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OPTIMAL BINARY CODING OF ORDERED NUMBERS*

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L. H. Harper [1] has considered the problem of assigning the integers $1, 2, \cdots, 2^n$ to the vertices of an n-cube so as to minimize $\sum \Delta_{ij}$, where the sum runs over all neighboring pairs of vertices and Δ_{ij} is the absolute value of the difference of numbers assigned to the vertices. Such an optimal assignment determines a binary code for 2^n consecutive integers which has a minimum average absolute error due to single errors in transmission. Harper showed that the following algorithm produces all such assignments: having assigned $1, \cdots, l$, assign $l + 1$ to an unnumbered vertex (not necessarily unique) which has the most numbered nearest neighbors. He also showed that the natural numbering is one of the assignments produced by this algorithm. (The natural numbering assigns the number i to the vertex which represents the binary expansion of $i - 1$.)

The following generalization of Harper's problem arises when one considers the problem of coding numbers which are not necessarily equally spaced: How should an arbitrary set of 2^n numbers be assigned to the vertices of an n-cube so as to minimize $\sum \Delta_{ij}$? The purpose of this note is to show that the optimal assignments described by Harper remain optimal in this more general situation.

THEOREM. *Let $k_1 \leqq k_2 \leqq \cdots \leqq k_{2^n}$ be arbitrary numbers. Then the following algorithm assigns these numbers to the vertices of an n-cube so as to minimize $\sum \Delta_{ij}$, and produces all such assignments: Assign k_1 to an arbitrary vertex; having assigned $k_1, \cdots, k_l$, assign k_{l+1} to an unnumbered vertex (not necessarily unique) which has the most numbered nearest neighbors.*

Proof. Call the vertex assigned k_i the ith vertex. Let r_i be the number of numbered nearest neighbors of the ith vertex when only the first i numbers have been assigned. Thus in the final assignment the ith vertex has r_i neighbors which are not greater than k_i and $n - r_i$ neighbors which are not smaller than k_i. It follows that when computing $\sum \Delta_{ij}$, k_i will have the coefficient $r_i - (n - r_i) = 2r_i - n$, so that

$$\sum \Delta_{ij} = \sum_{i=1}^{2^n} (2r_i - n)k_i. \tag{1}$$

Minimizing $\sum \Delta_{ij}$ is therefore equivalent to minimizing $\sum r_i k_i$. Since

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the k_i form a nondecreasing sequence and since

$$\sum_{i=1}^{2^n} r_i = n2^{n-1} = \text{total number of edges of the } n\text{-cube},$$

this is equivalent to maximizing $\sum_{i=1}^{l} r_i$ for every l. But $\sum_{i=1}^{l} r_i$ is the number of "connections" between numbered vertices when l vertices have been numbered and Harper has shown that the algorithm described in the theorem maximizes this quantity at each stage, and in fact produces all assignments which do this.

It follows from the algorithm that the r_i associated with all optimal assignments of the n-cube are the same. To find these r_i consider the natural numbering of the n-cube. Each "one" in the binary expansion of $i - 1$ corresponds to a numbered nearest neighbor of the ith vertex when only the first i numbers have been assigned. Similarly each "zero" in the binary expansion of $i - 1$ corresponds to an unnumbered nearest neighbor of the ith vertex. Hence

$$r_i = \text{weight of binary number } i - 1,$$

and the values of r_i do not depend on n. The minimum value of $\sum \Delta_{ij}$ may now be calculated from (1) using these values of r_i and the given values of k_i.

The problem of finding the minimum value of $\sum \Delta_{ij}$ when only k_1 and k_{2^n} are specified follows easily from the above results. A lower bound can first be established by noting that regardless of the values chosen for k_i, $1 < i < 2^n$, an optimal assignment will always be used and thus, without loss of generality, we can assume that k_1 and k_{2^n} are assigned to vertices $(00 \cdots 0)$ and $(11 \cdots 1)$ respectively. It is not difficult to show that there are exactly n disjoint paths on the n-cube between these two vertices and so we have that

$$\sum \Delta_{ij} \geqq n(k_{2^n} - k_1). \tag{2}$$

It follows from the above discussion of r_i that

$$(2r_{1+p} - n) = -(2r_{2^n-p} - n), \qquad 0 \leqq p < 2^n,$$

and thus (1) becomes

$$\sum \Delta_{ij} = (2r_1 - n)(k_1 - k_{2^n}) + \sum_{i=2}^{2^{n-1}} (2r_i - n)(k_i - k_{2^n-i+1}).$$

Since $r_1 = 0$ it follows that the lower bound of (2) can be achieved by choosing $k_i = c$ for $1 < i < 2^n$, where $k_1 \leqq c \leqq k_{2^n}$.

Harper also solved the problem of maximizing $\sum \Delta_{ij}$ when the k_i are 2^n consecutive integers. When the k_i are arbitrary, (1) can be maximized by minimizing $\sum_{i=1}^{l} r_i$ for each l. Hence Harper's maximization procedure holds in the general case as well.

THEOREM. *Let $k_1 \leqq k_2 \leqq \cdots \leqq k_{2^n}$ be arbitrary numbers. Then to maximize $\sum \Delta_{ij}$ for an n-cube label any vertex k_1 and put $k_2, k_3, \cdots, k_{2^{n-1}}$ at random on the vertices whose weight is equal modulo 2 to that of the first vertex when the vertices are numbered in accord with the natural numbering system. Then assign the remaining numbers again at random to the remaining vertices. This algorithm produces all maximizing assignments.*

It is clear that if the maximizing algorithm is used then $r_i = 0$ for $1 \leqq i \leqq 2^{n-1}$ and $r_i = n$ for $2^{n-1} + 1 \leqq i \leqq 2^n$. It then follows from (1) that in this case

$$\sum \Delta_{ij} = n\left[\sum_{i=2^{n-1}+1}^{2^n} k_i - \sum_{i=1}^{2^{n-1}} k_i\right].$$

This problem was originally discussed by Kautz [2] and extensions of Harper's results to non-binary alphabets have been obtained by Lindsey [3].

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