Circulant Markov Chains as Digital Signal Sources

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Abstract—This paper is concerned with the class of Markov chains with circulant transition matrices. We show that such chains generate random processes whose spectral densities are of a particularly simple form, and that they provide a partial solution to the problem of synthesizing Markov chains that generate processes with given spectral densities.

I. Introduction

Sittler [1] showed that if real numbers are associated with the states of a finite Markov chain, the chain generates a random process with a rational spectral density, the poles being at the eigenvalues of the transition matrix. Thus, a hardware implementation of a Markov chain can be used to generate a digital signal with the same spectral density as would be obtained by filtering white noise with an appropriate linear filter (but, in general, with a different amplitude probability density).

This suggests the following problem, which can be called the *approximation problem for Markov chains:* given a spectral density D, find an n-state Markov chain and an assignment of output numbers to states so that the associated random process has a spectral density as close to D (in some prescribed sense) as possible.

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In the case that D is approximated by a rational function D^*, we may also want to consider the *synthesis problem for Markov chains:* given a rational spectral density D^*, find a Markov chain and an assignment of output numbers to states so that the associated random process has a spectral density D^*.

This paper is concerned with the class of Markov chains with circulant transition matrices. We will show that such chains generate random processes whose spectral densities are of a particularly simple form, and that they provide a partial solution to the synthesis problem.

II. Preliminary Definitions

Let

$$\mathbf{p}^t = (p_0{}^t, \cdots, p_{n-1}{}^t) \tag{1}$$

be the row vector of probabilities of being in state i at time t; $i=0, \cdots, n-1$; and let $\overline{\mathbf{C}}$ be an $n \times n$ stochastic matrix. Then a Markov chain is defined by the transition equation

$$\mathbf{p}^{t+1} = \mathbf{p}^t\overline{\mathbf{C}} \tag{2}$$

Definition: A *circulant Markov chain* (CMC) is a Markov chain whose transition matrix satisfies

$$(\overline{\mathbf{C}})_{ij} = c_{(j-i) \bmod n}. \tag{3}$$

Thus, each row of $\overline{\mathbf{C}}$ is a circular right shift of the preceding row. The first row of $\overline{\mathbf{C}}$ will be denoted by the row vector

$$\mathbf{c} = (c_0, \cdots, c_{n-1}). \tag{4}$$

Definition: A *circulant Markov chain random process* (CMCRP) is the discrete-time random process $\{x^t\}$ defined by

$$x^t = a_i \tag{5}$$

if the chain is in state i at time t, where

$$\mathbf{a} = (a_0, \cdots, a_{n-1}) \tag{6}$$

is a row vector of real numbers whose sum is 0. A CMCRP is completely specified, then, by two row vectors **c** and **a**, and we sometimes express this fact by writing CMCRP (**c**, **a**). We will assume in what follows that $\overline{C}$ has a single eigenvalue at 1 and $n-1$ eigenvalues in the open unit disk. This will ensure that the CMCRP is ergodic [2].

Given a row vector **v**, we define the polynomial $v(x)$ by

$$v(x) = v_0 + v_1 x + \cdots + v_{n-1}x^{n-1}. \tag{7}$$

Thus, because $\overline{C}$ is a stochastic matrix,

$$c(1) = 1 \tag{8}$$

and because **a** is zero sum,

$$a(1) = 0. \tag{9}$$

If the rotation matrix $\overline{R}$ is defined by

$$\overline{R} = \left[\begin{array}{c|c} 0 & I \\ \hline 1 & 0 \end{array}\right], \tag{10}$$

we can write

$$\overline{C} = c(R). \tag{11}$$

Finally, the characteristic polynomial of $\overline{R}$ is $x^n - 1$, so its eigenvalues are the n distinct roots of unity

$$\omega^k, \qquad k = 0, \cdots, n-1 \tag{12}$$

where

$$\omega = \exp[j2\pi/n]. \tag{13}$$

III. The Spectral Representations of $\overline{R}$ and $\overline{C}$

The matrix $\overline{R}$ has the spectral representation [3], [4]

$$\overline{R} = \sum_{k=0}^{n-1} \omega^k \overline{L}_k(\overline{R}) \tag{14}$$

where $L_k(R)$ is the Lagrange polynomial

$$\overline{L}_k(\overline{R}) = \prod_{\substack{i=0\\ i\neq k}}^{n-1} (\overline{R} - \omega^i) \Big/ \prod_{\substack{i=0\\ i\neq k}}^{n-1} (\omega^k - \omega^i). \tag{15}$$

A straightforward calculation yields

$$\overline{L}_k(\overline{R}) = \frac{1}{n}\psi(\omega^k)\psi^*(\omega^k) \tag{16}$$

where $\psi(x)$ is the column vector

$$\psi(x) = \begin{bmatrix} 1 \\ x \\ \cdot \\ \cdot \\ \cdot \\ x^{n-1} \end{bmatrix} \tag{17}$$

and $\psi^*(x)$ is the conjugate transpose of $\psi(x)$.

Hence we have the spectral representation

$$R = \frac{1}{n}\sum_{k=0}^{n-1} \omega^k \psi(\omega^k)\psi^*(\omega^k). \tag{18}$$

For a wide class of functions $g(\cdot)$, which includes polynomials, we also have

$$g(R) = \frac{1}{n}\sum_{k=0}^{n-1} g(\omega^k)\psi(\omega^k)\psi^*(\omega^k) \tag{19}$$

and hence

$$\overline{C} = c(R) = \frac{1}{n}\sum_{k=0}^{n-1} c(\omega^k)\psi(\omega^k)\psi^*(\omega^k). \tag{20}$$

IV. The Correlation Function and Spectral Density of CMCRP(c, a)

The spectral representation allows us to obtain a simple expression for the correlation function of CMCRP(**c**, **a**). Sittler [1] shows that the correlation function of CMCRP(**c**, **a**) is

$$r(\tau) = \mathbf{a}\overline{M}\overline{C}^{|\tau|}\mathbf{a}' \tag{21}$$

where

$$\overline{M} = \text{diag}(\mu_i) \tag{22}$$

and μ_i is the steady-state probability of being in state i. In the case of a CMC,

$$\overline{M} = \frac{1}{n}\overline{I}. \tag{23}$$

From (19) we also have that

$$\overline{C}^{|\tau|} = \frac{1}{n}\sum_{k=0}^{n-1} [c(\omega^k)]^{|\tau|}\psi(\omega^k)\psi^*(\omega^k). \tag{24}$$

Substituting (23) and (24) in (21), we get

$$r(\tau) = \frac{1}{n^2}\sum_{k=0}^{n-1} [c(\omega^k)]^{|\tau|} \,|\, \mathbf{a}\psi(\omega^k) \,|^2 \tag{25}$$

$$= \frac{1}{n^2}\sum_{k=0}^{n-1} [c(\omega^k)]^{|\tau|} \,|\, a(\omega^k) \,|^2. \tag{26}$$

The sequence

$$\frac{1}{n}a(\omega^k), \qquad k = 0, \cdots, n-1 \tag{27}$$

is the inverse discrete Fourier transform of the sequence of output numbers

$$a_i, \qquad i = 0, \cdots, n-1. \tag{28}$$

We denote this by writing

$$A_k = \frac{1}{n}a(\omega^k), \qquad k = 0, \cdots, n-1. \tag{29}$$

Denote the eigenvalues of $\overline{C}$ by

$$\gamma_k = c(\omega^k), \qquad k = 0, \cdots, n-1. \tag{30}$$

Hence we can write

$$r(\tau) = \sum_{k=1}^{n-1} |\, A_k \,|^2 \gamma_k^{|\tau|} \tag{31}$$

where we have used the fact that $\hat{A}_0 = 0$ from (9). We see from this that the poles of the spectral density are determined solely by the transition probability vector c, and their relative weights by the magnitude of the inverse discrete Fourier transform of the output vector a.

The spectral density of CMCRP(c, a) is then

$$\Phi(z) = \sum_{k=1}^{n-1} |A_k|^2 \left[\frac{1}{1-\gamma_k z^{-1}} + \frac{1}{1-\gamma_k z} - 1\right] \quad (32)$$

where z is on the unit circle. When n is odd

$$\Phi(z) = \sum_{k=1}^{(n-1)/2} |A_k|^2 2 \operatorname{Re} \cdot \left[\frac{1}{1-\gamma_k z^{-1}} + \frac{1}{1-\gamma_k z} - 1\right] \quad (33)$$

and when n is even

$$\Phi(z) = \sum_{k=1}^{(n/2)-1} |A_k|^2 2 \operatorname{Re}\left[\frac{1}{1-\gamma_k z^{-1}} + \frac{1}{1-\gamma_k z} - 1\right] + |A_{n/2}|^2\left[\frac{1}{1-\gamma_{n/2} z^{-1}} + \frac{1}{1-\gamma_{n/2} z} - 1\right]. \quad (34)$$

Incidentally, when n is even

$$\gamma_{n/2} = c(-1). \quad (35)$$

V. A Partial Solution to the Synthesis Problem

A general rational spectral density without repeated poles corresponds to a correlation function of the form

$$r(\tau) = \sum_{i=1}^{m} B_i \gamma_i^{|\tau|} \quad (36)$$

where some B_i may be negative and the complex γ_i occur in conjugate pairs. Hence we cannot hope to synthesize a general spectral density using CMCRP's, since (31) implies that the B_i are nonnegative. We can, however, state the following synthesis procedure for the restricted class of densities that correspond to correlation functions (36) with nonnegative weights B_i.

Theorem: A Markov chain can be constructed with any correlation function of the form (36) with $B_i \geq 0$, $i=1, \cdots, m$.

Proof: Consider the correlation function

$$r_i(\tau) = B_i[\gamma_i^{|\tau|} + (\gamma_i^*)^{|\tau|}] \quad (37)$$

where $B_i \geq 0$. Choose n_i large enough so that γ_i lies in the polygon with vertices at the n_i roots of unity. A probability vector c can then be chosen so that

$$\gamma_i = c(\omega), \qquad \omega = \exp[j2\pi/n_i]. \quad (38)$$

Choose the sequence a_j, $j=0, \cdots, n_i-1$ as the discrete Fourier transform of the sequence

$$A_j = \sqrt{B_i}(\delta_{1j} + \delta_{n_i-1,j}), \qquad j = 0, \cdots, n_i - 1. \quad (39)$$

Then from (31), this CMC, say C_i, has $r_i(\tau)$ given by (37). A similar procedure works if γ_i is real. Now implement each chain C_i with independent random numbers and add their outputs. Since the output random processes are independent, the spectral density of the sum of the outputs is simply (36). This new construction is a Markov chain that may be considered a direct product of component CMC's. Q.E.D.

This synthesis procedure may use many more states than necessary. However, circulant chains of high order can be implemented in a natural way with digital hardware, since the vector of transition probabilities represents a fixed set of transitions in a fixed table.

The following example demonstrates that correlation functions with negative coefficients can be obtained:

$$P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad (40)$$

$$a = (4 \quad 3 \quad -7) \quad (41)$$

$$r(\tau) = -\left(\frac{1}{2}\right)^{|\tau|} + \frac{77}{3}\delta(\tau). \quad (42)$$

Thus chains other than circulant chains must be considered in any attack on the general synthesis problem.

VI. Comments

Equation (31) can also be derived from (21) by using the fact that the discrete Fourier transform diagonalizes circulant matrices [5].

Prof. W. Schuessler of the University Erlangen-Nuernberg has brought to the authors' attention related work, which the reader may wish to pursue [6]–[11].

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