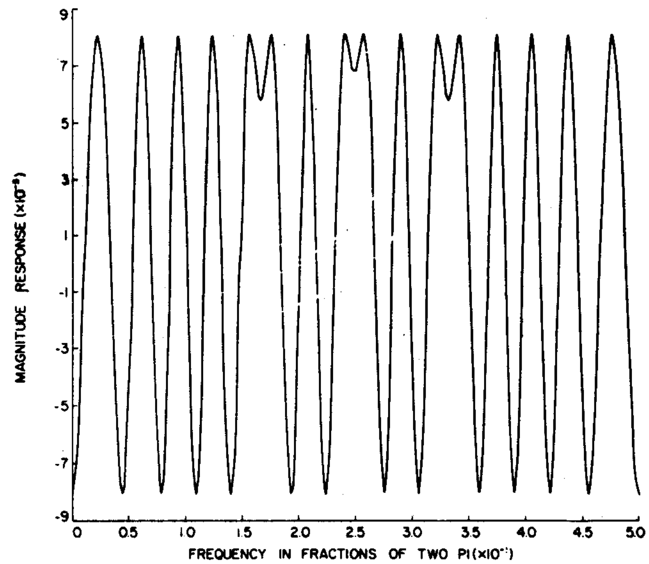

Fig. 13. Magnitude in decibels versus frequency for a sine-delay all-pass filter, $M = 61$, Example 3.

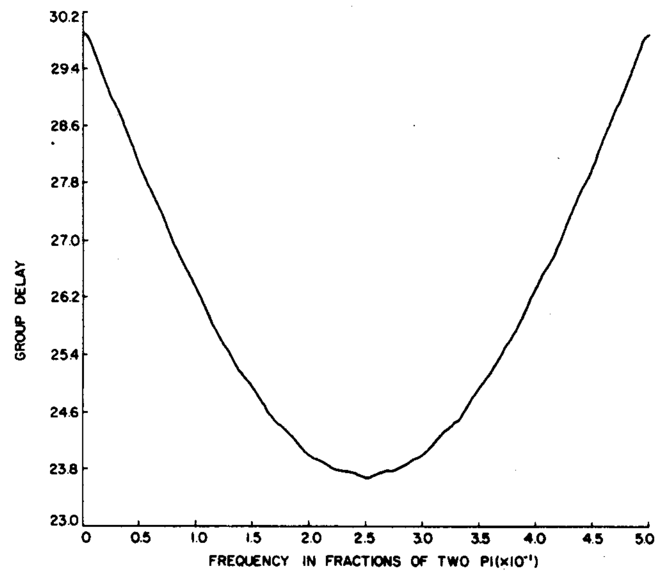

Fig. 14. Group delay versus frequency for Example 3.

linear programming deviation is 2.846 *D*-2 and actual magnitude deviation 2.885 *D*-2. Figs. 11 and 12 shows the resulting magnitude and group delay. Since the phase has a discontinuous derivative (the delay is itself discontinuous), this is a relatively difficult example.

Example 3 (Sine-Delay All-Pass)

As a last example, we approximate a unit amplitude and a phase characteristic

$$\operatorname{Arg} H(\omega) = -\left(\frac{M-1}{2}\right)\omega + 2\pi(1-\cos\omega) \tag{24}$$

corresponding to delay

$$\tau(\omega) = \left(\frac{M-1}{2}\right) - 2\pi \sin \omega. \tag{25}$$

The resultant magnitude and delay for $M = 61$ are shown in Figs. 13 and 14. The actual magnitude deviation is in this case 9.310 D-4, much lower than Example 2 because of the relatively smooth delay curve.

VI. Note on Timing

All jobs were run in double precision on an IBM 3033 computer at Princeton University, using Fortran H. The execution times were at most 1.3 s for $M = 15$ and 250 grid points, 9 s for $M = 31$ and 500 grid points, and 45 s for $M = 61$ and 610 grid points.

VII. Conclusions and Comments

We have considered here the desi of (near) all-pass FIR digital filters with prescribed phase sponse, the FIR counterparts to the usual IIR all-pass pha etworks. This nonlinear problem is well approximated by a ..ıear program—mathematically to within second-order terms in the deviation—and close enough for all practical considera ıons. This provides another example of the flexibility of the near programming approach to design optimization problems in digital signal processing, especially when the simplex algorithm is specially tailored to the problem.

In this paper two functions are being approximated simultaneously in a minimax sense, and the resultant approximants sometimes look quite different from those obtained on the more familiar linear-phase magnitude approximation problem. The author does not know whether the Remez algorithm can be applied successfully to the problem considered here.

It is also an open question whether the delay instead of the phase can be approximated in a near-minimax sense with such a linear programming formulation.

The FIR near all-pass filters obtained in this paper seem quite practical for phase equalization and chirp processing. The chirp filters of length 61, for example, require only 31 multiplications per point and have magnitude ripple of less than 0.00762 dB full band with comparable phase accuracy.

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Kenneth Steiglitz (S'57–M'64–SM'79–F'81) was born in Weehawken, NJ, on January 30, 1939. He received the B.E.E. degree in 1959, the M.E.E. degree in 1960, and the Eng.Sc.D. in 1963, all from New York University, New York, NY.

Since September 1963 he has been with the Department of Electrical Engineering and Computer Science, Princeton University, Princeton, NJ, where he is now Professor, teaching and conducting research in the computer and systems areas. He is an Associate Editor of the *Journal of the Association for Computing Machinery* and the journal *Networks*, and the author of an introductory text on discrete systems.

Dr. Steiglitz is a member of Eta Kappa Nu, Tau Beta Pi, and Sigma Xi. He is Vice-Chairman of the Digital Signal Processing Committee of the IEEE Acoustics, Speech, and Signal Processing Society, as well as Chairman of the Awards Committee, and a member of the AdCom of that Society.