constraint on the estimated state vector $\mathbf{y}$ which has been overlooked. Because the first K elements of $\mathbf{y}^{(s)}$ are the output, at K successive sampling instants, of a dynamic system described by a Kth order difference equation [(1) of the original paper], each is completely determined by the values of the K preceding outputs, the K preceding inputs, and the current input. These $2K+1$ values for each of the output variables in $\mathbf{y}^{(s)}$ are contained in $\mathbf{y}^{(s-1)}$ and the last K elements of $\mathbf{y}^{(s)}$. Therefore, while $\mathbf{y}^{(1)}$ can be selected arbitrarily to minimize the criterion D, *only the last K elements* of $\mathbf{y}^{(s)}$ $(s \neq 1)$ can be so chosen.

But the procedure described by Koopmans[2] requires that $\mathbf{y}^{(s)}$ be constrained *only by* the equation

$$\mathbf{y}^{(s)T}\gamma = 0 \quad \text{for } s = 1, 2, \cdots, S. \tag{2}$$

Hence, (4) of the paper, Koopmans' result of minimization with respect to $\mathbf{y}^{(s)}$, is inapplicable; the vectors $\omega^{(s)}$ and the corresponding D are not correct unless the unconstrained minimum of (4) happens to coincide with the constrained minimum.

The additional constraint can, of course, be ignored if the nonoverlapping observation sets $\lambda^{(s)}$ are chosen to be so widely separated that $\lambda^{(s)}$ is independent of $\lambda^{(s-1)}$. However, this choice means a much longer period of observation to obtain an estimate with a given variance, and implies an a priori assumption about the effective settling time of the system.

The proposed estimate, while apparently not optimal, may still be useful if the unconstrained minimum is close to the constrained (true) minimum. One way of checking this possibility for a particular set of observations would be to compute Koopmans' optimal state vector

$$\omega^{(s)} = \lambda^{(s)} - \frac{\hat{\gamma}^T\lambda^{(s)}}{\hat{\gamma}^T Z \hat{\gamma}} Z\hat{\gamma}$$

after finding the estimate $\hat{\gamma}$. These vectors $\omega^{(s)}$ specify an estimated set of inputs and outputs $c(i)$ and $r(i)$ which can be substituted into (1) along with the α's and β's specified by $\hat{\gamma}$.

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[2] Koopmans, T., *Linear Regression Analysis of Economic Time Series*, N.V. Haarlem, The Netherlands: DeErven T. Bohn, 1937.

Author's Comment[3]

It is correct, as L. E. McBride, Jr. observes, that in Section IV of the paper referred to,[1] the maximum likelihood estimates were determined without taking into account the linear constraints between elements of adjacent $\mu^{(s)}$ vectors. This was not overlooked by the author, but apparently the discussions of this point in Sections IV and VII require amplification.

The estimates of Section IV utilize only a part of the information available from the $x(n)$ and $y(n)$ sequences to estimate the pulse transfer function coefficient vector γ, namely the linear relation among the elements of each individual vector $\mathbf{y}^{(s)}$. If the further linear relations between disjoint $\mathbf{y}^{(s)}$ vectors could be included then the estimates would be improved but, unfortunately, no explicit solution is known. Since these constraints are omitted, the elements of the $\omega^{(s)}$ vectors are constrained to obey the linear relation (2) only for elements within each vector and hence are not particularly good estimates of the $\mathbf{y}^{(s)}$. However, this paper was directed only to estimates of γ which were shown to be equivalent to fitting the coefficients of a hyperplane to a set of observed points in a hyperspace. So far as I can see, McBride has not established any reason why the additional linear relations invalidate these estimates or their properties which were given.

As pointed out in Section V, if there were more than one linear constraint of the type (2) among the elements within each $\mathbf{y}^{(s)}$, then the hyperplane of best fit would not be well defined. However, it can be verified that the presence of additional linear constraints between elements of disjoint $\mathbf{y}^{(s)}$ does not, in general, produce this effect. It should be noticed that if for some reason it was desired to base the estimates on a set of observed points for which the linear constraints between the $\mathbf{y}^{(s)}$ were absent, then it would suffice to select every other nonoverlapping point; the settling time is immaterial.

It was mentioned in Section VII that all the linear relations (2) could be included by taking as the observed points all overlapping sets of points. However, the noise components are not uncorrelated but certain components of each noise vector $\zeta^{(s)}$ are identically equal to other components of $\zeta^{(s-1)}$. This suggests a somewhat different way of setting up the maximum likelihood estimates but, as before, an explicit solution is not known. If the constraints between disjoint $\zeta^{(s)}$ are omitted, then the resulting estimates of γ are plausible from the geometric point of view but the $\omega^{(s)}$ are not useful estimates of the $\mathbf{y}^{(s)}$.

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[3] Manuscript received November 3, 1964.

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Comments on the Statistical Design of Linear Sampled-Data Feedback Systems

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In the last few years several methods have been proposed for statistical design of sampled-data systems. Though these methods are based on Wiener's design philosophy they differ as to the approach used. Generally three basic methods are proposed, namely, 1) the Laplace transform theory [1], 2) the Z-transform Theory [2], and 3) finally the modified Z-transform method [3]–[7]. The basic philosophy of design using these methods is to minimize the mean square value of the error function at all times. As to be expected these methods yield the same optimization results; however, the Z-transform approach [2] seems to be more convenient and straightforward than the other two methods. In particular, when using the modified Z transform special care should be exercised in extending its definition to the two-sided form. In this connection, certain minor errors [4] have been committed using this approach which unfortunately been propagated in recent texts [6], this error has been clarified by T. Nishimura [7] in a correspondence item.

The evaluation of the mean square error at all times can be performed using either one of the previously mentioned methods. However, with the recent availability of tables of total square integrals [10], it seems that the modified Z transform is most amenable for easy use.

For instance the mean square error is given in the following three forms:

$$\overline{e(t)^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_{ee}(s)ds \tag{1}$$

$$\overline{e(t)^2} = \frac{1}{2\pi j}\oint_{\substack{\text{unit}\\ \text{circle}}} \frac{1}{T}\Phi_{rr}(z)Z[G(s)G(-s)]\frac{dz}{z} \tag{2}$$

$$\overline{e(t)^2} = \frac{1}{2\pi j}\oint_{\substack{\text{unit}\\ \text{circle}}} \overline{\Phi_{ee}(z, m)}\,\frac{dz}{z} \tag{3}$$

where

$$\overline{\Phi_{ee}(z, m)} = \int_0^1 \Phi_{ee}(z, m)dm.$$

The equivalence between (1) and (2) has been shown by Sklansky [8] and the equivalence between (2) and (3) is given by Mori's formulation [9] as follows:

$$|\overline{G(z, m)}|^2_{z=e^{j\omega t}} = \frac{1}{T}Z[G(s)G(-s)]\Big|_{s=j\omega}. \tag{4}$$

This formula can be generalized to give [10]

$$\overline{G(z, m)G(z^{-1}, m)} = \frac{1}{T}Z[G(s)G(-s)] \tag{5}$$

or [10],

$$\overline{G(z, m)H(z^{-1}, m)} = \frac{1}{T}Z[G(s)H(-s)]. \tag{6}$$

Further extension of these formulas can be found elsewhere [10], [11].

By utilizing (4) one can easily show the equivalence between (2) and (3). For table use, (1) is quite complicated because of the mixed form of (e^{Ts}) and (s) in this equation. The Z-transform approach using (2) is not very convenient for the table. This is because $Z[G(s)G(-s)]$ should be factored in the form $F(z)F(z^{-1})$. However, the third form is amenable to table use directly, for one can integrate with respect to "m" after using the table. For a higher-order system this approach is simpler than using (2).

In concluding this correspondence one may mention that for optimization procedure, i.e., in obtaining the form of the

optimum filter, the second approach, the Z transform, is most convenient in most practical cases. However, for calculating the mean square value of the error, the modified Z transform is more straightforward and easy to apply for table use.

The fact that the three approaches are equivalent can be readily ascertained from the equivalence of (1), (2) and (3).

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J. T. Tou[1]

The growing number of correspondence, articles, books, and reports concerning the statistical design of linear sampled-data feedback systems indicates the great interest and enthusiasm of engineers toward this relatively simple problem which was satisfactorily solved some four or five years ago. The three methods of design mentioned by Jury are quite straightforward and by now well known among control engineers. Further review and interpretation of these three design techniques appear to be superfluous. Would it be more valuable to devote the precious printed pages to new techniques for statistical design of nonlinear sampled-data feedback control systems? The modified Z-transform approach has been applied to statistical design of sampled-data feedback systems subject to power limitations [1]. It is hoped that these TRANSACTIONS will publish more articles on statistical design of nonlinear discrete-data feedback control systems.

The writer does not agree with Jury's statement that the Z-transform approach is more convenient than the other two methods. The modified Z-transform approach to statistical design [2], which the writer developed four years ago, is in fact at least as convenient as the Z-transform approach. The proposed method reduces the optimum design problem to simple integration and elementary algebraic manipulations, which is almost as simple as we can go in system design. As an illustration, consider a sampled-data system with unity feedback and

$$G_1(s) = \frac{1 - e^{-TS}}{s},\quad G_2(s) = \frac{10}{s + 10}\quad G_d(s) = 1.$$

The input signal and noise are uncorrelated and have spectral densities given by

$$\phi_{r_s r_s}(\omega) = \frac{4}{0.04 + \omega^2} \quad \text{and} \quad \phi_{nn}(\omega) = 0.1.$$

Following the writer's design procedure, the pertinent z-transforms are determined:

$$K_1^+(z) = \frac{-1.368(z + 0.368)}{z - 0.368}$$

$$K_1^-(z) = \frac{z - 1.005}{1 - 0.368z}$$

$$K_2(z^{-1}) = \frac{1.9z(1 + 1.41z)(1 - 1.05z)}{(1 - 0.68z)(z - 0.98)}$$

$$K_4(z^{-1}) = \frac{-1.73z(1 + 2.75z)(1 - 0.19z)}{(z - 0.368)(1 - 0.368z)}$$

$$\phi_{r_s r_s}(z) = \frac{-0.4z}{(z - 1.02)(z - 0.98)}$$

$$\phi_{n_1 n}(z) = \frac{-z}{z - 2.718}$$

$$\phi^+(z) = \frac{-10z(z - 0.809)(z - 0.049)}{(z + 0.987)(z - 0.368)}$$

$$\phi^-(z) = \frac{z - 1.2}{(z - 1.02)(z - 2.718)}.$$

Simple algebraic manipulations yield $W(Z)$ as

$$W(z) = \frac{0.778(z - 0.368)(z - 0.987)(z^2 - 1.85z + 0.906)}{(z + 0.049)(z + 0.368)(z - 0.809)(z - 0.98)^2}$$

from which the desired $D(z)$ is readily determined.

Apparently, Jury has neglected to mention the Kalman Method and the approach developed by Joseph and the writer [3], which are based upon the state space concept. These two approaches appear quite powerful and should be kept in the tool box of the control engineer.

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[1] Manuscript received December 11, 1964.

Messrs Steiglitz, Franaszek, and Haddad

We would like to call attention to an error which is easily overlooked in the statistical design of sampled-data systems [1], [2], [3]. The problem under consideration is the design of a digital compensator $D(z)$ which minimizes the mean square sampled error in a control system with plant $G(s)$, input signal spectral density $\phi_{r_s r_s}(z)$ and additive noise $\phi_{r_n r_n}(z)$.[1] The desired overall pulse transfer function is $G_d(z)$, and $W(z)G(z)$ is the overall pulse transfer function. In examining the analyticity of the integrand in Tou[2] one can easily overlook the factor z^{-1}, resulting in

$$W_1(z) = \frac{\left\{\dfrac{G(z^{-1})G_d(z)\left[\phi_{r_s r_s}(z) + \phi_{r_n r_n}(z)\right]}{\left[G(z)G(z^{-1})\right]^- \phi_{rr}^-(z)}\right\}_+}{\left[G(z)G(z^{-1})\right]^+ \phi_{rr}^+(z)} \quad (1)$$

rather than

$$W_2(z) = \frac{z\left\{\dfrac{G(z^{-1})G_d(z)\left[\phi_{r_s r_s}(z) + \phi_{r_n r_n}(z)\right]}{z\left[G(z)G(z^{-1})\right]^- \phi_{rr}^-(z)}\right\}_+}{\left[G(z)G(z^{-1})\right]^+ \phi_{rr}^+(z)}. \quad (2)$$

Thus, the expression for the sampled-data solution is not quite formally identical to the continuous data expression. The same kind of oversight can occur in the design of an optimum digital compensator for minimum mean square continuous error [1]–[3]. Accordingly Tou[3] should read

$$W(z) = \frac{z\left\{\dfrac{K_2(z^{-1})\left[\phi_{r_s r_s}(z) + \phi_{r_n r_n}(z)\right]}{zK_0^-(z)\phi_{rr}^-(z)}\right\}_+}{K_0^+(z)\phi_{rr}^+(z)}. \quad (3)$$

To illustrate the difference between solutions (1) and (2), consider the example where

$$G(z) = \frac{1}{1 - 0.5z^{-1}}$$

$$G_d(z) = 1$$

$$\phi_{r_n r_n}(z) = 1$$

$$\phi_{r_s r_s}(z) = \frac{0.98}{(1 - 0.2z)(1 - 0.2z^{-1})}$$

$$\phi_{r_n r_s}(z) = \phi_{r_s r_n}(z) = 0$$

$$\phi_{rr}(z) = \frac{2(1 - 0.1z)(1 - 0.1z)^{-1}}{(1 - 0.2z)(1 - 0.2z^{-1})}.$$

Then by (1)

$$W_1(z) = \frac{0.1z^{-1}(1 - 0.5z^{-1})}{1 - 0.1z^{-1}}$$

$$D_1(z) = \frac{W}{1 - WG} = \frac{0.1z^{-1}(1 - 0.5z^{-1})}{1 - 0.2z^{-1}}$$

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[1] See Tou, [1], Fig. 1.
[2] See Tou, [1], (25).
[3] Tou, [1], (46).