Testing for Cycles in Infinite Graphs with Periodic Structure†

(Extended Abstract)

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1. Introduction

A k-dimensional *dynamic graph* is obtained by repeating a basic cell in a k-dimensional orthogonal grid. The nodes in each cell are connected to a finite number of nodes in other cells, and, furthermore, the pattern of the inter-cell connections is the same for each cell. Thus, a dynamic graph is a finitely described infinite graph, with a periodic structure. In this paper we study the problem of deciding whether a given two-dimensional dynamic graph has a directed cycle.

A k-dimensional dynamic graph can be represented by a finite graph with k-dimensional labels on each edge, which is called a *static graph.* See Fig. 1 for an example. The cycle problem then becomes that of finding whether there is a (not necessarily simple) cycle in the static graph with component-wise sum of labels equal to $(0, 0, \ldots, 0)$.

Many VLSI applications involving a regular structure can be modeled with two-dimensional dynamic graphs. For example, the above cycle problem in two-dimensional dynamic graphs is associated with the problem of whether a VLSI circuit is free of a signal "circuit loop".

Our basic approach is to construct a semiring defined on the set of convex polygons in the plane. The two operations of the semiring are vector summation ($\bullet$), and taking convex hull of union ($+$). The Kleene closure algorithm on this semiring then essentially solves the problem.

Our main result is that for certain classes of static graphs, this Kleene closure algorithm is polynomial. One such class is a generalization of two-terminal series-parallel graphs obtained by allowing a backedge from node j to i if there is a path from i to j, what we call *backedged two-terminal series-parallel* graphs. We also show polynomiality when the labels are bounded, and when the dynamic graph is undirected or one-dimensional. *Bounded graphs* arise in VLSI applications where interconnections between regular basic cells are made locally, as in sys-

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tolic arrays. Polynomiality of zero-cycle testing for general static graphs is open.

We also show that the problem of whether there is a decomposition of a given convex polygon α into two convex polygons β and γ such that $\alpha = \beta \cdot \gamma$ is NP-complete, while computing $\cdot$ product can be done in linear time with respect to the number of edges involved.

In this extended abstract, we will omit proofs of lemmas and theorems unless they are specially related to the further discussion. We use n (resp. m) to denote the number of vertices (resp. edges) in a graph.

2. Two operations and a semiring

In this section, we define two operations on the set of convex polygons which form a closed semiring (Mehlhorn 1984a).

Let $S = \{ \alpha^+ \mid \alpha \in 2^{Z \times Z} \}$ where α^+ indicates the convex hull of α. That is, S is the set of all convex polygons whose vertices are integer points. We regard a point or a line segment as an element of S. Let $\varepsilon = \{ (0, 0) \} \in S$. For $\alpha, \beta \in S$, let $\alpha + \beta = (\alpha \cup \beta)^+$, that is, $\alpha + \beta$ is the convex hull of the union of α and β. Let $\alpha \cdot \beta$ be the polygon obtained by the vector summation of α and β, that is,

$$\alpha \cdot \beta = \{ (x, y) \mid \exists (a_x, a_y) \in \alpha, \exists (b_x, b_y) \in \beta \text{ such that } x = a_x + b_x, y = a_y + b_y \}.$$

By convention, we define $\alpha \cdot \varnothing = \varnothing \cdot \alpha = \varnothing$. See Fig. 2 for an example of $\alpha \cdot \beta$. Note that we can easily show that $\alpha \cdot \beta$ is a convex polygon, and thus S is closed under the $\cdot$ operation.

Theorem 2.1: The system $(S, +, \cdot, \varnothing, \varepsilon)$ is a closed semiring.

Proof: The following are the five defining properties of semirings:

1) $(S, +, \varnothing)$ is a commutative monoid.
2) $(S, \cdot, \varepsilon)$ is a monoid.
3) Multiplication $\cdot$ distributes over finite and countably infinite sums $+$.
4) Let $I = \{ i_1, i_2, \ldots, i_k \}$ be a finite non-empty index set. Let $\alpha_i \in S$ for all $i \in I$. Then
$$\sum_{i \in I} \alpha_i = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k}$$
where $\sum_{i \in I} \alpha_i$ is defined by $(\bigcup_{i \in I} \alpha_i)^+$. We also have $\sum_{i \in \varnothing} \alpha_i = \varnothing$ for empty index set $I = \varnothing$.
5) The result of summation does not depend on the ordering of the factors.

The proof is technically detailed but straightforward. 1) can be shown by using the fact that $(\alpha^+ \cup \beta) = (\alpha \cup \beta)^+$ for any $\alpha, \beta \in 2^{Z \times Z}$. 2) is straightforward. 4) can be proved by induction on k. We can prove 5) by using the fact that $(\bigcup_{i \in I} \alpha_i^+)^+ = (\bigcup_{i \in I} \alpha_i)^+$ for $\alpha_i \in 2^{Z \times Z}$, $i \in I$ where I is a countable index set. Note that if x is a point in the union of countably many convex polygons, then x is either a point in a finite sum of convex polygons, or the limit point of a sequence of points, each of which is in some convex polygon. Using this idea, we can prove 3). □

Let $E(\alpha)$ be the edge set of the convex polygon α. We regard an edge $e \in E(\alpha)$ as a vector e which is directed in such a way that α lies on the right-hand-side of that edge. Two edges are *equivalent* if their directions are the same. Let $\alpha_{vector} = \{ e \mid e \in E(\alpha) \}$. Let $A = \{ \alpha_i \}$ be a set of convex polygons. Then we define

$$| A | = \textit{the number of distinct vectors in } \bigcup_{\alpha_i \in A} (\alpha_i)_{vector}.$$

Theorem 2.2: (Lozano-Perez 1983; Guibas, Ramshaw, and Stolf 1983) Suppose two convex polygons $\alpha, \beta \in S$ are not points. Then every edge in $E(\alpha \bullet \beta)$ is equivalent to an edge in $E(\alpha) \cup E(\beta)$, and vice versa. This enables us to define an onto function $f = \varphi(e)$ from $E(\alpha) \cup E(\beta)$ to $E(\alpha \bullet \beta)$. (See Fig. 2.)

For a convex polygon α, we define a convex polygon α^m for nonnegative integer m as follows: 1) $\alpha^0 = \varepsilon$ and 2) $\alpha^m = \alpha \cdot \alpha^{m-1}$ for $m > 1$. Since a system $(S, +, \cdot, \varnothing, \varepsilon)$ is a closed semiring, we can define the convex polygon α^* by $\alpha^0 + \alpha^1 + \cdots = \sum_{i=0}^{\infty} \alpha^i$.

The following theorem shows how $\bullet$ and $+$ operations affect the number of distinct vectors.

Theorem 2.3: Let $\alpha, \beta, \alpha_i, \beta_i \in S$ for $i = 1, 2, \ldots, k$. Then we have

$$| \alpha \bullet \beta | \le | \{ \alpha, \beta \} | \le | \alpha | + | \beta |, \quad (2.1)$$

$$| \alpha + \beta | \le | \alpha | + | \beta |, \quad (2.2)$$

and

$$\left| \sum_{i=1}^{k} \alpha_i \bullet \beta_i^* \right| \le \left| \sum_{i=1}^{k} \alpha_i \right| + 1. \quad (2.3)$$

Proof: (2.1) is immediate from Theorem 2.2. Let $\alpha \in S$ and let $p_1, p_2, \ldots, p_k$ be points. Then we have

$$| \alpha + p_1 + \cdots + p_k | \le | \alpha | + k. \quad (2.4)$$

(2.2) is straightforward from (2.4). We can also show that

$$\alpha_1 \bullet \beta_1^* + \cdots + \alpha_n \bullet \beta_n^* = (\alpha_1 + \cdots + \alpha_n) \bullet \beta_1^* \bullet \cdots \bullet \beta_n^* \quad (2.5)$$

for $\alpha_i, \beta_i \in S$, $i = 1, 2, \ldots, n$. Then we have (2.3) immediately from (2.5) and $| \alpha \bullet \beta^* | \le | \alpha | + 1$. □

Theorem 2.4: Let $| \alpha | + | \beta | = m$. Then the operations $\bullet$, $+$ and $*$ can all be done in $O(m \log m)$ steps.

Proof: From Theorem 2.2, we know that the $\bullet$ operation takes $O(m)$ time. We have an algorithm which takes $O(m \log m)$ time for computing the convex hull of two convex polygons (Mehlhorn 1984b). The $*$ operation can be done in $O(\log m)$ time by an algorithm which computes $\alpha + \{ p \}$ in $O(\log(| \alpha |))$ steps where α is a convex hull and p is a point (Mehlhorn 1984b). □

A convex polygon α is said to be *decomposable* if and only if there exist two convex polygons β and γ such that $\alpha = \beta \bullet \gamma$ and neither β nor γ is a point and there are no equivalent edges in $E(\beta)$ and $E(\gamma)$. As illustrated in Fig. 3, a decomposition into irreducible convex polygons is not necessarily unique. The decomposability problem has been well studied (Kallay 1984; Shephard 1963; Meyer 1974). In contrast with the $\bullet$ operation, the following theorem shows that decomposition of a convex polygon is in general difficult.

Theorem 2.5: The problem determining whether a given convex polygon α is decomposable or not is NP-complete.

Proof: We can easily show that the decomposability problem can be formulated as follows:

Instance I_{DC}: Let I be a finite index. A set of two-dimensional integer vectors $\{ \mathbf{e}_i \mid i \in I \}$ such that $\mathbf{e}_i \neq \mathbf{e}_j$ for any $i, j \in I$ with $i \neq j$ and $\sum_{i \in I} \mathbf{e}_i = (0, 0)$.

Question: Is there a proper subset J of I such that $\sum_{j \in J} \mathbf{e}_j = (0, 0)$?

It is obvious that the problem DC is in NP, since we can guess a proper subset $J \subset I$ then we can check whether or not $\sum_{j \in J} \mathbf{e}_j = (0, 0)$ in polynomial time.

We reduce the following variation of the subset sum problem SS_1, which is NP-complete (Papadimitriou and Steiglitz 1982), to the problem DC.

Instance I_{SS_1}: $\{ a_k \in Z^+ \mid k \in K \}$ where K is a finite index and a_k's are different from each other. $B \in Z^+$.

Question: Is there a subset L of K such that $\sum_{l \in L} a_l = B$?

We reduce SS_1 to DC as follows: given the above instance I_{SS_1}, we construct an instance I_{DC} with the property that I_{SS_1} has a solution if and only if I_{DC} has a solution. Let

$$K = \{ 1, 2, \ldots, n \}$$

and

$$I = \{ 0, 1, \ldots, 2n + 3 \}.$$

Let

$$A = \sum_{k \in K} a_k$$

and

$$M = (n + 1)(A + B).$$

Then we define a set of vectors

$$\{ \mathbf{e}_i \mid i = 1, 2, \ldots, 2n + 3 \}$$

as follows:

$$\mathbf{e}_0 = (-B, 1),$$

$$\mathbf{e}_i = (a_i, 1) \quad \text{for } i = 1, 2, \ldots, n,$$

$$\mathbf{e}_{n+i} = (M, -i) \quad \text{for } i = 1, 2, \ldots, n, n + 1,$$

$$\mathbf{e}_{2n+2} = (-M, 0), \text{ and}$$

$$\mathbf{e}_{2n+3} = -\left(\sum_{i=0}^{2n+2} \mathbf{e}_i \right)$$

$$= (-nM + B - A, n(n + 1)/2).$$

The rest of the proof is straightforward. □

3. Application of the closed semiring $(S, +, \cdot, \emptyset, \varepsilon)$

From now on, we discuss a doubly weighted graph $G = (V, E)$ with a two-dimensional labeling T such that $T(e) = (e_x, e_y)$ for $e \in E$. We can naturally extend the definition of T to a path $W = e_1 e_2 \cdots e_k$ with $e_i \in E$ in such a way that $T(W) = \sum_{i=1}^{k} T(e_i)$. A (not necessarily simple) cycle W is called a *zero-sum* cycle if $T(W) = (0, 0)$. Then the *zero-sum cycle* problem is to ask whether there exists a zero-sum cycle in a given doubly weighted graph. Note that a doubly weighted digraph G can be regarded as a static graph. Hence we have the two-dimensional dynamic graph G^2 induced by the static graph G as illustrated in Fig. 1. Then the zero-sum cycle problem in G corresponds to the problem of whether the dynamic graph G^2 has a cycle or not.

Having established that the structure $(S, +, \cdot, \emptyset, \varepsilon)$ is a closed semiring, we can solve the zero-sum cycle problem with the Kleene closure algorithm (Aho, Hopcroft, and

Ullman 1974) as follows:

For a given digraph G, the algorithm ZSC answers "Yes" if the digraph G has a zero-sum cycle, otherwise it answers "No". We compute the convex hull α_{ij}^{k} for $1 \le i, j \le n$ and $0 \le k \le n$. The convex hull α_{ij}^{k} is the convex hull of the lengths of all paths from v_i to v_j such that all vertices on the path, except possibly the endpoints, are in the set $\{ v_1, v_2, \ldots, v_k \}$.

The algorithm ZSC is as follows:

procedure Zero-Sum Cycle
begin
1. **for** $1 \le i, j \le n$ **do**

$$\alpha_{ij}^{0} = \begin{cases} \{ T((v_i, v_j)) \} & \text{if } (v_i, v_j) \in E \\ \emptyset & \text{otherwise} \end{cases}$$

2. **for** $k = 1$ **to** n
   do
3. **for** $1 \le i, j \le n$ **do**
4. $$\alpha_{ij}^{k} = \alpha_{ij}^{k-1} + \alpha_{ik}^{k-1} \cdot (\alpha_{kk}^{k-1})^{*} \cdot \alpha_{kj}^{k-1};$$
5. **if** $1 \le \exists i \le n$ such that $(0, 0) \in \alpha_{ii}^{k}$
   then exit ("Yes");
   od
6. exit ("No");

end □

Theorem 3.1: Algorithm ZSC works correctly and uses $O(n^3)$ +, •, and * operations from the closed semiring defined above.

Proof: We can show that there is a zero-sum cycle W if and only if there is a vertex v_i such that $(0, 0) \in \alpha_{ii}^{n}$. Line 4 is executed n^3 times in total. □

4. Special cases of the zero-sum problem

In this section, we discuss the special cases of the zero-sum problem where 1) the graphs have one-dimensional labels, 2) the graphs are undirected, 3) the graphs have labels with magnitude at most M (we call these graphs the *M-bounded graphs*), and 4) we discuss the zero-sum *simple* cycle problem. The first three cases have low order polynomial algorithms, whereas the fourth is NP-complete.

Theorem 4.1: The one-dimensional zero-sum cycle problem can be solved in $O(n^3)$ time. (This result is implicit in Orlin 1984.)

Proof: We can apply our algorithm ZSC by ignoring the second labels. Note that every α_{ij}^{k} has at most two vertices, and, therefore, $| \alpha_{ij}^{k} | \le 2$. Hence from Theorem 3.1, the algorithm ZSC takes $O(n^3)$ time. □

Theorem 4.2: The two-dimensional undirected zero-sum cycle problem can be solved in $O(m \log m)$ time.

Proof: We only have to check whether the convex hull of $\{ T(e) \mid e \in E \}$ contains the origin or not, which takes $O(m \log m)$ time. Further discussion of the special cases is straightforward. □

In many VLSI applications, the communication between regular cells is made locally: that is, interconnections are made only to neighbors. For example, the $n \times n$ multiplier can be constructed from arrays of one-bit full adders with carry and sum signal connections to the neighbors of each cell. The parallel adder can also be constructed from one-bit full adders with the carry connection to the neighbor of each cell. Many systolic arrays are also implemented with interconnections to neighbors. In such VLSI applications, the associated static digraphs of the regular structure are all 1-bounded graphs. (Iwano and Steiglitz 1986.)

Theorem 4.3: The algorithm ZSC takes $O(n^4 M \log(nM))$ time for M-bounded graphs.

Proof: Let β_{ij}^k be defined in the same way as α_{ij}^k except that β_{ij}^k is defined for simple paths. In M-bounded graphs, the length of a simple path is at most nM in each dimension. Therefore, $|\beta_{ij}^k| \le 4nM + 2$, since β_{ij}^k is bounded by the rectangle

$$[-nM, nM] \times [-nM, nM].$$

From Theorem 2.3,

$$|\alpha_{ij}^k| \le |\beta_{ij}^k| + 1 \le 4nM + 3.$$

□

Theorem 4.4: The zero-sum simple cycle problem (ZSSC) is NP-complete.

Proof: Here we use a variant of Orlin's reduction from the subset sum to the directed path problem in one-dimensional dynamic graphs (Orlin 1984). The problem SS (Subset Sum) is defined as follows:

Input: $B \in Z^+$ and $\{ a_i \in Z^+ \mid i \in I \}$ where $I = \{1, 2, \ldots, n\}$.

Question: Is there a subset J of I such that $\sum_{j \in J} a_j = B$?

Given an instance I_{SS} of the subset sum problem, we construct an instance I_{ZSSC} of the zero-sum simple cycle problem as illustrated in Fig. 4. Then we can show that I_{SS} has a solution if and only if I_{ZSSC} has a zero-sum simple cycle. The zero-sum simple cycle is easily shown to be in NP. □

5. Backedged two-terminal series-parallel multidigraphs.

The class of Two-Terminal Series-Parallel (TTSP) graphs has been well studied (Valdes, Tarjan, and Lawler 1979; Adam 1961; Duffin 1965; Riordan and Shannon 1942; Weinberg 1971).

Definition: [Two-Terminal Series-Parallel Multidigraphs]. (Valdes, Tarjan, and Lawler 1979)

(1) A digraph consisting of two vertices joined by a single edge is in TTSP.

(2) If G_1 and G_2 are TTSP multidigraphs, so too is the multidigraph obtained by either of the following operations:

  (2.a) *Two terminal parallel composition:* identify the source of G_1 with the source of G_2 and the sink of G_1 with the the sink of G_2.

  (2.b) *Two terminal series composition:* identify the sink of G_1 with the source of G_2. □

Definition: [Backedged Two-Terminal Series-Parallel Multidigraph].

Let G be a TTSP graph. A multidigraph G_B is called a *BTTSP* (*Backedged Two-Terminal Series-Parallel*) graph if G_B is obtained from a TTSP graph G by adding any number of *backedges*. An edge (x, y) is called a *backedge* if there is a path from y to x in G. The graph G is called *the underlying*

TTSP graph of G_B. □

Let $TTSP(m)$ (resp. $BTTSP(m)$) be the class of TTSP (resp. BTTSP) multidigraphs which have m edges. Fig. 5 shows an example of a BTTSP graph G_B which consists of a backedge indicated by dotted lines and the underlying TTSP graph G.

For any $x, y \in V$, let $\alpha_{xy}(T)$ be the convex hull of all lengths of paths from x to y in G with the two-dimensional labeling T. For any multidigraph G, let $A(G) = \max_{x,y,T} |\alpha_{xy}(T)|$ and similarly for a class of graphs we write $A(\{G\})$. That is, $A(G)$ is the maximum number of edges in $\alpha_{xy}(T)$ when x, y, and T are arbitrary and G is fixed.

Theorem 5.1: $A(TTSP(m)) = m$.

Proof: Let G be in $TTSP(m)$ with source s and sink t, and let T be a two-dimensional labeling of G. Let x, y be arbitrary vertices in G such that $(x, y) \neq (s, t)$. Then there exists a two-dimensional labeling T' such that

$$|\alpha_{xy}(T)| \leq |\alpha_{st}(T')|.$$

Thus

$$A(TTSP(m)) = \max_T |\alpha_{st}(T)|.$$

Suppose G is obtained from G_1 and G_2 by a parallel (resp. series) composition. Let s_i be source and t_i be sink of G_i for $i = 1, 2$. Then

$$\alpha_{st} = |\alpha_{s_1t_1} + \alpha_{s_2t_2}|$$

$$(\text{or } |\alpha_{s_1t_1} \bullet \alpha_{s_2t_2}|)$$

$$\leq |a_{s_1t_1}| + |\alpha_{s_2t_2}|.$$

(from Theorem 2.3)

Then by induction on m, we can prove $A(TTSP(m)) \leq m$. On the other hand we have $A(TTSP(m)) \geq m$, since $A(L_m) = m$ where $L_m \in TTSP(m)$ is a graph consisting of m edges from source to sink. □

We can show the same result for the class of BTTSP multidigraphs.

Theorem 5.2: $A(BTTSP(m)) = m$.

Proof: Let B_{st} be an arbitrary $s-t$ path in a BTTSP graph G_B. Then we can show that every backedge in B_{st} lies on a cycle in B_{st}. Therefore, B_{st} can be expressed as union of a simple $s-t$ path in G and some cycles. Thus from Eq. (2.3), we have

$$A(BTTSP(m)) \leq A(TTSP(m-1)) + 1$$

$$= m.$$

Since

$$TTSP(m) \subset BTTSP(m),$$

we have

$$m = A(TTSP(m)) \leq A(BTTSP(m)).$$

□

From Theorem 3.1 and 5.2, we have the following corollary:

Corollary 5.1: For BTTSP, the algorithm ZSC runs in $O(n^3 m \log m)$ time.

We are now working on the following conjecture about the number of edges of the convex polygons which appear in the algorithm ZSC for general graphs:

Conjecture. Let G be a general graph and let T and $\alpha_{ij}(T)$ be defined in the same way as in the text. Then

$$A(G) = \max_{i,j,T} | \alpha_{ij}(T) | \leq m$$

where m is the number of edges in G.

If this conjecture is true, then algorithm ZSC runs in $O(n^3 m \log m)$ time on general graphs.

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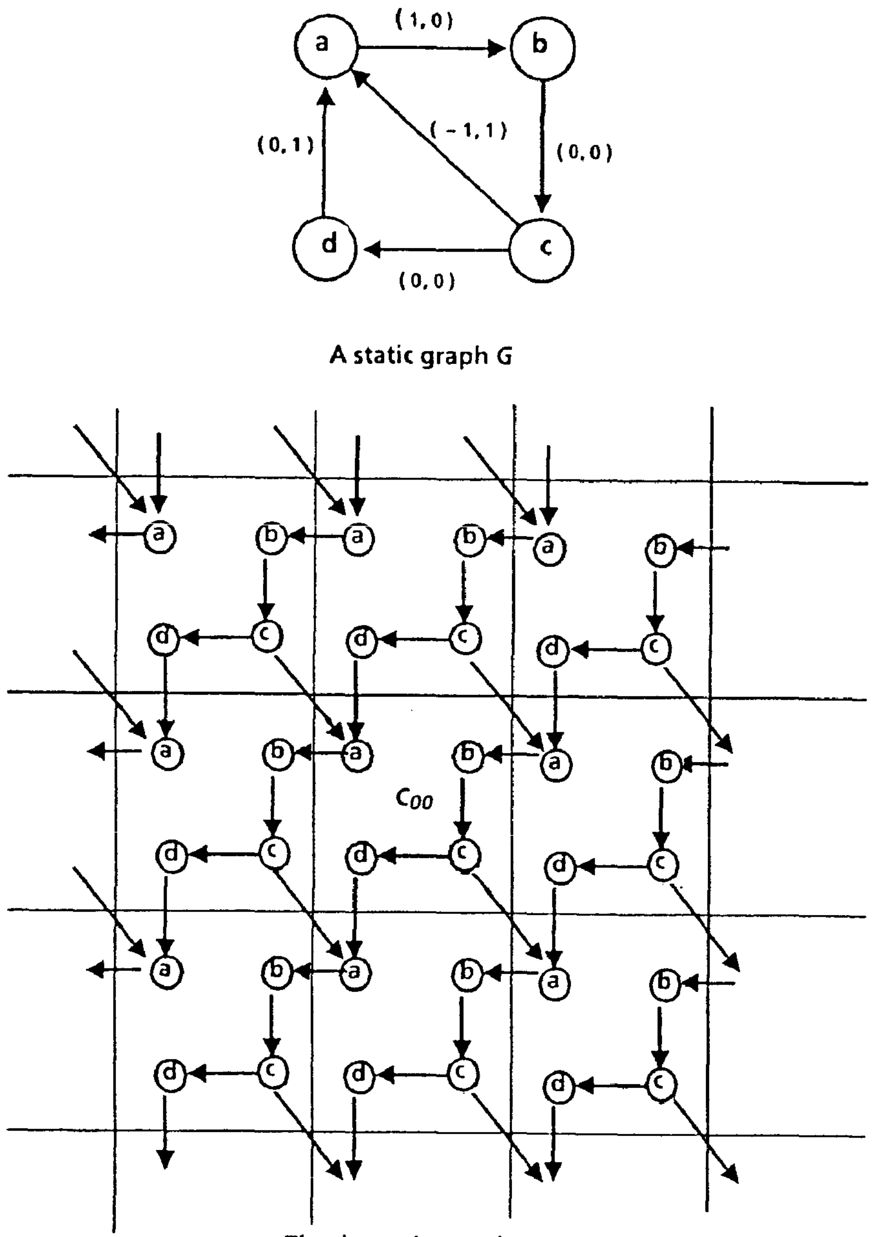

Figure 1. A static graph G shows how to connect the nodes in G^2. The shaded area shows the basic cell C_{00}.

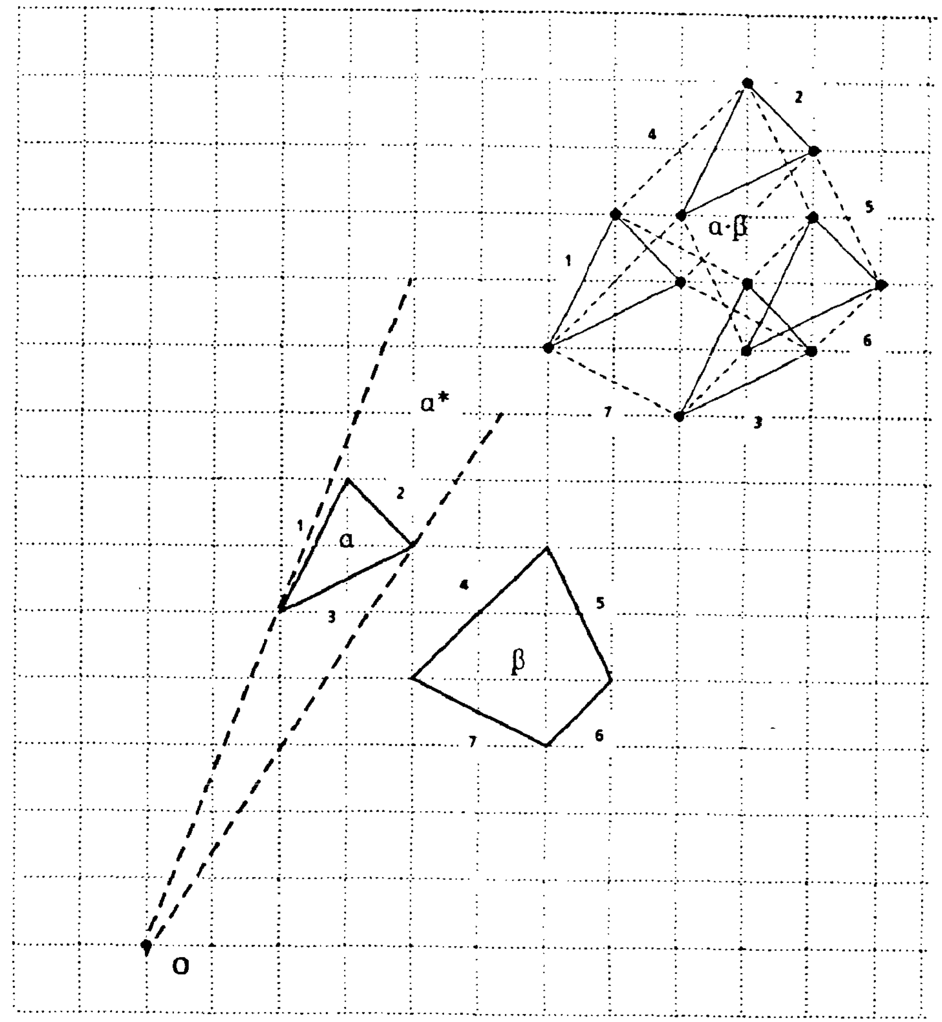

Figure 2. α·β is bounded by edges which are equivalent to the edges of α or β. The equivalent edges are shown by the same numbers.

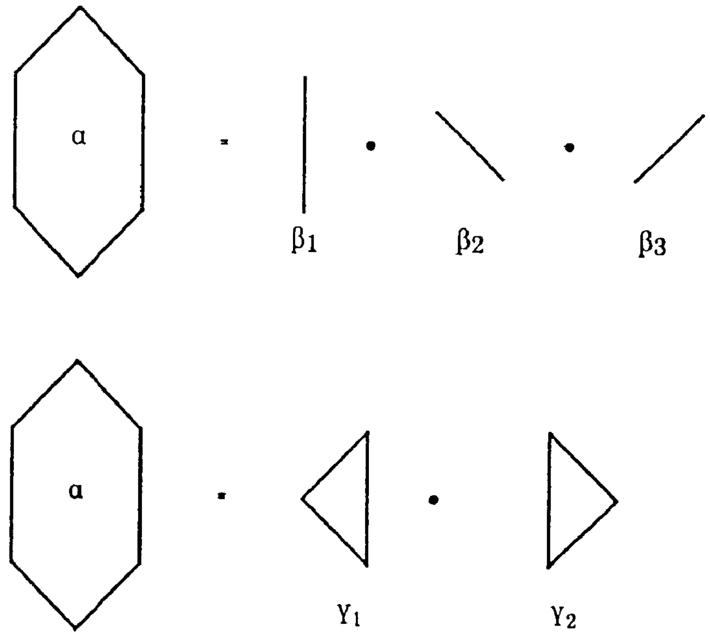

Figure 3. α is decomposed in two different ways.

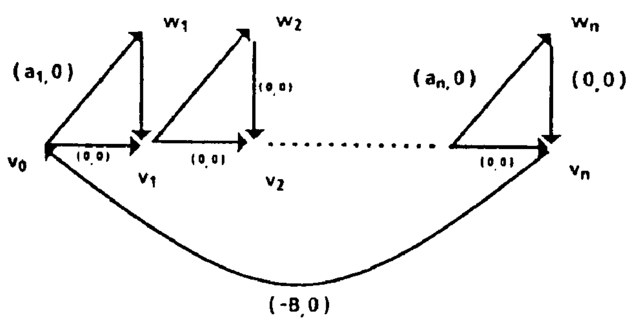

Figure 4. The graph above has a zero-sum simple cycle if and only if there exists a subindex $J \subset I = \{1, 2, \ldots, n\}$ such that $\Sigma_{j \in J} a_j = B$.

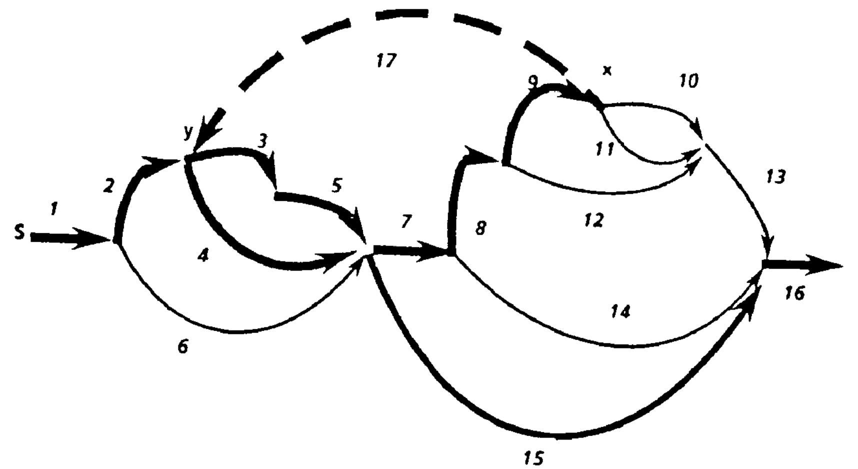

Figure 5. A BTTSP multidigraph G_B; a backedge 17 is indicated by the dotted line. The wide lines form an *s-t* path B_{st} with a backedge *17*. Note that B_{st} consists of an *s-t* path in *G* (*1-2-4-15*) and a cycle in G_B (*3-5-7-8-9-17*).