THE GENERAL THEORY OF DIGITAL FILTERS WITH APPLICATIONS TO SPECTRAL ANALYSIS

A THESIS

by

Kenneth Steiglitz
B.E.E. and M.E.E.

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APPROVAL OF READERS

This is to certify that I have read and approved the thesis entitled "The General Theory of Digital Filters with Applications to Spectral Analysis," submitted by Kenneth Steiglitz in partial fulfillment of the requirements for the degree of Doctor of Engineering Science, College of Engineering, New York University.

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(1) Sheldon S. L. Chang (Advisor)

(2)

(3) John R Ragazzini

(4) C. F. Rehberg

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ABSTRACT

Part I is devoted to the general theory of digital filters. The filtering theories for both continuous-time and discrete-time signals are formulated in terms of abstract Hilbert space, with the notion of a stable filter defined as a bounded linear operator. This abstract setting allows the z-transform to be defined with the same generality as the Fourier transform. A specific isomorphism is then constructed which connects the filtering theories for continuous-time and discrete-time signals, and in the linear time-invariant case the two theories are shown to be essentially identical. This means that many optimization problems can be solved simultaneously for continuous-time and digital systems.

In the second part, the isomorphism developed in Part I is used to reduce the approximation problem for digital filters to that for continuous-time filters. This allows the designer of digital filtering computer programs to use many of the concepts which have proven important to the communications engineer.

In the last part, the problem of estimating the power-spectral-density of a signal from equally spaced samples is discussed. It is shown that bandpass digital filters generate a class of spectral windows which produce always positive estimates of the power-spectral-density. The optimum bandwidth and shape of such a filter are then derived. Finally, a method for identifying unknown parameters in the power-spectral-density of a digital signal is presented.

PREFACE

Historically, methods for processing signals that are functions of continuous time were developed long before the advent of high speed digital computers. When high speed computing facilities did become available, the communications and control engineers were not the people who developed computing techniques. As a result, the filtering theory that had been highly developed for continuous-time signals was not applied in full force for the processing of digital signals.

The main purpose of this thesis is to tie together the theories of filtering digital and analog information. This will enable the data analyst to carry over effectively to his domain many of the concepts which have been important to network designers. In particular, all the approximation techniques developed for continuous-time filters become available for digital applications.

The strong link that is developed between the digital and continuous domains will also be of theoretical value. It will present to us a unified picture of signal and filtering theory, a picture that is equally applicable to digital and continuous signals.

PART I: THE GENERAL THEORY OF DIGITAL FILTERS

1. Introduction

It is easy to observe a parallel between signal theory for signals with a continuous time parameter and signal theory for discrete-time signals. In fact, it is common practice to develop in detail a filtering theory for continuous-time signals and to pay less attention to the discrete theory, with the assumption that the derivation in the discrete case follows the one for continuous-time signals without much change. Thus, without going into details,[1,2,3] the Wiener filter for a noise-corrupted continuous-time signal is

$$F_o(s) = \frac{1}{Y(s)}\left[\frac{\Phi_{r_1 r}(s)}{\overline{Y}(s)}\right]_{LHP} , \qquad Y\overline{Y} = \Phi_{r_1 r_1}(s) \qquad ;$$

and the optimum filter in the discrete case is

$$F_o(z) = \frac{1}{Y(z)}\left[\frac{\Phi_{r_1 r}(z)}{\overline{Y}(z)}\right]_{IN\,|z|=1} , \quad Y\overline{Y} = \Phi_{r_1 r_1}(z) \qquad ;$$

where r is the uncorrupted signal and r_1 is the corrupted signal. On the other hand, the two cases are always considered as distinct and essentially different situations.

This correspondence between continuous and discrete phenomena is far from accidental, however. In fact, when both theories are axiomatized in terms of Hilbert space theory (L_2 and l_2 theory), they are isomorphic. This simple fact is quite illuminating and leads to a more unified theory of filtering and prediction.

Usually, it is assumed that the signals of interest are of exponential order as t becomes infinite. This leads to two-sided Laplace transforms which converge in a strip in the s-plane, or double ended z-transforms which converge in an annulus of the z-plane. This is replaced in Hilbert space theory by mean convergence on the $j\omega$-axis and unit circle, respectively. In one sense the signal spaces L_2 and l_2 are more restrictive, because they do not include signals of positive exponential order. On the other hand, assuming that we are dealing with physically real signals, the spaces L_2 and l_2 are more general and intuitively satisfying: roughly, they include all signals whose total energy content is finite.

Our main purpose in this first part, then, will be to imbed the theory of continuous-time signals in L_2 theory and the theory of discrete-time signals in l_2 theory; and to show that the filtering theories for these two classes of signals are essentially the same. We will thus arrive at a definition of digital filter that is as general as the definition of continuous-wave filter, and we will show that many problems in the design of discrete-time systems need not be re-solved. As a by-product, we will see how well Hilbert space theory is suited to describe linear filtering theory for both continuous and discrete time.

While Youla, Castriota and Carlin,[4] and other network theorists have applied L_2 theory to continuous-time network theory, to the author's knowledge l_2 theory has not been applied to the z-transform, and the isomorphism between L_2 and l_2 has not been exploited by electrical engineers.

We begin with a review of the elements of Hilbert space theory.[5,6,7]

2. A Review of Hilbert Space Theory

We will adopt the widely accepted definition of abstract Hilbert space. That is: a set H of arbitrary elements f,g,... (sometimes called functions or vectors) is termed a Hilbert space if:

I. H is a linear space.

II. An inner product is defined in H as follows: to every pair of elements f,g there is associated a complex number (f,g) such that

   1) $(f,g) = \overline{(g,f)}$

   2) $(\alpha f,g) = \alpha(f,g)$

   3) $(f_1+f_2,g) = (f_1,g) + (f_2,g)$

   4) $(f,f) \overset{\geq}{=} 0$ with equality if and only if $f = 0$.

III. The space H is complete in the metric $\|f-g\| = (f-g,f-g)^{\frac{1}{2}}$.

IV. H is infinite dimensional; that is, for any integer n there are n linearly independent elements in H.

V. H is separable; that is, H contains a countable and dense set. (This condition is often omitted, allowing spaces of dimension higher than $\aleph_0$).

Thus, a Hilbert space is a complete, separable, infinite-dimensional Euclidean space.

Historically, two concrete realizations of Hilbert space play central roles. The first is the space $L_2(a,b)$, which is defined to be the set of all complex-valued Lebesgue measurable functions on (a,b) such that

$$\int_a^b |f(t)|^2 \, dt < \infty \quad .$$

The inner product in this space is defined by

$$(f,g) = \int_a^b f(t)\overline{g}(t) \, dt \quad .$$

Two functions in L_2 are considered equal if they differ only on a set of measure zero. Since the metric in this space is $(f-g,f-g)^{\frac{1}{2}}$, the sequence f_n will approach f if

$$\lim_{n \to \infty} \int_a^b |f_n - f|^2 \, dt = 0 \quad .$$

This will be called mean convergence and will be written

$$f = \underset{n \to \infty}{\text{l.i.m.}} \, f_n \quad .$$

The other Hilbert space is called l_2. It is defined to be the set of all sequences of complex numbers

$$x = \{x_1, x_2, \dots, x_n, \dots\}$$

satisfying the condition

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty \quad .$$

Here, the inner product is defined by

$$(x,y) = \sum_{n=1}^{\infty} x_n \overline{y_n} \quad .$$

(Sometimes it will be convenient to think of l_2 as containing double ended sequences: $\{\dots x_{-1}, x_0, x_1, x_2, \dots\}$. The theory is really the same).

For us, the space $L_2(-\infty, +\infty)$ will play the role of the space of continuous time signals, and l_2 will represent the space of discrete-time signals.

An <u>isomorphism</u> from one Hilbert space H_1 to another Hilbert space H_2 is a one-to-one linear transformation U from H_1 onto H_2 such that $(Ux,Uy) = (x,y)$ for every pair of vectors x,y in H_1. An isomorphism preserves all the structure embodied in the definition of Hilbert space and isomorphic Hilbert spaces are geometrically indistinguishable and for our purposes can be considered as identical.

The following theorem is central for our purposes:

Theorem 1. All Hilbert spaces are isomorphic.

The proof of this theorem is interesting and useful. We now review its main points.

1. Since H is separable, we can choose in H a countable dense set. From this set we can construct an orthonormal set $\{h_1, h_2, h_3, \ldots\}$ that is complete in H. That is,

$$(h_i, h_j) = \begin{cases} 0 \text{ is } i \neq j \\ 1 \text{ if } i = j \end{cases} ,$$

and linear combinations of the h_i are dense in H.

2. This implies that any element of H can be approximated with arbitrary accuracy by linear combinations of the h_i. If we define the partial sum of a generalized Fourier series by

$$S_n = \sum_{k=1}^{n} c_k h_k ,$$

then the distance between S_n and f in the metric of H is smallest when

$$c_k = (f, h_k) .$$

In that case, we have in fact

$$|| f - S_n ||^2 = (f, f) - \sum_{k=1}^{n} |c_k|^2 .$$

Now let n approach infinity. Since S_n is the best n-th order approximation to f, and since the orthonormal set $\{h_1, h_2, \ldots\}$ is complete, we must have

$$\lim_{n \to \infty} \| f - S_n \|^2 = 0 \quad ,$$

and hence

$$(f,f) = \sum_{k=1}^{\infty} |c_k|^2 \quad .$$

3. Conversely, let $c_1, c_2, \ldots$ be a sequence of numbers such that

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty \quad ,$$

and construct the sequence of partial sums

$$f_n = \sum_{k=1}^{n} c_k h_k \quad .$$

It then follows that

$$\| f_{n+p} - f_n \|^2 = \sum_{k=n+1}^{n+p} |c_k|^2 \quad .$$

As n approaches infinity the right side goes to zero. The left side must also go to zero, and this implies that the sequence f_n is fundamental. The fact that H is complete in its metric then implies that there is a limit function $f \in H$ such that

$$||f-f_n|| \longrightarrow 0$$

as $n \longrightarrow \infty$. It then follows easily that

$$c_k = (f,h_k)$$

and that

$$(f,f) = \sum_{k=1}^{\infty} |c_k|^2 \quad .$$

4. We now assign to each element in H the sequence $\{c_1, c_2, \ldots\}$ of its Fourier coefficients. By step 2 above this is an element in l_2. Furthermore, by step 3, for each element $\{c_1, c_2, \ldots\}$ in l_2 there is an f in H which has Fourier coefficients $\{c_1, c_2, \ldots\}$. This correspondence is linear, one-to-one, onto, and preserves norm. It is therefore an isomorphism, and we have therefore shown that any Hilbert space is isomorphic to l_2, and hence to any other Hilbert space. In the case $H = L_2(a,b)$ this procedure corresponds to mapping a function to the sequence of its coefficients in some orthogonal expansion on the interval (a,b); such as an ordinary Fourier series on $(0, 2\pi)$ or a Laguerre series on $(0, \infty)$, for example.

With this review we go on to apply these ideas to more familiar situations.

3. Axiomatization of Deterministic Signal Theory

In most deterministic situations encountered by engineers, the signals are either functions of a continuous time variable or a discrete time variable. In either case, the total energy contained in a signal is really finite, even though we make up models which deny this. For example, we say that a step input is applied to some system at $t = 0$ and we write

$$f(t) = \begin{cases} 0 & -\infty < t < 0 \\ 1 & 0 < t < \infty \end{cases} .$$

This is clearly not realistic. The definition

$$f(t) = \begin{cases} 0 & -\infty < t < 0 \\ 1 & 0 < t < T \\ 0 & T < t < \infty \end{cases}$$

where T is very large; or the definition

$$f(t) = \begin{cases} 0 & -\infty < t < 0 \\ e^{-\alpha t} & 0 < t < \infty \end{cases} ,$$

where α is very small, describe the situation just as well. Thus, without serious limitation, we can assume that any wave will have a finite total energy. With this assumption, Hilbert space $L_2(-\infty,\infty)$, with its convenient completeness and with its continuous Fourier transform, provides a neat setting for our discussion of deterministic signals which are functions of the continuous time parameter t.

Similarly, when a signal is a function of discrete times, the Hilbert space l_2 is a realistic model with many convenient mathematical properties. From now on, a function in $L_2(-\infty,\infty)$ will be called an analog signal, and a function in l_2 will be called a digital signal.

It is now rather startling and counterintuitive to the engineer that $L_2(-\infty,\infty)$ and l_2 are isomorphic. After all, any signal in l_2 could have been obtained by sampling at discrete times any one of an infinite number of analog signals. The problem here is that the mapping $L_2(-\infty,\infty) \longrightarrow l_2$ defined by sampling:

$$f(t) \longrightarrow \{\ldots,f(-2T),f(-T),f(0),f(T),\ldots\}$$

is not an isomorphism, since it is not one-to-one. Nevertheless, L_2 can be made isomorphic to l_2 by an appropriate choice of mapping; in the same way, for example, that the Abelian group of integers can be made isomorphic to the Abelian group of even integers.

4. The Transform Domains

Our next goal will be to construct a specific isomorphism which can serve as a concrete link between the analog and digital signal spaces. Naturally, we would like the mapping to have some intuitive significance. The very natural correspondence provided by sampling analog signals has been ruled out because it is not an isomorphism. It would still be desirable, however, to have the left half s-plane correspond to the interior of the unit circle in the z-plane, because these regions seem to play analogous roles, even when no signals have been sampled. To make these ideas precise, we must add the Laplace transform and the z-transform to our Hilbert space theory.

The key theorem for the construction of a transform domain for $L_2(-\infty,\infty)$ is called Plancherel's Theorem:[8,9]

Theorem 2. (Plancherel) If $f(t) \in L_2(-\infty,\infty)$, then

$$F(s) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} f(t)e^{-st}\, dt \qquad \text{(I-1)}$$

exists for $s = j\omega$, and $F(j\omega) \in L_2(-\infty,\infty)$.

Furthermore,

$$(f,f) = \int_{-\infty}^{+\infty} |f(t)|^2\, dt = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} |F(s)|^2\, ds \quad . \qquad \text{(I-2)}$$

and

$$f(t) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-jA}^{jA} F(s)e^{st}\, dt \quad . \qquad \text{(I-3)}$$

Analytic extension of $F(j\omega)$ to the rest of the s-plane (via (I-1) when it exists, for example) will give us the Laplace transform.

We will also use Parseval's Theorem:

Theorem 3. (Parseval) If $f, g \in L_2(-\infty,\infty)$, then

$$(f,g) = \int_{-\infty}^{\infty} f(t)\overline{g}(t)\, dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s)G(-s)\, ds \quad . \tag{I-4}$$

The theory required for the analogous construction of a z-transform for digital signals is really no more than the theory of Fourier series. Think of the original periodic function as the z-transform evaluated on the unit circle in the z-plane; and think of the Fourier coefficients as the values of our digital signal. The Riesz-Fischer Theorem[8,10] then reads:

Theorem 4. (F. Riesz-Fischer) If $\{f_n\}_{n=-\infty}^{\infty} \in l_2$, then

$$F(z) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{n=-N}^{N} f_n z^{-n} \tag{I-5}$$

exists for $z = e^{\underline{j\omega T}}$, and $F(e^{\underline{j\omega T}}) \in L_2(0,2\pi/T)$, where $\underline{\omega}$ is the independent variable of $L_2(0,2\pi/T)$, and this $\underline{\omega}$ is unrelated to the ω used in the s-plane.

Furthermore,

$$(\{f_n\},\{f_n\}) = \sum_{n=-\infty}^{+\infty} |f_n|^2 = \frac{1}{2\pi j} \oint_{|z|=1} |F(z)|^2 \frac{dz}{z} \quad , \tag{I-6}$$

and

$$f_n = \frac{1}{2\pi j} \oint_{|z|=1} F(z) z^n \frac{dz}{z} \quad . \tag{I-7}$$

As in the analog case, the analytic extension of $F(e^{\underline{j\omega T}})$ to the rest of the z-plane will coincide with the ordinary z-transform, which is usually defined only for digital signals of exponential order.

Parseval's relation also holds:

<u>Theorem 5. (Parseval)</u> If $\{f_n\}, \{g_n\} \in l_2$, then

$$(\{f_n\}, \{g_n\}) = \sum_{n=-\infty}^{+\infty} f_n \overline{g}_n = \frac{1}{2\pi j} \oint_{|z|=1} F(z) G(z^{-1}) \frac{dz}{z} \quad . \tag{I-8}$$

To summarize, we have defined an analog signal space $L_2(-\infty, \infty)$; together with its transform domain, which, when $s = j\omega$, is also $L_2(-\infty, \infty)$. Analogously, we have defined a digital signal space l_2; together with its transform domain, which, when $z = e^{\underline{j\omega T}}$, is $L_2(0, 2\pi/T)$. We now are in a position to define a specific isomorphism between the analog and digital signal spaces via their transform domains; a procedure which was hinted at before.

5. <u>A Specific Isomorphism, μ</u>

Remembering that we wish to map the $j\omega$-axis in the s-plane onto the unit circle in the z-plane, the familiar bilinear transformation

$$s = \frac{z-1}{z+1} \quad , \quad z = \frac{1+s}{1-s}$$

is a natural choice. There is an additional factor required so that the transformation will preserve norms. The image $\{\underline{f}_n\} \in l_2$ corresponding to $f(t) \in L_2(-\infty,\infty)$ will then be defined as the sequence with the z-transform

$$\underline{F}(z) = \frac{\sqrt{2}}{z+1} \; F\left(\frac{z-1}{z+1}\right)$$, where we use the underscore to denote digital domains.

Thus, the mapping $\mu: L_2(-\infty,\infty) \longrightarrow l_2$ is defined by a chain which goes from $L_2(-\infty,\infty)$ to $L_2(-\infty,\infty)$ to $L_2(0,2\pi/T)$ to l_2, as follows:

$$\mu: f(t) \longrightarrow F(s) \longrightarrow \frac{\sqrt{2}}{z+1} \; F\left(\frac{z-1}{z+1}\right) = \underline{F}(z) \longrightarrow \{\underline{f}_n\} \;. \tag{I-9}$$

The inverse mapping is easily defined, since each of these steps is uniquely reversible:

$$\mu^{-1}: \{\underline{f}_n\} \longrightarrow \underline{F}(z) \longrightarrow \frac{\sqrt{2}}{1-s} \; \underline{F}\left(\frac{1+s}{1-s}\right) = F(s) \longrightarrow f(t) \;.$$

The mapping μ and its relations to the various spaces are shown schematically in Figure 1.

To show that μ is indeed an isomorphism, we first verify that μ preserves the inner product. Let f and g be any two analog signals. By Theorem 3 (Parseval's relation for analog signals), we have

$$(f,g) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s)G(-s)\, ds \quad .$$

Letting $z = \frac{1+s}{1-s}$, this becomes, with some algebraic manipulation,

$$(f,g) = \frac{1}{2\pi j} \oint_{|z|=1} \underline{F}(z)\underline{G}(z^{-1}) \frac{dz}{z} \quad .$$

Then, using Theorem 5 (Parseval's relation for digital signals), we find that μ does preserve the inner product:

$$(f,g) = (\{\underline{f}_n\},\{\underline{g}_n\}) \quad .$$

μ is obviously linear and onto. We can now show that μ is one-to-one in the following way: if $f \neq g$, then $(f-g,f-g) = (\{\underline{f}_n\}-\{\underline{g}_n\},\{\underline{f}_n\}-\{\underline{g}_n\}) \neq 0$; which implies that $\{\underline{f}_n\} \neq \{\underline{g}_n\}$, and hence that μ is one-to-one. This establishes the fact that μ is an isomorphism.

We note here that under the isomorphisms μ and μ^{-1} functions with rational transforms are always matched with functions with rational transforms, this fact following from the nature of the transformation μ. This is a great convenience, since many of the functions commonly encountered in engineering problems have transforms which are rational functions of s or z.

6. The Orthonormal Expansion Attached to μ

In our review of Hilbert space theory we showed how a set of orthonormal functions generated an isomorphism between two Hilbert spaces. It should come as no surprise, then, to learn that the isomorphism μ could have been so generated. This section will be devoted to finding this orthonormal expansion.

We start with the z-transform of the digital signal $\{\underline{f}_n\}$ which is the image under μ of an arbitrary analog signal $f(t)$:

$$\underline{F}(z) = \frac{\sqrt{2}}{z+1} \; F\left(\frac{z-1}{z+1}\right) = \sum_{n=-\infty}^{\infty} \underline{f}_n \, z^{-n} \qquad .$$

By (I-7), the formula for the inverse z-transform, we have

$$\underline{f}_n = \frac{1}{2\pi j} \oint_{|z|=1} \frac{\sqrt{2}}{z+1} \; F\left(\frac{z-1}{z+1}\right) z^n \, \frac{dz}{z} \qquad .$$

Letting $z = \dfrac{1+s}{1-s}$, this integral becomes

$$\underline{f}_n = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(s) \cdot \frac{\sqrt{2}}{1+s} \left(\frac{1+s}{1-s}\right)^n ds \qquad . \qquad \text{(I-11)}$$

By Parseval's relation (I-4), this can be rewritten in terms of time functions as

$$\underline{f}_n = \int_{-\infty}^{\infty} f(t)\,\lambda_n(t)\,dt \quad , \qquad \text{(I-12)}$$

where the $\lambda_n(t)$ are given by the inverse Laplace transform of the factor appearing in the integrand of (I-11) with s replaced by -s. Thus:

$$\lambda_n(t) = \mathcal{L}^{-1}\left[\frac{\sqrt{2}}{1-s}\left(\frac{1-s}{1+s}\right)^n\right] \quad .$$

We see immediately that, depending on whether $n > 0$ or $n \leq 0$, $\lambda_n(t)$ vanishes for negative time or positive time, respectively. By manipulating a standard transform pair involving Laguerre polynomials, we find:

(I-13)

$$\lambda_n(t) = \begin{cases} (-1)^{n-1}\,\sqrt{2}\,\,e^{-t}\,L_{n-1}(2t)\,u(t) \quad , \; n = 1,2,3,\ldots, \\ (-1)^{-n}\,\sqrt{2}\,\,e^{t}\,L_{-n}(-2t)\,u(-t) \;, \; n = 0,-1,-2,\ldots, \end{cases}$$

(I-14)

where $u(t)$ is the Heaviside unit step function, and $L_n(t)$ is the Laguerre polynomial of degree n, defined by:

$$L_n(t) = \frac{e^t}{n!}\,\frac{d^n}{dt^n}\,(t^n e^{-t}),\; n = 0,1,2,\ldots \qquad .$$

The set of functions $\{\lambda_n\}_{n=1}^{\infty}$ is a complete, orthonormal set on $(0,\infty)$, and are called Laguerre functions. They have been employed by Lee,[11] Wiener,[12] and others for network synthesis; and are tabulated in Wiener,[12] and, with a slightly different normalization, in Head and Wilson.[13] The functions $\{\lambda_{-n}\}_{n=0}^{\infty}$ are similarly complete and orthonormal on $(-\infty,0)$, so that the orthonormal expansion of $f(t)$ corresponding to (I-12) is

$$f(t) = \sum_{n=-\infty}^{\infty} \underline{f}_n \, \lambda_n(t) \quad . \tag{I-15}$$

We see then, that the values of the digital signal for $n > 0$ correspond to the coefficients in the Laguerre expansion of $f(t)$ for positive t; and that the values of the digital signal for $n \leq 0$ correspond to the coefficients in the Laguerre expansion of $f(t)$ for negative t.

There follows from this representation the fact that the isomorphism μ matches one-sided functions with one-sided functions. That is, $f(t) = 0$ for $t < 0$ if and only if $\{\underline{f}_n\} = 0$ for $n \leq 0$; and similarly, $f(t) = 0$ for $t > 0$ if and only if $\{\underline{f}_n\} = 0$ for $n > 0$.

Other orthonormal expansions, such as the Hermite, for example, will also generate isomorphisms; but these will not be as convenient and as simple for our purposes as the Laguerre expansion. In particular, Kautz,[14] Gabor,[15] Huggins,[16] and others have considered the construction of orthonormal functions for signal representation.

The fact that the mapping μ is equivalent to a Laguerre expansion can sometimes lead to a quick way of expanding a given time function in a Laguerre series. One need only find $\underline{F}(z)$ from the Laplace transform of f(t) and expand this in a power series in z. To illustrate this, and the way that the mapping μ works in general, consider the function

$$f(t) = e^{-t} \sin t \, u(t) \quad .$$

This function is in L_2 and its Laplace transform is analytic in the half-plane $\mathrm{Re}(s) > -1$. Thus,

$$F(s) = \frac{1}{(s+1)^2 + 1} \quad ,$$

and

$$\underline{F}(z) = \frac{\sqrt{2}\,(z+1)}{5z^2 + 2z + 1}$$

$$= \sqrt{2}\left[\frac{1}{5} z^{-1} + \frac{3}{25} z^{-2} - \frac{11}{125} z^{-3} + \frac{7}{625} z^{-4} + \dots\right]$$

Thus, by (I-15),

$$f(t) = \sqrt{2}\left[\frac{1}{5}\lambda_1(t) + \frac{3}{25}\lambda_2(t) - \frac{11}{125}\lambda_3(t) + \frac{7}{625}\lambda_4(t) + \dots\right]$$

These coefficients can be checked by carrying through the integrations indicated in (I-12).

7. Stable Filters as Bounded Linear Operators

We come now to the problem of incorporating within our framework the concept of "filter" or "transfer function." That is, we wish to formalize the notion of a device which transforms one element of Hilbert space into another. Such a device might be a network of resistors, inductors, and capacitors which transforms one analog signal into another; or it might be a digital computer which transforms one digital signal into another such signal. We assume, mostly because we must to achieve any generality, that such filters are linear. It is also reasonable to expect that if we limit the energy content of the input function to a stable filter, that the energy content of the output will be limited.

Fortunately, operators with such properties have been studied widely in connection with Hilbert space.[5,6,7] An operator A in a Hilbert space H is defined as a transformation which attaches to each element f in H some element Af which is also in H. An operator A is said to be linear if

$$A(\alpha f + \beta g) = \alpha Af + \beta Ag$$

for any f,g in H and any complex numbers α and β. Lastly, corresponding to our energy requirement, a linear operator is said to be bounded if there is a positive real number M such that

$$\|Af\| \leq M \|f\|$$

for all f in H. The norm of the linear operator A is the infimum of all such values of M, and is written $\|A\|$. Equivalently, the norm of A can be defined as

$$\|A\| = \sup_{f \in H} \frac{\|Af\|}{\|f\|} \quad . \tag{I-16}$$

One example of a bounded linear operator is the fourier transform. By (I-2) this operator has a norm equal to $\sqrt{1/2\pi}$. As another example, consider a simple low-pass RC section with the transfer function

$$\frac{\tau}{s+\tau}$$

If an input wave f(t) is applied to this network, the total energy in the output will be

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} |F(s)|^2 \left| \frac{\tau^2}{\omega^2 + \tau^2} \right| ds \leq \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} |F(s)|^2 \, ds \quad ;$$

so that the norm of this operator cannot exceed 1. Since this is a passive network, it is to be expected that the total output energy cannot exceed the input energy.

We are thus led to adopt the following terminology: A bounded linear operator on the space L_2 will be called a <u>(linear) analog filter</u> and a bounded linear operator on l_2 will be called a <u>(linear) digital filter</u>.

It is now a direct consequence of our axiomatic setup that any bounded linear operator is continuous in the metric of Hilbert space. That is, if $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in the Hilbert space H, and if f is a function in H such that

$$\lim_{n \to \infty} \|f_n - f\| \longrightarrow 0 \quad ,$$

then

$$\lim_{n \to \infty} \|Af_n - Af\| \longrightarrow 0 \quad ,$$

where A is a bounded linear operator. This follows immediately from the fact that

$$\|Af_n - Af\| \leq \|A\| \cdot \|f_n - f\| \quad .$$

Continuity is a desirable property of operators. In L_2, for instance, it means that if input functions to an analog filter A approach a function f in the mean, then the output will approach Af in the mean. This convenience is bought at the price of considering only functions in L_2 and using mean convergence as the convergence criterion. If we insist on thinking in terms of pointwise convergence, for instance, we lose continuity; as the following example shows: Let a set of input functions to some network approach the delta function. The pointwise limit of the input functions is then 0 almost everywhere. But in general the output will not approach 0, so that filters will not be continuous in this framework. In a way, our convergence criterion is more natural than pointwise convergence:

for a sequence f_n to approach f in the mean we demand only that the total energy of $f_n - f$ approach zero.

Since μ can be thought of as a bounded linear operator in the abstract Hilbert space H, μ is continuous. Similarly, the Fourier transform is continuous. Also, since l_2 and $L_2(0, 2\pi/T)$ are isomorphic Hilbert spaces, the z-transform as defined in Theorem 4 is also continuous. We summarize these facts in the following Theorem:

Theorem 6. All bounded linear operators are continuous in the metric of Hilbert space. In particular, the following bounded linear operators are continuous:

1. Analog filters
2. Digital filters
3. The Fourier transform
4. The z-transform
5. The isomorphism μ.

8. The Mapping μ for Filters and the Transforms of Filters

Since our signal spaces are now equipped with operators, it is natural to extend our isomorphism μ so that it matches operators that act equivalently in the two spaces L_2 and l_2. More precisely, if A is an analog filter, we define its image $\mu(A) = \underline{\underline{A}}$ in the following way: let $\underline{x}$ be any digital signal. Then there corresponds to $\underline{x}$ a unique analog signal $\mu^{-1}(\underline{x})$. The result of operating on this analog signal by the analog filter A, $A\mu^{-1}(\underline{x})$, is also well-defined. This new analog signal can then be mapped by μ into a unique digital signal $\mu A\mu^{-1}(\underline{x})$, which we designate as the result of operating by $\underline{\underline{A}}$ on $\underline{x}$. Thus, we define $\underline{\underline{A}}$ to be the composite operator .

$$\underline{\underline{A}} = \mu A \mu^{-1} \tag{I-17}$$

To avoid confusion between digital filters and z-transforms of digital signals, we use the double underscore.

It is easy to see that the mapping μ for operators is linear, one-to-one, and onto. Given a digital filter $\underline{\underline{A}}$, its corresponding analog filter is $\mu^{-1}\underline{\underline{A}}\,\mu$. To show that the norm of an operator is preserved under the matching μ, we need only carry out the following calculation:

$$\begin{aligned} \|\underline{\underline{A}}\| &= \sup_{\underline{x}\in l_2} \frac{\|\underline{\underline{A}}\underline{x}\|}{\|\underline{x}\|} = \sup_{\underline{x}\in l_2} \frac{\|\mu(A\mu^{-1}(\underline{x}))\|}{\|\underline{x}\|} \\ &= \sup_{\underline{x}\in l_2} \frac{\|A\mu^{-1}(\underline{x})\|}{\|\mu^{-1}(\underline{x})\|} = \sup_{x\in L_2} \frac{\|Ax\|}{\|x\|} = \|A\| \quad . \end{aligned} \tag{I-18}$$

It should be pointed out that in one sense there is really no problem here. The spaces L_2 and l_2 are isomorphic; -- an analog filter and its digital image under μ are just two names for the same abstract object.

Having defined the effect of μ on filters, we should now like to do the same for the Fourier and z-transforms. This can be done in an equally natural way. Suppose A is an analog filter, a bounded linear operator on the space L_2 of analog signals. The analog signal space is mapped by the Fourier transform operator, say $\mathfrak{F}(\ )$, into a new space L_2, the space of Fourier transforms. The Fourier transform of the operator A, denoted by $\mathfrak{F}(A)$, will then be defined as an operator on this transform space so that if A maps f to g, then $\mathfrak{F}(A)$ maps F(s) to G(s). Analogously to (I-17) above, we require

$$\mathfrak{F}(A) = \mathfrak{F} A \mathfrak{F}^{-1} \quad , \tag{I-19}$$

where $\mathfrak{F}(\ )$ denotes the Fourier transform of an analog signal as well as an analog filter. Going through the same calculation that we performed for μ, we find that the Fourier transform preserves the norm of a filter:

$$\| \mathfrak{F}(A) \| = \| A \| \quad . \tag{I-20}$$

Similarly, we define the z-transform of a digital filter $\underline{\underline{A}}$ by

$$\mathfrak{Z}(\underline{\underline{A}}) = \mathfrak{Z} \underline{\underline{A}} \mathfrak{Z}^{-1} \tag{I-21}$$

where $\mathfrak{Z}(\ )$ denotes the z-transform of a digital signal or filter. Again, the norms of filters are preserved:

$$\| \mathfrak{Z}(\underline{\underline{A}}) \| = \| \underline{\underline{A}} \| \tag{I-22}$$

We have now generalized μ so that it pertains to filters as well as to signals, and we have defined the transforms of filters. Thus, a diagram analogous to Figure 1 can be drawn for filters, and this is shown in Figure 2. The connection between the Fourier transforms of analog filters and the z-transforms of digital filters is well-defined by the three legs of the diagram, but nothing more than that can be said at this time.

9. Some Familiar Classes of Filters

In this section we will show how the preceding theory of filters applies to many situations that are commonly encountered in engineering. For instance, time-invariant filtering is usually expressed by convolution in the time domain and by multiplication in the transform domain. Such time-invariant filtering is described in the analog case by the following theorem:

Theorem 7. Let $a(t)$ be a measurable function satisfying

$$\int_{-\infty}^{\infty} |a(t)| \, dt < \infty \quad . \tag{I-23}$$

That is, let $a(t)$ belong to $L_1(-\infty,\infty)$. Let the operator A be defined by the following convolution integral:

$$Af(t) = \int_{-\infty}^{\infty} f(\tau)a(t-\tau)\, d\tau \quad . \tag{I-24}$$

Then A is an analog filter with norm

$$\|A\| \leq \int_{-\infty}^{\infty} |a(t)|\, dt \quad .$$

Furthermore, the Fourier transform of the operator A is multiplication by the function A(s), the Fourier transform of a(t).

The proof of this theorem is a direct consequence of Schwarz's inequality and can be found in detail in Titchmarsh,[9] section 3-13.

This theorem applies to any linear time-invariant filter whose impulse response satisfies (I-23). Thus, any stable RLC network is an analog filter. Theorem 7 can also apply to the case where A is the identity operator Af = f, provided we are willing to admit the delta function as a sifting function satisfying (I-23). It is hardly necessary to introduce distributions or other generalized functions on this account, however, since the identity operator is clearly a bounded linear operator in its own right.

The following theorem for time-invariant digital filters can be obtained in exactly the same way as Theorem 7:

Theorem 8. Let $\{a_n\}$ be a sequence of complex numbers satisfying

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty \quad , \tag{I-25}$$

and let the operator $\underline{\underline{A}}$ be defined on the space of digital signals by the following convolution sum:

$$\underline{\underline{A}}\,\{\underline{f}_n\} = \left\{ \sum_{i=-\infty}^{\infty} \underline{f}_i \, a_{n-i} \right\} \quad . \tag{I-26}$$

Then $\underline{\underline{A}}$ is a digital filter with norm

$$\|\underline{\underline{A}}\| \leq \sum_{n=-\infty}^{\infty} |a_n| \quad .$$

Furthermore, the z-transform of the operator $\underline{\underline{A}}$ is multiplication by $\underline{\underline{A}}(z)$, the z-transform of the sequence $\{a_n\}$.

Now consider the case where the digital filter $\underline{\underline{A}}$ in Theorem 8 is the image under μ of the analog filter A of Theorem 7. Since the Fourier transform of Af is $A(s)F(s)$, the z-transform of the digital signal $\mu(Af)$ is

$$\frac{\sqrt{2}}{z+1}\, A\left(\frac{z-1}{z+1}\right)\, F\left(\frac{z-1}{z+1}\right) = \underline{\underline{A}}(z)\underline{F}(z) \quad .$$

Therefore

$$\underline{\underline{A}}(z) = A\left(\frac{z-1}{z+1}\right) \quad . \tag{I-27}$$

Thus, the transforms of filters which are equivalent under the isomorphism μ are related by a simple change of variable. This observation will be useful when we consider the approximation problem for digital signals.

By reversing the roles of the time and transform domains in Theorems 7 and 8 we come to consider the operation of multiplication by bounded time functions, or, in electrical engineering terms, amplitude modulation. More specifically, we have the following pair of theorems:

Theorem 9. Let $a(t)$ be a bounded measurable function of time, and let the operator A be defined on the class of analog signals by multiplication:

$$Af(t) = a(t)f(t) \quad . \tag{I-28}$$

Then A is an analog filter with norm

$$\|A\| \leq \sup_{\text{all } t} |a(t)| \quad .$$

Theorem 10. Let $\{a_n\}$ be a bounded sequence of complex numbers and let the operator $\underline{\underline{A}}$ be defined on the class of digital signals by multiplication

$$\underline{\underline{A}}\,\{\underline{f}_n\} = \{a_n\underline{f}_n\} \tag{I-29}$$

Then $\underline{\underline{A}}$ is a digital filter with norm:

$$\|\underline{\underline{A}}\| \leq \sup_{\text{all } n} |a_n| \quad .$$

The proofs follow immediately from the relations

$$\int_{-\infty}^{\infty} |a(t)f(t)|^2 \, dt \leq \left[\sup_{\text{all } t} |a(t)|\right]^2 \cdot \int_{-\infty}^{\infty} |f(t)|^2 \, dt \quad ,$$

and

$$\sum_{n=-\infty}^{\infty} |a_n f_n|^2 \leq \left[\sup_{\text{all } n} |a_n|\right]^2 \cdot \sum_{n=-\infty}^{\infty} |f_n|^2 \quad .$$

When the transforms of $a(t)$ and $\{a_n\}$ are in $L_1(-\infty,\infty)$ and $L_1(0,2\pi/T)$, respectively, it can be shown that the transforms of these multiplication operators are convolution operators on the $j\omega$-axis or unit circle. This representation is not important for us, however.

It is now easy to see that bounded linear operators are not in general commutative. That is, if A and B are two bounded linear operators, then it is not necessarily true that $B(Af) = A(Bf)$. Take, for example, the case where A is multiplication by $u(t)$, and B is an RLC filter. A and B do commute, however, in the special cases when A and B are both time-invariant filters as in Theorems 7 or 8; or when A and B are both multiplications as in Theorems 9 or 10. In general, we have the following results concerning combinations of operators:[7]

<u>Theorem 11</u>. If A and B are bounded linear operators, then A+B and AB are also bounded linear operators, and

$$\|A+B\| \leq \|A\| + \|B\| \quad , \quad .$$

and

$$\|AB\| \leq \|A\| \cdot \|B\| \quad .$$

10. A General Matrix Representation for Filters

We have seen in the last section how certain classes of filters can be represented in the time domain by convolution with time-invariant weighting functions or by multiplication. It would be desirable, however, to have a representation valid for any bounded linear operator. Such a representation can be constructed in the same way that matrices can be constructed from linear operators on a finite dimensional vector space; - that is, by examining the effect of an operator on a set of elements which forms a basis. Thus, if A is a linear operator in a finite dimensional vector space of dimension n, and if $\{e_1, e_2, \ldots, e_n\}$ is a basis, we can assemble the following array of equations:

$$\begin{aligned} Ae_1 &= a_{11}e_1 + a_{12}e_2 + \ldots + a_{1n}e_n \\ Ae_2 &= a_{21}e_1 + a_{22}e_2 + \ldots + a_{2n}e_n \\ &\ldots \\ Ae_n &= a_{n1}e_1 + a_{n2}e_2 + \ldots + a_{nn}e_n \end{aligned} \qquad \text{(I-30)}$$

In this way, every linear operator is associated with a unique nxn matrix $\{a_{ij}\}$. Conversely, every nxn matrix determines a linear operator; for if x is a vector with components $\{x_1, x_2, \ldots x_n\}$,

$$Ax = A\sum_{i=1}^{n} x_i e_i = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} x_i a_{ij}\Big) e_j \quad . \qquad \text{(I-31)}$$

This procedure allows us to characterize bounded linear operators in the infinite dimensional case, provided we impose an appropriate condition on the elements of the matrices involved:

<u>Definition</u>: The infinite matrix $\{a_{ij}\}_{i,j=-\infty}^{\infty}$ is said to be <u>bounded</u> if for some constant M we have

$$\left| \sum_{j=-r}^{s} \sum_{i=-p}^{q} a_{ij} x_i \overline{y}_j \right|^2 \leq M \sum_{i=-p}^{q} |x_i|^2 \cdot \sum_{j=-r}^{s} |y_j|^2 \qquad \text{(I-32)}$$

for any numbers $x_{-p}, x_{-p+1}, \ldots, x_o, x_1, \ldots, x_q$ and $y_{-r}, y_{-r+1}, \ldots, y_o, y_1, \ldots, y_s$.

We then have the following result, which is proved in Akhiezer and Glazman:[6]

<u>Theorem 12</u>. Let $\{e_i\}_{i=-\infty}^{\infty}$ be an orthonormal basis for the Hilbert space H. Then every bounded linear operator determines a unique bounded infinite matrix $\{a_{ij}\}$ by

$$Ae_i = \sum_{j=-\infty}^{\infty} a_{ij} e_j \quad , \qquad i = \ldots, -1, 0, 1, 2, \ldots \quad . \qquad \text{(I-33)}$$

Conversely, every bounded infinite matrix determines a bounded linear operator in the following way: If $f \in H$ has the orthonormal expansion

$$f = \sum_{i=-\infty}^{\infty} f_i e_i ,$$

put

$$Af = \sum_{j=-\infty}^{\infty} \Big(\sum_{i=-\infty}^{\infty} f_i a_{ij} \Big) e_j \quad . \tag{I-34}$$

For a fixed basis, we write $A \sim \{a_{ij}\}$ whenever the bounded linear operator A admits the bounded infinite matrix representation $\{a_{ij}\}$. In analogy with the finite dimensional case, it can be shown that if $A \sim \{a_{ij}\}$ and $B \sim \{b_{ij}\}$, then

$$A+B \sim \{a_{ij}+b_{ij}\} \quad , \tag{I-35}$$

and

$$BA \sim \Big\{ \sum_{k=-\infty}^{\infty} a_{ik} b_{kj} \Big\} \quad , \tag{I-36}$$

where $BA(f) = B(A(f))$. We have thus constructed a matrix-mechanical representation of signal filters, very much like that employed in quantum mechanics. Sometimes it will be convenient to think of a filter as being disconnected from both the input and the output for negative time. In this case we need only consider the lower right quandrant of the matrix: $\{a_{ij}\}_{i,j=1}^{\infty}$.

The interpretation of this matrix representation in the digital case is rather simple, mostly because the particular basis we have chosen has an easily grasped physical significance. The n-th basis element for the digital signal space is just

$$\{\underline{e}_n\} = \{\ldots, 0, 1, 0, \ldots\} \quad , \quad n = \ldots, -1, 0, 1, 2, \ldots ,$$

where the one is in the n-th place. This is the image under μ of the n-th orthonormal Laguerre function $\lambda_n(t)$. Thus, the z-transform of $\{\underline{e}_n\}$ is z^{-n}, and the z-transform of a digital signal, written as a power series in z, is a formal representation of its orthonormal expansion. The element a_{ij} in the matrix representation of a digital filter $\underline{\underline{A}}$ then corresponds to the output of the filter at time j when $\{\underline{e}_i\}$ has been applied. If any signal $\{\underline{f}_n\}$ is applied, the output signal will be

$$\underline{\underline{A}}\,\{\underline{f}_n\} = \left\{\sum_{i=-\infty}^{\infty} \underline{f}_i\, a_{in}\right\} \quad , \tag{I-37}$$

by (I-34). In the time-invariant case, we can write

$$a_{ij} = a_{j-i}$$

and then the effect of a digital filter can be described by the familiar convolution formula

$$\underline{\underline{A}}\,\{\underline{f}_n\} = \left\{\sum_{i=-\infty}^{\infty} \underline{f}_i\, a_{n-i}\right\} \quad . \tag{I-38}$$

This can be written in the z-transform domain as

$$\underline{\underline{A}}\{\underline{f}_n\} = \underline{\underline{A}}(z)\underline{F}(z) \quad , \tag{I-39}$$

where

$$\underline{\underline{A}}(z) = \sum_{i=-\infty}^{\infty} a_i z^{-i} \tag{I-40}$$

is the z-transform of the digital filter $\underline{\underline{A}}$. From (I-33) we see that

$$\sum_{i=-\infty}^{\infty} |a_i|^2 < \infty \quad ,$$

so that the weighting sequence $\{a_i\}$ is in l_2 and (I-40) is the z-transform of a digital signal.

A common type of digital filter is the so-called recursive or autoregressive filter defined by:

$$\underline{\underline{A}}\{\underline{f}_n\} = \left\{ \sum_{k=o}^{N} b_k \underline{f}_{(n-k)} - \sum_{k=1}^{M} c_k \underline{Af}_{(n-k)} \right\} \quad . \tag{I-41}$$

This filter produces each output by taking a linear combination of past outputs, and past and present inputs. It is time-invariant and its z-transform is

$$\underline{\underline{A}}(z) = \frac{\sum_{k=o}^{N} b_k z^{-k}}{1 + \sum_{k=1}^{M} c_k z^{-k}} = \sum_{i=o}^{\infty} a_i z^{-i} \quad . \tag{I-42}$$

Since the one-sided sequence $\{a_i\}$ is in l_2, $\underline{\underline{A}}(z)$ must be analytic outside the unit circle.

11. Relationship to the Weighting Function Representation in the Analog Case

The mapping μ does not affect the matrix representation of a filter, since it maps basis elements into basis elements. Thus, if A is an analog filter and $\underline{\underline{A}}$ is its digital image under μ, we have

$$A\lambda_i(t) = \sum_{j=-\infty}^{\infty} a_{ij}\lambda_j(t) \quad ,$$

and

$$\underline{\underline{A}}\{\underline{e}_i\} = \sum_{j=-\infty}^{\infty} a_{ij}\{\underline{e}_j\} \quad .$$

The interpretation of the matrix representation $\{a_{ij}\}$ is somewhat more difficult in the analog case, however, because we do not usually think of an analog signal as being represented by the coefficients in its Laguerre expansion; while we do think of a digital signal as being made up of its values at the discrete observation times. Furthermore, we usually think of an analog filter as being defined in terms of the convolution integral

$$Af(t) = \int_{-\infty}^{\infty} f(\tau)a(t,\tau)\, d\tau \quad , \tag{I-43}$$

where a(t) is some weighting function. We are thus faced with the problem of relating this representation to the matrix representation $\{a_{ij}\}$. When A is the identity operator, a(t) is the delta function, so we see immediately that we cannot expect a(t) to be a proper function in the general case. We can proceed formally, however, in the following manner: Letting

$$f(t) = \sum_{i=-\infty}^{\infty} f_i \lambda_i(t) \qquad ,$$

we can write

$$\begin{aligned} Af(t) &= \sum_{i=-\infty}^{\infty} f_i A\lambda_i(t) \\ &= \sum_{i=-\infty}^{\infty} f_i \sum_{j=-\infty}^{\infty} a_{ij}\lambda_j(t) \qquad , \end{aligned}$$

since the bounded operator is continuous. Replacing f_i by

$$\int_{-\infty}^{\infty} f(\tau)\lambda_i(\tau)\, d\tau \qquad ,$$

and interchanging the orders of integration and summation, we have

$$Af(t) = \int_{-\infty}^{\infty} f(\tau) \Big[\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij}\lambda_i(\tau)\lambda_j(t) \Big] d\tau \qquad .$$

We therefore have the formal equivalence ,

$$a(t,\tau) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij}\lambda_i(\tau)\lambda_j(t) \quad . \qquad \text{(I-44)}$$

The problem is: if A is a bounded operator, what kind of function will (I-44) be. We would expect, in general, that $a(t,\tau)$ will be a distribution, but a theorem to this effect does not exist in the mathematical literature and is certainly not obvious. We will therefore content ourselves with the formal connection between the bounded matrix $\{a_{ij}\}$ and the weighting function $a(t,\tau)$ given by (I-44).

The formula inverse to (I-44) can be derived as follows: The effect of A on $\lambda_i(t)$ is

$$A\lambda_i(t) = \int_{-\infty}^{\infty} \lambda_i(\tau)a(t,\tau)\, d\tau = \sum_{j=-\infty}^{\infty} a_{ij}\lambda_j(t) \quad . \qquad \text{(I-45)}$$

Therefore,

$$a_{ij} = \int_{-\infty}^{\infty} \lambda_j(t) \int_{-\infty}^{\infty} \lambda_i(\tau)a(t,\tau)\, d\tau\, dt. \qquad \text{(I-46)}$$

Here we have assumed that (I-45) is in L_2 and hence can be expanded in a Laguerre series.

When $a_{ij} = a_{j-i}$, the digital filter $\underline{\underline{A}}$ will have as its z-transform multiplication by $\underline{\underline{A}}(z)$. Hence, the Fourier transform of the analog filter A will be multiplication by $\underline{\underline{A}}\left(\frac{1+s}{1-s}\right)$, and therefore $a(t,\tau) = a(t-\tau)$. Conversely, if $a(t,\tau) = a(t-\tau)$, the Fourier transform of A will be multiplication by $A(s)$, and hence the z-transform of $\underline{\underline{A}}$ will be multiplication by $A\left(\frac{z-1}{z+1}\right)$. This implies that $a_{ij} = a_{j-i}$. We see that an analog or a digital filter will be time-invariant when and only when a_{ij} can be written a_{j-i}.

Those time-invariant filters which are physically realizable are of great importance in many fields. A time-invariant analog filter A is called realizable if $Af = 0$ for $t < 0$ whenever $f = 0$ for $t < 0$. Similarly, a time-invariant digital filter $\underline{\underline{A}}$ is called realizable if $\underline{\underline{A}}\{\underline{f}_n\} = 0$ for $n \leq 0$ whenever $\{\underline{f}_n\} = 0$ for $n \leq 0$. It is an important property of the mapping μ that it always matches time-invariant realizable filters with time-invariant realizable filters. To see this, suppose first of all that A is a time-invariant realizable analog filter. Let $\{\underline{f}_n\}$ be any digital signal for which $\{\underline{f}_n\} = 0$ for $n \leq 0$. Then its analog image $f(t)$ is such that $f(t) = 0$ for $t < 0$. Thus $Af = 0$ when t is negative, and this implies that $\underline{\underline{A}}\{\underline{f}_n\} = 0$ for $n \leq 0$. This shows that $\underline{\underline{A}}$ is a realizable digital filter. The same argument works the other way, and this establishes

Theorem 13. The mapping μ for filters always matches time-invariant realizable filters with time-invariant realizable filters.

A time-invariant digital filter is realizable if and only if $a_{ij} = a_{j-i} = 0$ when $i > j$. Hence, it follows that a time-invariant analog filter is realizable when and only when $a_{ij} = a_{j-i} = 0$ if $i > j$.

We can thus characterize all time-invariant realizable filters by upper triangular infinite matrices of the form

$$\begin{matrix} & & \cdots & & & \\ & a_0 & a_1 & a_2 & a_3 & a_4 & \\ & 0 & a_0 & a_1 & a_2 & a_3 & \\ \vdots & 0 & 0 & a_0 & a_1 & a_2 & \vdots \\ & 0 & 0 & 0 & a_0 & a_1 & \\ & 0 & 0 & 0 & 0 & a_0 & \\ & & \cdots & & & \end{matrix} \qquad \text{(I-47)}$$

Thus, a time-invariant analog filter is determined completely by its response to any $\lambda_i(t)$; just as a time-invariant digital filter is determined completely by its response to any $\underline{e}_n = \{\ldots 0,0,1,0,0,\ldots\}$. It follows also that the response of a realizable time-invariant analog filter to $\lambda_i(t)$ will have no λ_j components when $j < i$. That is, the output vector in response to $\lambda_i(t)$ is orthogonal to $\lambda_j(t)$ when $j < i$. For example, if we apply $\lambda_2(t)$ to a realizable time-invariant analog filter A, we would expect

$$\int_{-\infty}^{\infty} \lambda_1(t) \left[\int_0^t a(t-\tau)\lambda_2(\tau)\, d\tau\right] dt = 0 \quad .$$

Using Parseval's relation (I-4) and writing $\Lambda_i(s)$ for the Laplace transform of $\lambda_i(t)$, this becomes

$$\int_{-j\infty}^{j\infty} \Lambda_2(s)A(s)\Lambda_1(-s)\, ds = 0 \qquad ,$$

which is indeed true, since the integrand is analytic in the right-half s-plane.

In the time-varying case, on the other hand, μ does not preserve realizability. To see this, consider the bounded operator with the matrix

$$a_{12} = 1$$

$$a_{ij} = 0, \text{ otherwise.}$$

This corresponds to the digital filter which delays $\underline{f}_1$ one unit but has zero output at other times. Thus, $\underline{\underline{A}}$ is a realizable digital filter. The analog filter A, however, is given by (I-43) and (-44).

$$Af(t) = \lambda_2(t) \int_{-\infty}^{\infty} f(\tau)\lambda_1(\tau)\, d\tau \qquad ,$$

and is not realizable.

To illustrate the relationship between the matrix and the weighting function representation of a time-invariant analog filter, we will take up as an example the all-pass function

$$A(s) = \frac{1-s}{1+s} \qquad .$$

Multiplication by $A(s)$ in the transform domain defines a bounded operator; in fact, the analog filter A leaves the energy content of any signal invariant. The digital filter $\underline{\underline{A}}(z)$ corresponding to $A(s)$ is z^{-1}, a unit delay. Hence, the matrix representation of these operators is

$$a_{ij} = 1 \text{ if } j = i + 1$$
$$a_{ij} = 0 \text{ otherwise.}$$

According to (I-44), then, the weighting function $a(t,\tau)$ is

$$a(t,\tau) = \sum_{i=-\infty}^{\infty} \lambda_i(\tau)\lambda_{i+1}(t) \quad . \tag{I-48}$$

Assume, without real loss of generality, that $t \geq 0$ and $\tau \geq 0$. (I-48) then becomes

$$a(t,\tau) = \sum_{i=1}^{\infty} \lambda_i(\tau)\lambda_{i+1}(t) \quad . \tag{I-49}$$

We now need the following two identities, which are given in Head and Wilson:[13]

$$\sum_{s=1}^{\infty} \lambda_s(\tau)\lambda_s(t) = \delta(t-\tau) \quad , \tag{I-50}$$

$$\sum_{s=1}^{\infty} \lambda_s(\tau)\lambda_{s+n+1}(t) = - \sum_{s=1}^{\infty} \lambda_s(\tau)\lambda_{n+s}(t) + \sqrt{2}\, \lambda_{n+1}(t-\tau) \quad . \tag{I-51}$$

These identities are very useful for putting $a(t,\tau)$ in closed form when the filter is time-invariant. Putting n=0 in (I-51), we get finally

$$a(t,\tau) = a(t-\tau) = -\,\delta(t-\tau) + 2e^{-(t-\tau)}\,u(t-\tau) \quad .$$

This checks with the inverse Laplace transform of $A(s) = \frac{1-s}{1+s}$. (I-46) can be checked similarly: the integral

$$\int_{-\infty}^{\infty} \lambda_i(\tau)a(t,\tau)\,d\tau$$

is equal to $\lambda_{i+1}(t)$, and hence

$$a_{ij} = \int_{-\infty}^{\infty} \lambda_j(t) \int_{-\infty}^{\infty} \lambda_i(\tau)a(t,\tau)\,d\tau\,dt$$

is 1 when $j = i + 1$, and zero otherwise.

It should be noted that the complexity of the representation of analog filters as compared with digital filters is reflected in the identification and synthesis problems. To identify a time-varying digital filter, one need only apply signals $\{\underline{e}_i\}$ and read the coefficients a_{ij} from the output at times j. Synthesis involves only the setting of coefficients in a digital computation program. In the analog filter case, however, there is no such natural and convenient basis. If we apply $\lambda_i(t)$ as a testing function, we must then resolve the output into a Laguerre series and use a formula like (I-44) to arrive at a weighting function. Even then, we are left with the problem of realizing $a(t,\tau)$ in the general case.

Still, it is a significant and not widely mentioned fact that any analog filter, even a time-varying one, can be characterized by an infinite matrix of numbers. More important, this fact has not been put to full use in the development of identification techniques. In fact, most identification techniques, whether they are based on a weighting function representation, a differential equation model, a time-varying transfer function model, or an orthogonal filter expansion, usually assume that the system is stationary over short measurement segments. The matrix representation, on the other hand, is a very general representation, valid even for fast varying systems.

12. Optimization Problems for Systems with Deterministic Signals

We are now in a position to see how some optimization problems can be solved simultaneously for analog and digital signals. Suppose, for example, that a certain one-sided analog input signal $r(t)$ is corrupted by a known one-sided noise signal $n(t)$, and that we are required to filter out the noise with a stable, realizable time-invariant filter H whose Laplace transform is, say, $H(s)$. If we adopt a least-mean-square error criterion, we require that

$$\int_0^\infty (r-H(r+n))^2 \, dt = \min. \qquad \text{(I-52)}$$

As described by Chang,[1] this can be transformed by Parseval's relation to the requirement

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} [R-H(R+N)][\overline{R-H(R+N)}]\, ds = \text{min.} \quad , \qquad \text{(I-53)}$$

where R,H, and N are functions of s, and the overscore indicates that s is replaced by -s. It can be then shown, using an adaptation of the calculus of variations, that the realizable solution for H(s), say $H_o(s)$, is given by

$$H_o(s) = \frac{1}{Y}\left[\frac{\overline{(R+N)}R}{\overline{Y}}\right]_{LHP} \quad , \qquad \text{(I-54)}$$

where

$$Y\overline{Y} = (R+N)(\overline{R+N}) \quad ,$$

and Y has only left-half plane poles and zeros, and $\overline{Y}$ has only right-half plane poles and zeros. The notation $[\ \ ]_{LHP}$ indicates that a partial fraction expansion is made and only the terms involving left-half plane poles are retained.

The fact that a least-mean-square error criterion is used means that the optimization criterion (I-52) can be expressed in the axiomatic framework of Hilbert space. Thus, in the Hilbert space $L_2(-\infty,\infty)$, (I-52) becomes

$$\| r - H(r+n) \| = \text{min.} \qquad \text{(I-55)}$$

If we now apply the mapping μ to the signal r-H(r+n), we see that

$$\left\| r-H(r+n) \right\| = \left\| \mu(r-H(r+n)\ ) \right\| = \left\| \underline{r}-\underline{\underline{H}}(\underline{r}+\underline{n}) \right\| \quad , \tag{I-56}$$

since μ preserves norm. Hence $\underline{\underline{H}}_o$ is a solution to the optimization problem

$$\left\| \underline{r}-\underline{\underline{H}}(\underline{r}+\underline{n}) \right\| = \text{min.} \tag{I-57}$$

Furthermore, since μ matches one-sided analog signals with one-sided digital signals, and since μ matches realizable time-invariant analog filters with realizable time-invariant digital filters, we see that $\underline{\underline{H}}_o$ is the solution to a digital optimization problem that is completely analogous to the original analog problem. In addition, the general solution (I-54) can be translated into digital terms by replacing the left-half plane by the unit circle in an appropriate way. Thus,

$$\underline{\underline{H}}_o(z) = \frac{1}{\underline{Y}}\left[\frac{\overline{(\underline{R}+\underline{N})}\,\underline{R}}{\overline{\underline{Y}}}\right]_{IN} \quad , \tag{I-58}$$

where

$$\underline{Y}\overline{\underline{Y}} = (\underline{R}+\underline{N})\overline{(\underline{R}+\underline{N})} \quad .$$

Now $\underline{R}$, $\underline{\underline{H}}_o$, and $\underline{N}$ are functions of z; the overscore indicates that z is replaced by z^{-1}; $\underline{Y}$ and $\overline{\underline{Y}}$ have poles and zeros inside and outside of the unit circle, respectively; and the notation $[\ \ ]_{IN}$ indicates that only the terms in a partial fraction expansion with poles inside the unit circle have been retained.

In other optimization problems we may wish to minimize the norm of some error signal while keeping the norm of some other system signal within a certain range. In a feedback control system, for instance, we may want to minimize the norm of the error with the constraint that the norm of the input to the plant be less than or equal to some predetermined number. Using Lagrange's method of undetermined multipliers, this problem can be reduced to the problem of minimizing a quantity of the form

$$\|e\|^2 + k^2 \|i\|^2 \quad , \tag{I-59}$$

where e is an error signal, i is some energy limited signal, and both e and i depend on an undetermined filter function H. Again, if $H_o(s)$ is the time-invariant realizable solution to such an analog problem, then $\underline{\underline{H}}_o(z)$ will be the time-invariant realizable solution to the analogous digital problem.

It is almost always important to us that the solution to an optimization problem be realizable, but we may want to allow as a solution a time-varying filter. Unfortunately, the isomorphism μ does not necessarily match realizable time-varying analog filters with realizable time-varying digital filters. We thus cannot show that optimization problems which allow time-varying solutions are equivalent in the analog and digital cases. Furthermore, since any known isomorphisms involve orthogonal expansions of the analog signals over semi-infinite or infinite ranges of time, it appears that an isomorphism between $L_2(-\infty,\infty)$ and l_2 which preserves the realizability of filters cannot be constructed. Thus, in order to show the equivalence of an analog with a digital optimization problem, we require that the allowed class of filters, say $\mathcal{C}$, be invariant under a particular isomorphism. Another way of looking at the problem is to say that we are really demanding that the entire optimization problem be expressible in the terms of abstract Hilbert space. Thus, when the class is the class of time-invariant realizable filters, we can characterize $\mathcal{C}$ in abstract Hilbert space as the class of all bounded linear operators A for which $a_{ij} = a_{j-i}$ and $a_{j-i} = 0$ for $i > j$.

We can therefore state that any optimization problem which can be expressed solely in terms of abstract Hilbert space can be solved simultaneously for analog and digital systems. In particular we can state:

Theorem 14. Let ν be an isomorphism between $L_2(-\infty,\infty)$ and l_2. Further, let the following optimization problem be posed in the analog signal space $L_2(-\infty,\infty)$: Find analog filters $H_1, H_2, \ldots, H_n$ which minimize some function of some norms in a given analog signal transmission system and which are in a class of filters $\mathcal{C}$. Then if the class of filters $\mathcal{C}$ is invariant under ν, the corresponding digital problem is equivalent to the original analog problem in the sense that if one can be solved, so can the other.

As we have seen, the case where $\mathscr{C}$ is the class of time-invariant realizable filters, and ν is μ, is an important application of this result. In this situation we have the following correspondences:

s	$\longleftrightarrow$	z
Left-Half Plane	$\longleftrightarrow$	Inside Unit Circle
$j\omega$-axis	$\longleftrightarrow$	Unit Circle
Right-Half Plane	$\longleftrightarrow$	Outside Unit Circle
$\dfrac{1}{2\pi j}\displaystyle\int_{-j\infty}^{j\infty} (\quad)\, ds$	$\longleftrightarrow$	$\dfrac{1}{2\pi j}\displaystyle\oint_{\lvert z\rvert=1} (\quad)\, \dfrac{dz}{z}$

13. Random Signals and Statistical Optimization Problems

While the consideration of systems with deterministic signals is important for many theoretical and practical reasons, it is more often the case that the design engineer knows only the statistical properties of the input and disturbing signals. For this reason, the design of systems on a statistical basis has become increasingly important in recent years. In this section we shall show that the idea of linking continuous theory with discrete theory can be extended to a broad class of random phenomena; namely, stationary, ergodic processes with well-behaved correlation functions and spectra.

Because a complete axiomatization of random processes is a very complex affair, we will simplify matters by approaching the subject through the correlation function. This is not nearly so restrictive as it might first appear, because physical stochastic processes almost always have correlation functions that are of exponential order, and their spectra are almost always bounded and continuous. For a more complete discussion of random signal theory and generalized harmonic analysis, the reader is referred to Wiener.[8,12] Accordingly, we assume that random signals are stationary, ergodic, and have zero mean. If $x(t)$ and $y(t)$ are two such random signals, we assume further that the cross-correlation function

$$\phi_{xy}(\tau) = E[x(t)y(t+\tau)] \tag{I-60}$$

dies down exponentially with increasing $|\tau|$. The notation E[ ] means "ensemble average of". Since the processes are ergodic, (I-60) can be expressed as a time average:

$$\phi_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)y(t+\tau)\, dt. \tag{I-61}$$

Now let $x_T(t)$ and $y_T(t)$ be the same signals as $x(t)$ and $y(t)$ for $0 \leq t \leq T$, but zero outside of this range; and let $X_T(s)$ and $Y_T(s)$ be their Laplace transforms. The cross-spectral-density function is then defined by

$$\Phi_{xy}(s) = \lim_{T \to \infty} \frac{1}{T} E[X_T(-s)Y_T(s)] \quad . \tag{I-62}$$

It is a classical result of generalized harmonic analysis, called Wiener's theorem, that $\phi_{xy}(t)$ and $\Phi_{xy}(s)$ are transform pairs:

$$\Phi_{xy}(s) = \int_{-\infty}^{\infty} \phi_{xy}(t)e^{-st}\, dt \quad , \tag{I-63}$$

and

$$\phi_{xy}(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{xy}(s)e^{st}\, ds \quad . \tag{I-64}$$

In the important case when x=y, the variance of x is given by (I-64):

$$E[x^2] = \phi_{xx}(0) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{xx}(s)\, ds \quad . \tag{I-65}$$

As we might expect, a parallel theory exists in the digital case. Here, if x_i and y_i are two discrete stationary, and ergodic random processes, the cross-correlation function is defined by

$$\phi_{xy}(n) = E[x_i x_{i+n}]. \tag{I-66}$$

Again, with ergodicity, we have

$$\phi_{xy}(n) = \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{i=1}^{N} x_i x_{i+n} \right\} \quad . \tag{I-67}$$

The cross-spectral-density is a function of z, defined by

$$\Phi_{xy}(z) = \lim_{N \to \infty} \frac{1}{N} E[X_N(z^{-1})Y_N(z)] \quad , \tag{I-68}$$

where $X_N(z)$ and $Y_N(z)$ are the z-transforms of signals which coincide with x_i and y_i for $0 \leq i \leq N$, and which are zero outside this range. As in the analog case, $\phi_{xy}(n)$ and $\Phi_{xy}(z)$ are transform pairs:

$$\Phi_{xy}(z) = \lim_{N \to \infty} \sum_{n=-N}^{N} \phi_{xy}(n)z^{-n} \quad , \tag{I-69}$$

which exists on the unit circle if we assume that the correlation function dies off exponentially as $n \to \infty$; and

$$\phi_{xy}(n) = \frac{1}{2\pi j} \oint_{|z|=1} \Phi_{xy}(z)z^{n} \frac{dz}{z} \tag{I-70}$$

The variance of the signal x_i, in analogy with (I-65), is

$$E[x_i^2] = \phi_{xx}(0) = \frac{1}{2\pi j} \oint_{|z|=1} \Phi_{xx}(z) \frac{dz}{z} \quad . \tag{I-71}$$

The parallel with the deterministic case is so strong when the random theory is put in the above form, that the introduction of the mapping μ presents no problem. Consider (I-62), for example.

If we map the transformed analog signal $X_T(s)$ to $[\sqrt{2}/(z+1)]X_T\left(\frac{z-1}{z-1}\right)$, we should map $X_T(-s)$ to

$$\frac{\sqrt{2}}{z^{-1}+1}\; X_T\left(\frac{z^{-1}-1}{z^{-1}+1}\right) \quad .$$

Similarly, $Y_T(s)$ should map to $[\sqrt{2}/(z+1)]\; Y_T\left(\frac{z-1}{z+1}\right)$. In accordance with (I-68), we define the mapping μ by

$$\mu:\; \phi_{xy}(t) \longrightarrow \Phi_{xy}(s) \longrightarrow \frac{2z}{(z+1)^2}\; \Phi_{xy}\left(\frac{z-1}{z+1}\right) = \underline{\Phi}_{xy}(z) \longrightarrow \underline{\phi}_{xy}(n) \quad . \tag{I-72}$$

The reverse mapping goes

$$\mu^{-1}:\; \underline{\phi}_{xy}(n) \longrightarrow \underline{\Phi}_{xy}(z) \longrightarrow \frac{2}{1-s^2}\; \Phi_{xy}\left(\frac{1+s}{1-s}\right) = \Phi_{xy}(s) \longrightarrow \phi_{xy}(t) \quad . \tag{I-73}$$

We have thus defined a mapping which maps analog to digital cross-correlation functions. The important invariants under μ are the quantities

$$\phi_{xy}(0) = E[x(t)y(t)] \quad ,$$

and

$$\underline{\phi}_{xy}(0) = E[x_n y_n] \quad ,$$

which correspond to the inner product in the deterministic case. To verify that these are preserved under μ, put $t=0$ in (I-64):

$$\phi_{xy}(0) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{xy}(s)\, ds \quad .$$

If we now make the change of variable $s = \frac{z-1}{z+1}$, we get

$$\phi_{xy}(0) = \frac{1}{2\pi j} \oint_{|z|=1} \Phi_{xy}\left(\frac{z-1}{z+1}\right) \frac{2}{(z+1)^2}\, dz$$

$$= \frac{1}{2\pi j} \oint_{|z|=1} \Phi_{xy}(z) \frac{dz}{z}$$

$$= \underline{\phi}_{xy}(0) \quad .$$

Since all of the steps in (I-72) and (I-73) are reversible and give unique results, the mapping μ is clearly one-to-one and onto.

It is easy to see from (I-62) that if $x(t)$ is passed through the time-invariant filter $H(s)$ and $y(t)$ is passed through the time-invariant filter $G(s)$, the resulting cross-spectral-density is[1]

$$\Phi_{Hx,Gy}(s) = H(-s)G(s)\Phi_{xy}(s) \quad . \tag{I-74}$$

Also, if a digital signal x_n is passed through the time-invariant digital filter $H(z)$ and y_n through $G(z)$, we have[1]

$$\Phi_{Hx_n,Gy_n}(z) = H(z^{-1})G(z)\Phi_{x_ny_n}(z) \quad . \tag{I-75}$$

Hence, the mapping μ can be extended to time-invariant filters by mapping $F(s)$ to $\underline{\underline{F}}(z) = F\left(\frac{z-1}{z+1}\right)$, as in the deterministic case.

With the mapping μ defined for correlation functions, we are now in a position to show the equivalence of certain analog and digital statistical optimization problems. Suppose, for instance, that in a particular analog signal transmission system we wish to choose a realizable time-invariant filter H which minimizes the mean-square-value of some error signal $e(t)$. Thus, we require

$$E[e^2] = \phi_{ee}(0) = \min. \quad , \tag{I-76}$$

where $\phi_{ee}(0)$ depends, of course, on H. If H_o is the solution to this problem, then $\underline{\underline{H}}_o$ will be the solution to the analogous digital problem: that is, the digital problem with correlation functions $\Phi_{xy}(z)$ instead of $\Phi_{xy}(s)$, and with filters $\underline{\underline{F}}(z)$ instead of $F(s)$. More generally, the problems of minimizing some function of some mean-square-values by appropriate choices of realizable time-invariant filters are equivalent in the analog and digital cases.

We now conclude the specific discussion of the mapping μ. It has proven useful in tying together signal and filter theory for the analog and digital cases. When the filters involved are realizable and time-invariant, when the deterministic signals involved are given for all time, and when the random signals are stationary and ergodic, we have in fact shown that the analog and digital cases are just two interpretations of the same mathematical theory.

14. Data Reduction Filters

It is often desirable in the processing and analysis of digital time series to reduce a large number of data points to a digital signal with less frequent data points. This section is devoted to a discussion of digital filters for which the output data rate is lower than the input data rate.

Suppose that a digital signal has data points occurring every T seconds, and that it is desirable to reduce this signal to one with data points every NT seconds, where N is an integer. Let A^N denote a realizable, bounded, linear filter which accomplishes this, let $f_i (i=\ldots,-1,0,1,2,\ldots)$ denote the input data point occurring at $t=iT$, and let $g_i (i=\ldots,-NT,0,NT,2NT,\ldots)$ denote the output data point occurring at $t=iT$. By the general matrix representation of such a filter, we can write

$$g_j = \sum_{i=-\infty}^{j} f_i a_{ij}^N \quad , \qquad j = \ldots,-NT,0,NT,2NT,\ldots, \tag{I-77}$$

where $a_{ij} = 0$ for $i > j$. a_{ij} represents the output at time j due to an input at time i, provided that j is a multiple of N. In the time-invariant case, we have

$$g_j = \sum_{i=-\infty}^{j} f_i a_{j-i}^N \quad , \qquad j = \ldots,-NT,0,NT,2NT,\ldots, \tag{I-78}$$

where $a_n^N = 0$ when $n < 0$. Now the filter of (I-78) can be represented as the ~~series~~ cascade connection of two elements: first the ordinary time-invariant filter A defined by $a_n = a_n^N$, and then a switch which closes only when t is an integral multiple of NT. Thus, we have proved

<u>Theorem 15</u>. Every bounded, linear, time-invariant data reduction filter A^N can be written as the ~~series~~ cascade combination of a bounded, linear, time-invariant digital filter A, and a bounded, linear digital filter B^N defined by

$$b_{ij}^N = \begin{cases} 1 \text{ if } i=j=kNT, \text{ where k is an integer} \\ 0 \text{ otherwise.} \end{cases}$$

If we were to use a data reduction filter, we would usually be interested in the frequency response of the output as compared with the input. Since the effect of the time-invariant digital filter A is well-known, we need only consider the effect of the intermittant switch B^N. With this end, let g(t) be any analog signal with sample points $g(iT) = g_i$. The frequency function associated with the digital signal g_i (really G(z) with $z = e^{j\omega T}$) is then[3]

$$G^T(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} G(j\omega + jn2\pi/T) \quad , \tag{I-80}$$

where $G(j\omega)$ is the Fourier transform of g(t), and $G^T(j\omega)$ denotes the Fourier transform of the sampled function

$$\sum_{i=-\infty}^{\infty} g(iT)\delta(t-iT). \tag{I-81}$$

Also, in the same way, the frequency function associated with the output function $B^N g_i$ is

$$G^{NT}(j\omega) = \frac{1}{NT} \sum_{n=-\infty}^{\infty} \mathcal{G}(j\omega + jn2\pi/NT) \quad . \qquad \text{(I-82)}$$

From (I-80) and (I-82) we see that

$$G^{NT}(j\omega) = \frac{1}{N} \sum_{k=o}^{N-1} G^T(j\omega + jk2\pi/NT) \quad . \qquad \text{(I-83)}$$

Equation (I-83) can be thought of as representing a "sub-aliasing" operation. In order that there be no overlapping and loss of information, it is sufficient, for example, that $G^T(j\omega)$ have significant magnitude only in the ranges $-\pi/NT \leq \omega \pm k2\pi/T \leq \pi/NT$, where k is an integer. In this case, $G^T(j\omega)$ can be reconstructed from $G^{NT}(j\omega)$ with a digital filter having a frequency response with magnitude unity in these intervals and zero outside these intervals. This is a digital analog to the familiar sampling theorem. We see then that if a digital signal is prefiltered with appropriate bandpass digital filters, it can be reduced to a signal with a lower data rate while retaining desired parts of its frequency function.

In the random case, the relation

$$\Phi^{NT}(j\omega) = \frac{1}{N} \sum_{k=o}^{N-1} \Phi^T(j\omega + jk2\pi/NT)$$

can be derived in exactly the same way, where $\Phi^{T}(j\omega)$ is the power-spectral-density of the random digital signal with data points every T seconds, and $\Phi^{NT}(j\omega)$ is the power-spectral-density of the reduced random signal.

The case of data reduction is but one example of the need for a method of designing digital filters with a prescribed frequency response. This problem is treated in the next part.

PART II: THE APPROXIMATION PROBLEM FOR DIGITAL FILTERS*

1. Introduction

The problem of finding a realizable transfer function with a prescribed magnitude or phase angle on the $j\omega$-axis has been an important one to the designers of continuous wave filters. The Butterworth and Tchebycheff filters, together with their high-pass and bandpass transformations, are well-known solutions to common approximation problems. The possibility of designing digital systems in terms of frequency response, however, has remained relatively unexplored since the early work of Salzer,[17] despite the advent of sampled-data systems and the increasingly wide use of digital computers in control and measurement. In the following sections we shall explore some approximation techniques for digital filters.

An immediate application of such techniques is the design of effective prefilters for data reduction and reconstruction programs for digital systems. In Part III we shall see how such approximation techniques bear on the problem of the power spectrum analysis of discrete data. Also, when some time delay is tolerable, it may sometimes be practical to use a digital filter in tandem with a data reconstruction device to filter an analog signal. With the development of faster, more compact computers and data transmission

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* The results in this part were reported by the author in reference 22.

systems with higher bit rates, the possibility of substituting digital processing for analog processing of signals is becoming more attractive.

2. Equivalence to the Approximation Problem for Analog Filters

Having seen the close correspondence between digital and analog filter theory, especially in the time-invariant case, we should suspect that the respective approximation problems can somehow be connected. This is indeed the case. In fact, the mapping μ is precisely the tool that can provide the link. It should be noted that Lewis[18] used the mapping μ to show how digital filters can be synthesized with networks of open- and short-circuited transmission lines. It was not appreciated at that time, however, just how powerful a link a bilinear frequency transformation provides between digital and analog theory.

In order to avoid confusion between variables, we shall use $s = \sigma + j\omega$ for the frequency variable for analog filters, and $z = e^{(\underline{\sigma}+j\underline{\omega})T}$ for digital filters. When we make the transformation

$$\mu: \quad s = \frac{z-1}{z+1} \quad , \tag{II-1}$$

the entire $j\omega$-axis is mapped onto the unit circle in the z-plane, and the entire ω-axis is mapped into intervals on the $\underline{\omega}$-axis $2\pi/T$ wide (see Fig. 3). When $s = j\omega$, we have, in fact, the following connection between ω and $\underline{\omega}$:

$$z = e^{j\underline{\omega}T} = \frac{1+j\omega}{1-j\omega} \quad , \tag{II-2}$$

or

$$\underline{\omega} = \frac{2}{T}\arctan \omega.$$

Suppose now that we are given some periodic function of $\underline{\omega}$, $C(\underline{\omega})$ say, that is to be the desired characteristic (magnitude, phase angle, real or imaginary part) of a digital filter. $C\left(\frac{2}{T}\arctan \omega\right)$ will then be the corresponding characteristic for an analog filter. We can then approximate $C\left(\frac{2}{T}\arctan \omega\right)$ as an analog filter characteristic, using any one of the many procedures available for analog filters. We thus arrive at a rational function of s, say $A(s)$. Then $\underline{\underline{A}}(z) = A\left(\frac{z-1}{z+1}\right)$ will be a digital filter whose characteristic approximates the desired one. Since the left-half s-plane is mapped inside the unit circle in the z-plane, stable poles of the analog filter $A(s)$ will become stable poles of the digital filter $\underline{\underline{A}}(z)$.

Loosely speaking, we have taken the interval $|\underline{\omega}| \leq \pi/T$ and stretched it out; done our approximation for an analog filter; and then squeezed the ω-axis back into the original interval. Although the ω-axis is compressed, many of the widely used approximation criteria, such as equal ripple, maximal flatness, etc., carry over directly to the digital filter case. If an analog filter $A(s)$ has magnitude $M(\omega)$, phase angle $\Phi(\omega)$, real part $R(\omega)$, and imaginary part $I(\omega)$; then the corresponding digital filter $\underline{\underline{A}}(z)$ will have magnitude $M(\tan \underline{\omega}T/2)$, phase angle $\Phi(\tan \underline{\omega}T/2)$ in $|\underline{\omega}| \leq \pi/T$, real part $R(\tan \underline{\omega}T/2)$, and imaginary part $I(\tan \underline{\omega}T/2)$.

As an illustrative example, suppose we wish to approximate the ideal low-pass characteristic shown as a dashed line in Figure 4. We have taken the cutoff frequency to be at $\underline{\omega} = \pi/2T$, one-half the Nyquist frequency. The analog filter $A(s)$ should therefore, by (II-3), have an ideal cutoff at

$$\omega = \tan \underline{\omega}T/2 = 1 \quad . \tag{II-4}$$

Let us now use for $A(s)$ a third-order maximally flat Butterworth low-pass filter[19] with unit cutoff frequency:

$$A(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \quad .$$

When we let $s = \frac{z-1}{z+1}$, this becomes the digital filter

$$\underline{\underline{A}}(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{3 + z^{-2}} \tag{II-5}$$

whose normalized magnitude is shown plotted as curve B in Figure 4. $\underline{\underline{A}}(z)$ is now a maximally flat digital filter. Its response is zero at the Nyquist frequency $\underline{\omega} = \pi/T$, this point corresponding to infinite frequencies for the analog filter $F(s)$. The filter $\underline{\underline{A}}(z)$

can be implemented in a hand or machine computation according to (I-41), section 10. Thus, if f_i and g_i, respectively, are the input and output digital signals;

$$g_i = \frac{1}{3}(f_i + 3f_{i-1} + 3f_{i-2} + f_{i-3} - g_{i-2}). \tag{II-6}$$

A typical application of such a smoothing operation would be to remove high frequency noise prior to halving the number of data points.

As a more elaborate example of a smoothing routine, suppose we wish a low-pass filter with a sharp cutoff at one-quarter the Nyquist frequency, $\underline{\omega} = \pi/4T$. This corresponds under the mapping μ to the frequency

$$\omega = \tan \underline{\omega}T/2 = \tan \pi/8 = 0.4142 \quad . \tag{II-7}$$

Suppose further that we desire the digital filter to have equal ripple in the pass band. We might then start off with the fourth-order Tchebycheff filter having about 10% ripple ($\epsilon^2 = 1/5$), and with a cutoff frequency at $\omega = 1$:

$$A(s) = \frac{1}{s^4 + 1.034s^3 + 1.535s^2 + 0.8306s + 0.3062} \ . \tag{II-8}$$

If we substitute $(s/0.4142)$ for s, we get

$$A_1(s) = \frac{1}{s^4 + 0.4284s^3 + 0.2633s^2 + 0.05903s + 0.009011} \ , \tag{II-9}$$

which has a cutoff frequency at $\omega = 0.4142$. We then substitute $s = \frac{z-1}{z+1}$ to obtain the desired digital filter:

$$\underline{\underline{A}}_1(z) = \frac{1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4}}{1.760 - 4.703z^{-1} + 5.527z^{-2} - 3.225z^{-3} + 0.7849z^{-4}} \quad . \tag{II-10}$$

Figure 5 shows the normalized magnitude of this Tchebycheff equal ripple digital filter. If this filter were used prior to a one-half data reduction, noise at frequencies greater than half the Nyquist frequency would affect the resulting signal very little. If the power-spectral-density of the resulting reduced digital signal were measured, it would be desirable to correct for the ripples in the frequency characteristic of the filter $\underline{\underline{A}}(z)$. The design of a high-pass or a bandpass digital filter follows the same pattern.

We have thus seen how the mapping μ allows us to reduce the approximation problem for digital filters to that for analog filters. The technique described allows the designer of digital information processing systems to deal with signals in the frequency domain in much the same way that the communications engineer deals with analog signals.

3. Comparison with Fourier Series Techniques

Guillemin[19] has suggested the use of Fourier series for the approximation of magnitude characteristics of analog filters. His approximation procedure consists of using the mapping μ to convert the desired characteristic to one that is a periodic function of frequency, using a truncated Fourier series in $\underline{\omega}$ to approximate this, and then inverting the transformation μ to give a rational function of ω. Since we deal directly with periodic magnitude characteristics as a function of $\underline{\omega}$, we can use Fourier series directly. Thus the use of Fourier series is a natural choice for the design of digital filters, and Guillemin reversed our program and used it for the design of analog filters.

Suppose, then, that we are given the desired magnitude characteristic $M(\underline{\omega})$ of some digital filter. Since this is an even function of $\underline{\omega}$ with period $2\pi/T$, we can approximate it in a least mean-square-error sense with the truncated Fourier series

$$M(\underline{\omega}) \cong \sum_{n=-K}^{K} c_n \, e^{-jn\underline{\omega}T} \quad , \tag{II-11}$$

where

$$c_n = c_{-n} = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} M(\underline{\omega}) \, e^{jn\underline{\omega}T} \, d\underline{\omega} \quad . \tag{II-12}$$

The realizable digital filter

$$\underline{\underline{A}}(z) = \sum_{n=o}^{2K} c_{n-K} \, z^{-n} = z^{-K} \sum_{n=-K}^{K} c_n \, z^{-n} \tag{II-13}$$

will then have a magnitude characteristic which approximates $M(\underline{\omega})$, because when $z = e^{\underline{j\omega T}}$

$$|\underline{\underline{A}}(z)| = \left| \sum_{n=-K}^{K} c_n z^{-n} \right| \cong M(\underline{\omega}) \quad . \tag{II-14}$$

This technique is particularly valuable for two reasons. First, since the series (II-11) is a cosine series, the only phase distortion is that caused by the delay factor z^{-K}. Thus, if the delay of KT is tolerable, there is essentially no phase distortion. Second, these filters are polynomials in z^{-1} and have no denominator. Therefore, their implementation

$$g_i = c_{-K}f_i + c_{-K+1}f_{i-1} + \ldots + c_K f_{i-2K} \tag{II-15}$$

does not require the storage of outputs. This leads to programs which can be easily effected by simple special purpose computers and which require a relatively small amount of storage capacity.

On the other hand, the fact that these filters are polynomials in z^{-1} means that there is a loss of several degrees of freedom. This usually leads to magnitude characteristics that have the ripple and overshoot characteristic of Fourier series approximations. Looking at this problem in another way, we can consider these Fourier series filters, or any other polynomial filters, as power series approximations to rational functions, since by (I-40) any time-invariant digital filter can be written as an infinite series:

$$\underline{\underline{A}}(z) = \sum_{i=-\infty}^{\infty} a_i z^{-i} \quad . \tag{II-16}$$

We would therefore expect a finite polynomial in z^{-1} to have more ripple and overshoot than a properly designed rational function, whose power series does not terminate.

To illustrate these points, suppose we again want a low-pass digital filter with a cutoff frequency at one-half the Nyquist frequency. $M(\underline{\omega})$ is the ideal characteristic shown as a dashed line in Figure 6. Equation (II-12) then yields the following Fourier coefficients:

$$c_n = c_{-n} = \begin{cases} 1/2 \ , & n = 0 \\ \dfrac{(-1)^{(n-1)/2}}{n\pi} \ , & n = 1,3,5,\ldots \\ 0 \ , & n = 2,4,6,\ldots \ . \end{cases} \tag{II-17}$$

The normalized magnitude characteristics of the first three of the resulting digital filters are plotted in Figure 6:

(II-18)

Curve A: $\underline{\underline{A}}(z) = \frac{1}{\pi} + \frac{1}{2}z^{-1} + \frac{1}{\pi}z^{-2}$

Curve B: $\underline{\underline{A}}(z) = \frac{1}{3\pi} + \frac{1}{\pi}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{\pi}z^{-4} - \frac{1}{3\pi}z^{-6}$

Curve C: $\underline{\underline{A}}(z) = \frac{1}{5\pi} - \frac{1}{3\pi}z^{-2} + \frac{1}{\pi}z^{-4} + \frac{1}{2}z^{-5} + \frac{1}{\pi}z^{-6} - \frac{1}{3\pi}z^{-8} + \frac{1}{5\pi}z^{-10}$.

We note the ripple and overshoot described above. One way to alleviate this difficulty would be to use Fejer means[20,21] of the coefficients c_n.

This would produce smooth approximations, but at the expense of having a slower cutoff and poorer rejection in the stop-band. In any event, if we need a digital filter with a magnitude characteristic that is both close to ideal and smooth, we must use either polynomials of very high order or rational functions.

4. Comparison with z-Transforms of Analog Filters

We take up now another approximation method; one that at first appears natural, but is actually not very promising. Suppose that instead of taking the μ-transform of an analog filter, $A(s)$, we take the ordinary z-transform. In this case, the resulting digital filter is given by

$$\underline{\underline{A}}(e^{\underline{j\omega T}}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} A(j\omega + jn2\pi/T) \quad . \tag{II-19}$$

Typically, $A(s)$ would be designed so that it approximates the desired digital filter characteristic for $|\omega| < \pi/T$, and is small in magnitude outside this range. If $A(s)$ then has all its poles inside the left-half plane, $\underline{\underline{A}}(z)$ will be a stable digital filter with approximately the desired characteristic. The main difficulty with this method is the addition of unwanted terms in (II-19) due to aliasing of the filter function. To use this idea, we must start with analog filters which have carefully tailored characteristics with sharp cutoffs and good rejection in stop-bands, and all this leads to high order filters.

Furthermore, finding the z-transform of a high order filter involves a great deal more work than just letting $s = \frac{z-1}{z+1}$; and when we are done, we must recalculate the magnitude or phase characteristic of the result to assess the errors introduced by the aliasing of the original characteristic. All in all, the μ-transform is much better suited for the purpose of converting analog filters into useful digital filters.

To illustrate the above, consider again the third-order maximally flat Butterworth low-pass filter that we considered in section 2:

$$A(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \quad . \qquad \text{(II-19)}$$

The z-transform of this filter is

$$\underline{\underline{A}}(z) = \frac{0.3703z^{-1} + 0.1346z^{-2}}{1 - 0.3981z^{-1} + 0.2474z^{-2} - 0.04321} \quad . \qquad \text{(II-20)}$$

The normalized magnitude characteristic of this digital filter is plotted as curve A in Figure 4, which also shows the magnitude characteristic of the μ-transformed filter. Because of the relatively high cutoff frequency of A(s) (one-half the Nyquist frequency), and because of the low order of A(s), the effects of the aliasing of the filter characteristic are quite pronounced -- the cutoff is not sharp and the rejection is poor.

In summary, the mapping μ converts the approximation problem for digital filters to the approximation problem for analog filters. This latter problem has received a great deal of attention over the past fifty years, and we are fortunate to be able to use it to our purposes.

5. Building Analog Filters with Digital Computers

We conclude part II with a discussion of the possibility of constructing an analog filter from a sampler, a digital filter, and a data reconstruction device. Such a system would probably be implemented in real time using a digital computer. The advantages of using a digital computer as an analog filter are the flexibility, accuracy, and stability which can be readily obtained, and which are practically impossible to achieve with analog hardware. The coefficients in a digital computer program can be set to a high degree of accuracy, can be changed very fast, and are not subject to unwanted variation with temperature or age. Furthermore, with the use of pulse-code modulation for the low noise transmission of signals over large distances, the availability of signals already in digital form can make it more feasible to filter in real time with a digital computer. Ultimately, however, whether such a scheme is practical depends on the state of computer technology.

Suppose then that we sample an analog signal f(t), pass the resulting digital signal through a digital filter $\underline{\underline{A}}(z)$, and then reconstruct an analog signal with a data reconstruction circuit $H(s)$. The Laplace transform of the output signal is

$$G(s) = \underline{\underline{A}}(e^{sT})H(s)F^*(s) \quad , \tag{II-21}$$

where $F^*(s)$ is the Laplace transform of the sampled input. We can thus write a transfer function with respect to the sampled input:

$$\frac{G(s)}{F^*(s)} = \underline{\underline{A}}(e^{sT})H(s) \quad . \tag{II-22}$$

We assume now that we have sampled at a frequency at least twice as great as the bandwidth of $f(t)$. Then, in the range $|\omega| \leq \pi/T$, the transfer function (II-22) represents the effect of the system on the original signal, and outside this range represents spurious harmonics of the input signal caused by imperfect data reconstruction. These upper sidebands can be removed with a simple low-pass analog post-filter having a cutoff frequency near the Nyquist frequency.

As an example, suppose that $H(s)$ is a zero-order hold:

$$H(s) = \frac{1 - e^{-sT}}{sT} \quad , \tag{II-23}$$

$$|H(j\omega)| = \left| \frac{\sin \omega T/2}{\omega T/2} \right| \quad . \tag{II-24}$$

$|H(j\omega)|$ has its first zero at twice the Nyquist frequency, and has lobes of appreciable magnitude well-outside the range $|\omega| \leq \pi/T$. Hence, the overall transfer function (II-22) will have spurious responses at high frequencies unless these are filtered out. Suppose now, as an example, that we use the Tchebycheff filter (II-10) of section 2 as our digital filter $\underline{\underline{A}}(z)$. The normalized magnitude of the resulting digital filter-hold combination is shown in Figure 7. We note that the shape of the digital filter $\underline{\underline{A}}(z)$ is slightly distorted in the pass-band by being multiplied by $\left|\frac{\sin \omega T/2}{\omega T/2}\right|$. If necessary, the magnitude characteristic of A(z) can be compensated to correct for this distortion.

It is interesting to note that the filtering characteristic of our final system can be changed as fast as the coefficients in the digital computer program can be changed. If we used bandpass digital filters, for example, we might then be able to use the system to replace a bank of fixed filters or a frequency sweeping system.

PART III: APPLICATIONS TO POWER SPECTRUM MEASUREMENT

1. Introduction

The concept of power-spectral-density has become an important tool for the analysis and synthesis of many types of physical systems. As a result, there is a pressing need for ways to estimate the power-spectral-density of a signal from a finite record of that signal. Originally, analog methods provided the only practical way to do this. These methods usually involve the selection of a narrow band of frequencies with a bandpass analog filter, and then a measurement of the power density of the signal in this band. Too wide a pass-band results in an averaging of the spectral density over an excessively wide range of frequencies, with a resulting decrease in resolution; while too narrow a pass-band results in excessive statistical fluctuations of the estimates. In 1954, Chang[23] derived an expression for the optimum bandwidth for the spectrum analyzer and showed that the optimum shape for the spectrum analyzer was semicircular.

In recent years, when high speed digital computers became available, methods for spectrum analysis based on equally spaced samples of the signal of interest were developed. These methods were at first divorced from the concept of a bandpass filter, until the concept of a spectral window was introduced. Still, the connection between the analog and digital methods of spectrum estimation has remained obscure. One goal of this part will be the clarification of this connection. We begin with a review of the methods for spectrum analysis of equally spaced data, based mostly on the work of Blackman, Tukey, and Press.[24,25,26]

First of all, if a random signal is sampled, the sampled power spectrum of the resulting digital signal is related to the original spectrum by

$$\Phi_{xx}(e^{j\omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \Phi_{xx}(j\omega + nj\frac{2\pi}{T}), \qquad \text{(III-1)}$$

where T is the sampling interval. We see immediately that we must sample at a rate fast enough to reduce undesirable aliasing of the original spectrum. Otherwise, the spectrum we measure, $\Phi_{xx}(z)$ will not be an accurate reflection of the spectrum of the original signal.

Assuming that we have sampled fast enough, and have prefiltered the original analog signal to reduce high frequency noise if necessary, we can compute estimates of the autocorrelation function. We assume throughout this part that we have observed samples $x_1, x_2, \ldots x_N$ of the original signal $x(t)$, and that N is so large that we can neglect end effects. Thus we compute the (m+1) mean lagged products (III-2)

$$f_k = \frac{1}{N-|k|} \sum_{i=1}^{N-|k|} x_i x_{i+|k|} \quad , \quad -m \leq k \leq m \quad .$$

These f_k are unbiased estimates of the autocorrelation function $\phi_{xx}(k)$:

$$\lim_{N \to \infty} f_k = \phi_{xx}(k) \quad . \tag{III-3}$$

Since the power spectrum is given by (I-69):

$$\Phi_{xx}(z) = \lim_{N \to \infty} \sum_{k=-N}^{N} \phi_{xx}(k) z^{-k} \quad , \tag{III-4}$$

we are led to the estimate

$$\hat{\Phi}_{xx}(z_o) = \sum_{k=-m}^{m} f_k \, z_o^{-k} \quad , \tag{III-5}$$

where $z_o = e^{j\omega_o T}$, and ω_o is the frequency of interest. This estimate is known as the periodogram. These estimates are statistically unstable because they give equal weights to all the f_k, while the f_k for larger k are much less reliable. This suggests weighting the sum (II-5) in the following manner:

$$\hat{\Phi}_{xx} = \sum_{k=-m}^{m} w_k f_k z_o^{-k} \tag{III-6}$$

where

$$w_k = w_{-k} \quad .$$

The expected value of this estimate is

$$E\hat{\Phi}_{xx}(\omega_o) = \sum_{k=-m}^{m} w_k \, \phi_{xx}(k) z_o^{-k}$$

$$= \sum_{k=-m}^{m} w_k \left(\frac{1}{2\pi j} \oint_{|z|=1} \Phi_{xx}(z) z^k \frac{dz}{z} \right) z_o^{-k}$$

$$= \frac{1}{2\pi j} \oint_{|z|=1} \Phi_{xx}(z) \left(\sum_{k=-m}^{m} w_k z^k z_o^{-k} \right) \frac{dz}{z}$$

$$= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \Phi_{xx}(\omega) \, W(\omega - \omega_o) \, d\omega \quad , \qquad \text{(III-7)}$$

where

$$W(z) = \sum_{k=-m}^{m} w_k z^k \qquad \text{(III-8)}$$

is a weighting function which determines an estimate of $\Phi_{xx}(\omega)$ and is called the spectral window. By a convenient abuse of notation we write $\Phi_{xx}(\omega)$ instead of $\Phi_{xx}(e^{j\omega T})$. The problem of choosing a good spectral window has received much attention. A good evaluation of many spectral windows can be found in Grenander and Rosenblatt.[27]

2. A Class of Windows Generated by Digital Filters

If we now try to mimic the analog method for spectrum analysis, using digital filters instead of analog filters, we are led to a special class of estimates involving a special class of spectral windows. Suppose then that we design a bandpass digital filter that is tuned to the particular frequency ω_o, say $D(z)$. Let us pass the digital signal $x_1, \ldots, x_N$ through this filter to obtain an output sequence $y_1, \ldots, y_N$. The power density of this output signal is then the average energy:

$$\hat{\Phi}_{xx}(\omega_o) = \frac{1}{N} \sum_{i=1}^{N} y_i^2 \quad . \tag{III-9}$$

The expected value of this estimate is

$$\begin{aligned} E\hat{\Phi}_{xx}(\omega_o) &= E(y_i^2) = \phi_{yy}(0) \\ &= \frac{1}{2\pi j} \oint D(z)D(z^{-1}) \, \Phi_{xx}(z) \, \frac{dz}{z} \\ &= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} |D(\omega)|^2 \, \Phi_{xx}(\omega) \, d\omega \quad , \end{aligned} \tag{III-10}$$

so that these estimates correspond to the weighting function

$$W(\omega-\omega_o) = |D(\omega)|^2 \quad . \tag{III-11}$$

Thus, this special class of estimates has the desirable property of having a weighting function that is always positive. This means that no matter what the shape of the original spectrum, the estimates will always be positive, a situation that is not always true for the more general estimates using windows $W(\omega-\omega_o)$. We therefore have eliminated the problem of negative power leaking through a side lobe of the weighting function.

In general, the implementation of the estimate (III-9) will necessitate running the digital signal x_i through the digital filter $D(z)$ for each frequency of interest. This is a decided disadvantage, because of the long time that this would take on a computer. In the special case when $D(z)$ is a polynomial, however, we can compute the estimates directly from the f_k. To see this, write

$$\hat{\Phi}_{xx}(\omega_o) = \frac{1}{N} \sum_{i=1}^{N} y_i^2 = \frac{1}{2\pi j} \oint_{|z|=1} D(z)D(z^{-1}) \frac{X(z)X(z^{-1})}{N} \frac{dz}{z} , \tag{III-12}$$

where we define

$$X(z) = x_1 + x_2 z^{-1} + x_3 z^{-2} + \ldots + x_N z^{-N} . \tag{III-13}$$

Assuming that

$$D(z) = \sum_{k=o}^{K} d_k z^{-k} ,$$

(III-12) becomes

$$\hat{\Phi}_{xx}(\omega_o) = \sum_{k,\ell=o}^{K} d_k d_\ell f_{k-\ell} , \tag{III-14}$$

which is just as easy a quantity to calculate on a computer as (III-6). The coefficients d_k will, of course, be different for each frequency of interest.

Whether we use this last method and restrict D(z) to be a polynomial or we use a rational function of z and run the signal through the filter for each measurement, we can now use the approximation methods discussed in part 2 to design spectral windows. It is very easy to use different windows for different parts of the spectrum, for we have complete control over the shape of the window at all times. We have thus seen how the measurement of power-spectral-density for discrete signals can be thought of in terms of filtering and energy measurements, just as in the analog case.

3. The Mean-Square-Error of These Estimates

With a view towards deriving the optimum digital filter for these estimates in a manner similar to Chang's,[23] we will now calculate the mean-square-error:

$$\epsilon^2 = E[\Phi_{xx}(\omega_o) - \hat{\Phi}_{xx}(\omega_o)]^2 \quad . \qquad \text{(III-15)}$$

This mean-square-error can be broken up into two parts; first the square of the bias:

$$(\text{bias})^2 = [\Phi_{xx}(\omega_o) - E\hat{\Phi}_{xx}(\omega_o)]^2 \quad , \qquad \text{(III-16)}$$

and second the variance

$$\text{variance} = E[\hat{\Phi}(\omega_o)]^2 - [E\hat{\Phi}_{xx}(\omega_o)]^2 \quad , \tag{III-17}$$

Thus

$$\epsilon^2 = (\text{bias})^2 + \text{variance} \quad . \tag{III-18}$$

From (III-10), we have

$$\text{bias} = \Phi_{xx}(\omega_o) - \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} |D(\omega)|^2 \, \Phi_{xx}(\omega) \, d\omega \quad . \tag{III-19}$$

We see from this that the bias error is due entirely to the fact that the weighting function is not a δ-function. For this reason we may call it the "blurring" error, after Chang.[23]

The variance is somewhat more difficult to calculate. From (II-9) we have first

$$E[\hat{\Phi}(\omega_o)]^2 = \frac{1}{N^2} \sum_{n,m=1}^{N} E[y_n^2 y_m^2] \quad . \tag{III-20}$$

In order to evaluate these fourth-order moments, we now assume that the original signal has a normal probability distribution function. With this assumption, the digital signal y_i is also normally distributed, and using the characteristic function for the y_i, we get

$$E[y_n^2 y_m^2] = \phi_{yy}^2(0) + 2\,\phi_{yy}^2(m-n) \quad . \tag{III-21}$$

Thus,

$$E[\hat{\Phi}(\omega_o)]^2 = \phi_{yy}^2(0) + \frac{2}{N^2} \sum_{n,m=1}^{N} \phi_{yy}^2(m-n) \quad . \tag{III-22}$$

Also, we have

$$[E\hat{\Phi}(\omega_o)]^2 = \phi_{yy}^2(0) \quad , \tag{III-23}$$

so that

$$\text{variance} = \frac{2}{N^2} \sum_{n,m=1}^{N} \phi_{yy}^2(m-n) \quad . \tag{III-24}$$

This can be put in terms of the power-spectral-density of the y_i by using formula (I-70):

$$\phi_{yy}(n) = \frac{1}{2\pi j} \oint_{|z|=1} \Phi_{yy}(z) z^n \frac{dz}{z} = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \Phi_{yy}(\omega)\, e^{jn\omega T}\, d\omega \quad . \tag{III-25}$$

Hence, we can write

$$\text{variance} = \frac{2}{N^2}(T/2\pi)^2 \int_{-\pi/T}^{\pi/T} \int_{-\pi/T}^{\pi/T} \left| \sum_{\nu=1}^{N} e^{-j\nu(\omega_1-\omega_2)T} \right|^2 \Phi_{yy}(\omega_1)\Phi_{yy}(\omega_2)\, d\omega_1 d\omega_2 \quad . \tag{III-26}$$

Using the identity

$$\left| \sum_{\nu=1}^{N} e^{-j\nu(\omega_1-\omega_2)T} \right|^2 = \frac{\sin^2 \frac{N}{2}(\omega_1-\omega_2)T}{\sin^2 \frac{1}{2}(\omega_1-\omega_2)T} \quad , \tag{III-27}$$

this becomes (III-28)

$$\text{variance} = \frac{2}{N^2}(T/2\pi)^2 \int_{-\pi/T}^{\pi/T} \int_{-\pi/T}^{\pi/T} \frac{\sin^2 \frac{N}{2}(\omega_1-\omega_2)T}{\sin^2 \frac{1}{2}(\omega_1-\omega_2)T} \Phi_{yy}(\omega_1)\Phi_{yy}(\omega_2)\, d\omega_1 d\omega_2 \quad .$$

The inner integral in this expression is known as Fejer's integral, and is discussed in Carslaw,[21] and in Titchmarsh.[10] The essential result is that

$$\lim_{N\to\infty} \frac{\sin^2 \frac{1}{2}N\omega T}{\sin^2 \frac{1}{2}\omega T} = \frac{2\pi}{T} N\, \delta(\omega) \quad ; \qquad \text{(III-29)}$$

That is, the Fejer kernel tends to a δ-function with increasing N. Since in our case N is very large (usually larger than 50 or 100 for a meaningful spectral analysis), we can write as a good approximation

$$\text{variance} = \frac{2}{N}\frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \Phi_{yy}^2(\omega)\, d\omega \quad .$$

$$= \frac{2}{N}\frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} |D(\omega)|^4\, \Phi_{xx}^2(\omega)\, d\omega \quad . \qquad \text{(III-30)}$$

Thus the variance is inversely proportional to N, which is in agreement with Grenander and Rosenblatt,[27] who used a different derivation that applies to spectral windows that are not necessarily generated by digital filters. Furthermore, if we normalize by the square of the area under the window:

$$\left[\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}|D(\omega)|^2\,d\omega\right]^2 \quad , \tag{III-31}$$

the variance is inversely proportional to the length of the record NT, which agrees with the analog case. We thus have derived an expression for the mean-square-error of the digital filter estimates:

$$\epsilon^2 = \left[\Phi_{xx}(\omega_o) - \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}|D(\omega)|^2\,\Phi_{xx}(\omega)\,d\omega\right]^2$$
$$+ \frac{2}{N}\,\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}|D(\omega)|^4\,\Phi^2_{xx}(\omega)\,d\omega \quad . \tag{III-32}$$

4. The Optimum Digital Filter

Our program now is to find the digital filter shape that minimizes this mean-square-error, thus following the derivation that Chang[23] presented for analog filters. Accordingly, we represent the digital filter characteristic $|D(\omega)|^2$ by $U^2(\Omega/\Delta)$, where $\Omega = \omega-\omega_o$, ω_o is the center frequency to which $D(\omega)$ is tuned, and Δ is some kind of bandwidth such that $U^2(\Omega/\Delta)$ is small for $|\Omega| > \Delta$. We thus focus our attention on only one main lobe of the digital filter characteristic, at $\omega = \omega_o$. We assume also that the filter $D(z)$ is sharply tuned to ω_o, so that we can write to a good approximation

$$\text{variance} = \frac{2}{N}\,\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} |D(\omega)|^4\,\Phi_{xx}^2(\omega)\,d\omega$$

$$= \frac{2}{N}\,\frac{T}{2\pi}\,\Phi_{xx}^2(\omega_o)\int_{-\pi/T}^{\pi/T} |D(\omega)|^4\,d\omega\ . \tag{III-33}$$

This becomes in our new notation:

$$\text{variance} = \frac{4}{N}\,\frac{T}{2\pi}\,\Phi_{xx}^2(\omega_o)\int_{-\infty}^{\infty} U^4(\Omega/\Delta)\,d\Omega \tag{III-34}$$

To express the bias simply, we expand the spectral density $\Phi_{xx}(\omega)$ in a power series about ω_o:

$$\Phi_{xx}(\omega) = \Phi_{xx}(\omega_o) + \Phi'_{xx}(\omega_o)\,(\omega-\omega_o)$$
$$+ \frac{1}{2}\,\Phi''_{xx}(\omega_o)(\omega-\omega_o)^2 + \ldots\ . \tag{III-35}$$

Assuming that the area under the filter characteristic is one, or, equivalently, that our estimates are adjusted by dividing by the area under the filter characteristic, the bias term (III-19) becomes

$$\text{bias} = \frac{1}{2}\,\frac{T}{2\pi}\,\Phi''_{xx}(\omega_o)\int_{-\pi/T}^{\pi/T} |D(\omega)|^2\,(\omega-\omega_o)^2\,d\omega$$

$$= \frac{T}{2\pi}\,\Phi''_{xx}(\omega_o)\int_{-\infty}^{\infty} U^2(\Omega/\Delta)\,\Omega^2\,d\Omega\quad , \tag{III-36}$$

where we have also assumed that $U^2(\Omega/\Delta)$ is an even function; that our filter has a symmetrical magnitude characteristic about the resonance frequency.

We seek to minimize the normalized mean-square-error, given by

$$\epsilon^2_{norm} = \frac{\epsilon^2}{\left[\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} |D(\omega)|^2 \, d\omega\right]^2} = \frac{\epsilon^2}{(T/2\pi)^2\left[2\int_{-\infty}^{\infty} U^2(\Omega/\Delta)\, d\Omega\right]^2} \quad . \tag{III-37}$$

For convenience of notation, we now define the following integrals:

$$I = \int_{-\infty}^{\infty} U^2(\Omega/\Delta)\, d(\Omega/\Delta) \quad , \tag{III-38}$$

$$J = \int_{-\infty}^{\infty} U^2(\Omega/\Delta)(\Omega/\Delta)^2\, d(\Omega/\Delta) \quad , \tag{III-39}$$

$$K = \int_{-\infty}^{\infty} U^4(\Omega/\Delta)\, d(\Omega/\Delta) \quad . \tag{III-40}$$

With these notations, the normalized mean-square-error becomes:

$$\epsilon^2_{norm} = \frac{1}{4}[\Phi''_{xx}(\omega_o)]^2 \frac{J^2}{I^2}\Delta^4 + \frac{2\pi}{NT}\Phi^2_{xx}(\omega_o)\frac{K}{I^2}\frac{1}{\Delta} \, , \tag{III-41}$$

which is exactly the same as Equation (23) in Chang's paper,[23] which was derived for an analog filter instead of a digital filter. Hence, the rest of the derivation is identical to that for the analog case, and we are done.

Thus, the optimum bandwidth, obtained by differentiating (III-41) with respect to Δ and setting the result equal to zero, is

$$\Delta^5 = \left(\frac{2\pi}{NT}\right) \frac{\Phi^2_{xx}(\omega_o)}{[\Phi''_{xx}(\omega_o)]^2} \frac{K}{J^2} \quad . \qquad \text{(III-42)}$$

We have therefore shown that the optimum bandwidth is inversely proportional to the length of the record available for spectral analysis. This optimum bandwidth was derived by Grenander and Rosenblatt for two specific windows; we have here shown that it holds in general for windows generated by digital filters.

The normalized mean-square-error in the case that the bandwidth is chosen optimally is

$$\epsilon^2_{norm} = \frac{5}{4} (2\pi/NT)^{.8} [\Phi_{xx}(\omega_o)]^{1.6} [\Phi''_{xx}(\omega_o)]^{.4} \frac{K^{.8}J^{.4}}{I^2} \quad . \qquad \text{(III-43)}$$

Since

$$\sqrt{\epsilon^2_{norm}} = \frac{\sqrt{5}}{2} \frac{K^{.4}J^{.2}}{I} [\Phi_{xx}(\omega_o)]^{.8} \left[\Phi''_{xx}(\omega_o) \left(\frac{2\pi}{NT}\right)^2\right]^{.2} \qquad \text{(III-44)}$$

we can define the error coefficient

$$K_\epsilon = \frac{\sqrt{5}}{2} \frac{K^{.4}J^{.2}}{I} \quad , \qquad \text{(III-45)}$$

which is a quantitative measure of how small an expected error can be achieved with filter characteristics of different shapes. As Chang shows, the optimum shape for the function U is given by

$$\begin{aligned} U(\Omega/\Delta) &= 0 \quad , \qquad \text{for } |\Omega| \geq \Delta \\ U^2(\Omega/\Delta) &= A\{1 - (\Omega/\Delta)^2\} \quad , \qquad \text{for } |\Omega| \leq \Delta . \end{aligned} \tag{III-46}$$

This being obtained by setting the first order variation of K_ϵ with respect to U equal to zero. This semicircular filter shape gives $K_\epsilon = 0.66$. The shape of the filter is actually not too critical, providing that the bandwidth has been chosen well, and providing that the side lobes of the digital filter characteristic in the region $|\Omega| \geq \Delta$ are small. Thus, the ideal rectangular filter shape

$$\begin{aligned} U(\Omega/\Delta) &= 0 \quad , \text{ for } |\Omega| \geq \Delta \\ U(\Omega/\Delta) &= 1 \quad , \text{ for } |\Omega| \leq \Delta \end{aligned} \tag{III-47}$$

yields $K_\epsilon = 0.68$, which is close to optimum. Thus, the bandpass transformations of a low-pass Butterworth or Tchebycheff filter would serve well as spectral windows, although the Fourier series filters would be easier to implement using (III-14).

5. Prewhitening Techniques

We thus see how the approximation techniques described in part 2 can be applied to the design of spectral windows. These approximation techniques are also especially useful in prewhitening spectra before the above estimation methods are applied. The idea of prewhitening has been strongly advocated by Blackman and Tukey[26] for a few reasons, one of which can be seen by examining the expression for the mean-square-error (III-41). This error depends directly on the second derivative of the spectrum at the measurement point, which appears in the bias term. If we could somehow flatten the spectrum before measurement and then compensate for this after the estimates have been computed, we would reduce the bias term without affecting the variance. Another advantage of measuring an essentially flat spectrum is that there is then little possibility of an unreasonable contribution from a peak in the spectrum that happens to correspond to a minor lobe in the spectral window.

Therefore, if we have a rough idea of the shape of the spectrum we are measuring, we can approximate this shape with a digital filter $D(z)$, so that

$$|D(\omega)|^2 \cong \Phi_{xx}(\omega) \quad . \tag{III-48}$$

We can then pass the original signal x_i through a digital filter $1/D(z)$, producing a signal with a relatively flat spectrum. Estimates of this power-spectral-density are then computed in the usual way, and then corrected by multiplying by $|D(\omega)|^2$. The techniques described in part 2 are well-suited to accomplish this prewhitening in an organized way.

6. The Identification of Power Spectrum Parameters*

Suppose now that a system designer needs to know the power-spectral-density of some signal. Assuming that he has an idea of the bandwidth of the signal, he can obtain samples of it, calculate the mean lagged products f_k, and then use some spectral window to estimate the spectral-density. What he gets after this procedure are estimates at points along the frequency axis, usually equally spaced. If the results of this spectral analysis are going to be used for anything besides a visual presentation, the designer will have to put this in some closed analytical form. One way to do this is suggested by the mapping μ. The points of the power spectrum can be transformed by the mapping μ as in Equations (I-72) and (I-73). The measurement points now represent the spectrum of an alalog signal, and this can be put in the form of a rational function of s by using Bode's method of semi-infinite slopes. The reverse mapping μ^{-1} will then yield a rational function of z, which is a form which can be used for explicit design. This procedure leaves much to be desired. First of all, it involves two consecutive approximations and the accuracy of the final result is difficult to gauge. Second, it is not easily

* The results in the remainder were reported by the author in reference 28.

mechanized on a computer, and is hence ill-suited for real time application as an identification method for adaptive systems.

It would therefore be desirable to have a method of measuring power-spectral-density that yields an analytical form for the answer. The technique of prewhitening suggests the following method of accomplishing this: Suppose we pass the signal of interest x_i through a digital filter $D(z)$ of some canonic form, and then adjust the coefficients of $D(z)$ so that the output is in some sense most nearly white noise. Then we have

$$|D(\omega)|^2 \; \Phi_{xx}(\omega) \cong 1 \qquad , \qquad \text{(III-49)}$$

so that

$$\Phi_{xx}(z) \cong \frac{1}{D(z)D(z^{-1})} \qquad \text{(III-50)}$$

can be used as an analytical expression for the unknown power-spectral-density.

If this program is to be carried out, the following consideration is important: The most time consuming, and hence expensive, step in a spectral analysis is always the computation of the mean lagged products f_k. Hence, we would like to calculate only one set of these for each spectral analysis. If we assume that $D(z)$ is a polynomial in z^{-1}, the output mean lagged products can be expressed in terms of the input mean lagged products rather easily. On the other hand, if $D(z)$ has even one pole, it becomes intractable to express the output mean lagged products in terms of those of the input, and the mean lagged products of the output must be recalculated for each choice of coefficients in $D(z)$; and this becomes impractical. Hence, the procedure outlined is only practical when $D(z)$ is a polynomial in z^{-1}.

At first, the following method was tried on a computer. $D(z)$ was assumed to have the form

$$D(z) = 1 + az^{-1} + bz^{-2} \quad , \tag{III-51}$$

the input mean lagged products were computed, and the appropriate mean lagged products of the output of $D(z)$ were computed from these. The criterion for whiteness was that the autocovariance determinant of the output signal be maximum. By the method of steepest descent, the coefficients a and b were found. The method converged nicely, but gave good results only when the unknown power spectrum was of the appropriate form:

$$\Phi_{xx}(z) = \frac{\beta^2}{D(z)D(z^{-1})} \quad . \tag{III-52}$$

Furthermore, the extremal seeking procedure becomes less reliable when more unknown coefficients are introduced.

It was then found that the above problem is equivalent to a well-known problem in mathematical statistics: That of estimating the coefficients of an autoregressive scheme. The solution of this latter problem can be found in the literature; a good discussion is given by Hannan,[29] for example. Thus, when D(z) is assumed to have the form

$$D(z) = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots + \alpha_p z^{-p} \quad , \qquad \text{(III-53)}$$

a completely analytical expression for the coefficients α_i can be derived in terms of the mean lagged products of the input signal. The method cannot be extended to the case where D(z) has poles, for essentially the same reason described above. Thus, it is the responsibility of the experimenter to ensure that the unknown spectrum can in fact, be represented closely by the form (III-52). Some ways of getting around this problem will be discussed later. We now present the solution to the identification problem described above when D(z) is a polynomial in z^{-1}.

7. Statement of the Problem

We make the following assumptions:

1. N points of the signal of interest are available:

   $$x_1, x_2, \ldots, x_n \quad ,$$

   and N is large enough so that end effects can be neglected.
2. The signal is normally distributed with zero mean, and is stationary and ergodic.
3. The signal has a power-spectral-density which can be closely represented by

$$\Phi_{xx}(z) = \frac{\beta^2}{D(z)D(z^{-1})} \quad , \qquad \text{(III-54)}$$

where

$$D(z) = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots + \alpha_p z^{-p} \qquad \text{(III-55)}$$

has all of its zeros inside the unit circle in the z-plane.

The problem is to estimate the parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$, and β^2; given the N observed points of the signal.

8. The Most Likely Estimates

The solution given here will be essentially the same as that given by Hannan,[29] except that our argument will be in terms of power spectra.

Define a new signal y_i by passing x_i through the digital filter $D(z)$. That is, put

$$y_i = x_i + \alpha_1 x_{i-1} + \alpha_2 x_{i-2} + \ldots + \alpha_p x_{i-p} \quad , \qquad \text{(III-56)}$$

or, in z-transform notation

$$Y(z) = D(z)\ X(z) \quad .$$

The stochastic variable y_i is normally distributed. Furthermore, its power-spectral-density is

$$\Phi_{yy}(z) = D(z)D(z^{-1})\ \Phi_{xx}(z) = \beta^2 \quad , \qquad \text{(III-57)}$$

so that the signal y_i is gaussian distributed white noise with mean square value β^2. The joint probability density function of the observed sample $(y_1, y_2, \ldots, y_N)$ is then

$$p(y_1, y_2, \ldots, y_N) = \frac{1}{(2\pi)^{N/2}\beta^N} \exp\left(-\frac{1}{2\beta^2} \sum_{i=1}^{N} y_i^2\right) \quad . \tag{III-58}$$

The maximum likelihood estimates of the unknown parameters; denoted by $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_p$, and $\hat{\beta}^2$; are obtained by maximizing this probability. Thus the following set of equations must be solved:

$$\frac{\partial \log p(y_1, y_2, \ldots, y_N)}{\partial \alpha_j} = 0, \; j = 1, 2, \ldots, p; \tag{III-59}$$

and

$$\frac{\partial \log p(y_1, y_2, \ldots, y_N)}{\partial \beta^2} = 0 \quad . \tag{III-60}$$

When (III-56) is substituted in (III-58) and the indicated operations are carried out, the most likely estimates result:

$$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3 \\ \cdot \\ \cdot \\ \cdot \\ \hat{\alpha}_p \end{bmatrix} = - \begin{bmatrix} f_o & f_1 & f_2 & \cdots & f_{p-1} \\ f_1 & f_o & f_1 & \cdots & f_{p-2} \\ f_2 & f_1 & f_o & \cdots & f_{p-3} \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ f_{p-1} & \cdots & & & f_o \end{bmatrix}^{-1} \times \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ \cdot \\ \cdot \\ f_p \end{bmatrix} , \tag{III-61}$$

and

$$\hat{\beta}^2 = \sum_{i,j=o}^{p} \hat{\alpha}_i \hat{\alpha}_j f_{|i-j|} = f_o + \hat{\alpha}_1 f_1 + \hat{\alpha}_2 f_2 + \hat{\alpha}_3 f_3 + \ldots + \hat{\alpha}_p f_p \quad , \tag{III-62}$$

where the f_j are the mean lagged products

$$f_j = \frac{1}{N-j} \sum_{i=1}^{N-j} x_i x_{i+j} \qquad (j \geq 0) \qquad . \tag{III-63}$$

In summary, then, the following computations are performed:

1. From the N sample points of the signal, the mean lagged products $f_o, f_1, \ldots, f_p$ are calculated in accordance with (III-63).
2. The pxp matrix $(f_{|i-j|}; \; i,j = 1,\ldots,p)$ is formed and inverted.
3. $\hat{\alpha}_j$ $(j = 1,\ldots,p)$ are calculated from (III-61).
4. $\hat{\beta}^2$ is calculated from (III-62).

These computational steps are shown diagramatically in Figure 8.

9. Variability of the Estimates

If this identification method is to be used for any practical purpose, some knowledge is required about the accuracy of the estimates for a given N. It can be shown[29] that the vector $\underline{\hat{\alpha}} - \underline{\alpha}$ defined by

$$\underline{\hat{\alpha}} - \underline{\alpha} = \begin{bmatrix} \hat{\alpha}_1 - \alpha_1 \\ \hat{\alpha}_2 - \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{\alpha}_p - \alpha_p \end{bmatrix} \tag{III-64}$$

is asymptotically normally distributed with zero mean and covariance matrix

$$\frac{\beta^2}{N-p}\begin{bmatrix} \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_{p-1} \\ \phi_1 & \phi_0 & \phi_1 & \cdots & \phi_{p-2} \\ \phi_2 & \phi_1 & \phi_0 & \cdots & \phi_{p-3} \\ \vdots & & & \ddots & \vdots \\ \phi_{p-1} & \cdots & & & \phi_0 \end{bmatrix}^{-1} \qquad \text{(III-65)}$$

This can be estimated conveniently by

$$\frac{\hat{\beta}^2}{N-p}\begin{bmatrix} f_0 & f_1 & f_2 & \cdots & f_{p-1} \\ f_1 & f_0 & f_1 & \cdots & f_{p-2} \\ f_2 & f_1 & f_0 & \cdots & f_{p-3} \\ \vdots & & & \ddots & \vdots \\ f_{p-1} & \cdots & & & f_0 \end{bmatrix}^{-1}, \qquad \text{(III-66)}$$

which does not use any quantities which have not already been calculated.

The distribution of the estimate $\hat{\beta}^2$ is difficult to calculate since it is a more complicated function of the f_j's. It is easy, however, to derive the distribution of

$$\frac{1}{N}\sum_{i=1}^{N} y_i^2 = \sum_{i,j=0}^{p} \alpha_i\alpha_j \, f_{|i-j|} \quad , \qquad \text{(III-67)}$$

and this will give some (optimistic) indication of the variability of $\hat{\beta}^2$. With this in mind consider the random variable

$$\sum_{i=1}^{N} (y_i/\beta)^2 \quad . \qquad \text{(III-68)}$$

This is the sum of squares of independent, normally distributed random variables whose means are zero and whose variances are 1. This random variable is therefore χ^2-distributed with N degrees of freedom. Cramér[30] shows that with increasing N the χ^2 distribution becomes asymptotically normal with mean N and standard deviation $\sqrt{2N}$. Therefore, the random variable (III-67) is asymptotically normally distributed with mean β^2 and standard deviation $\sqrt{2/N}\,\beta^2$; and hence $\sqrt{2/N}\,\hat{\beta}^2$ can be used as a low estimate of the standard deviation of $\hat{\beta}^2$.

10. Extension to Spectra with Zeros

As mentioned before, the assumption that the unknown spectrum does not have any zeros is rather restrictive, and the derivation breaks down when a more general form for $\Phi_{xx}(z)$ is assumed. There are some situations when something can be done to extend the method, and these will now be discussed.

Suppose that the signal of interest has an unknown power spectrum of the form

$$\beta^2 \, \frac{N(z)N(z^{-1})}{D(z)D(z^{-1})} \qquad \text{(III-69)}$$

and that the locations of the zeros are known (at least approximately). Then the signal can be prefiltered by a digital filter $1/N(z)$ (or an equivalent analogue filter). The resultant signal will then be of the requisite form and the method described in this paper can be used to determine the pole locations and β^2.

As another example, suppose that the signal of interest, x_n, is the sum of two independent signals; one of which has a known power spectrum (such as white noise of a given amplitude), and the other of which has only poles in its power spectrum. That is, suppose

$$\Phi_{xx}(z) = \Phi^*_{xx}(z) + \frac{\beta^2}{D(z)D(z^{-1})} \quad . \tag{III-70}$$

The autocorrelation function of the signal is, then, the sum of known and unknown components:

$$\phi_{xx}(n) = \phi^*_{xx}(n) + \phi^{\dagger}_{xx}(n) \quad . \tag{III-71}$$

The known components can be subtracted from the computed f_n and the resulting mean lagged products

$$f_o - \phi^*_{xx}(0),\ f_1 - \phi^*_{xx}(1),\ \dots,\ f_p - \phi^*_{xx}(p) \tag{III-72}$$

can then be used to estimate $D(z)$ and β^2.

Other situations suggest themselves. Some pole locations may be known in advance, for instance. These poles can be removed before analysis by a digital or analog filter. Alternatively, the maximum likelihood equations (III-59) and (III-60) can be reworked.

11. An Example

To demonstrate the method, a sequence of 210 independent normal random numbers was passed through the digital filter $1/(1-.5z^{-1})$. The resultant time series then had a power spectrum

$$\frac{1/252}{(1-.5z^{-1})(1-.5z)} \quad . \tag{III-73}$$

Thus for this signal, assuming $p = 2$,

$$\begin{aligned} \alpha_1 &= -.5 \\ \alpha_2 &= 0.0 \\ \beta^2 &= .00397 \quad . \end{aligned} \tag{III-74}$$

Three mean lagged products were computed:

$$\begin{aligned} f_0 &= .00624, \\ f_1 &= .00307, \\ f_2 &= .00147, \end{aligned} \tag{III-75}$$

and (III-61) used to give the estimates

$$\begin{aligned} \hat{\alpha}_1 &= -.495, \\ \hat{\alpha}_2 &= .0070, \\ \hat{\beta}^2 &= .00473 \quad . \end{aligned} \tag{III-76}$$

The estimated covariance matrix of the $\hat{\alpha}_j$ was calculated from (III-66):

$$\begin{bmatrix} .0048 & -.0024 \\ -.0024 & .0048 \end{bmatrix} \quad , \tag{III-77}$$

and it is seen that the $\hat{\alpha}_j$ are well within one standard deviation of the α_j. The optimistically estimated standard deviation of $\hat{\beta}^2$ is

$$\sqrt{2/N}\,\hat{\beta}^2 \cong (1/10)\hat{\beta}^2 \quad , \tag{III-78}$$

so that the 16 percent actual deviation is not unreasonable.

Figure 9 shows plots of the actual and the estimated power spectrum. Also shown are the results of a conventional spectrum analysis[26] using a Hamming window and 7 mean lagged products. Note that more than twice as many multiplications were required by the conventional method to produce similar accuracy, and that the results are not in a form that is suited for direct use.

12. Applications of the Identification Method

The above identification method is especially promising for use in an adaptive loop; first because it can be implemented in real time by a computer, and second because it gives direct estimates of parameters that can characterize a signal or a plant. Thus, the following method of self-optimizing control is suggested: A controller is designed whose optimum or near-optimum operation depends on the knowledge of the parameters $\alpha_1,\ldots,\alpha_p$ and β^2 of the power spectrum of some signal in the system. From a record of this signal of length N the estimates $\hat{\alpha}_1,\ldots,\hat{\alpha}_p$ and $\hat{\beta}^2$ are periodically calculated by a digital computer and used to adjust the controller. In a particular application, the choice of N is an important problem. N must be chosen large enough so that the estimates of the power spectrum parameters are accurate enough to be useful. On the other hand, N should not be so large that the system reacts to obsolete information.

The identification method described above may also be used as a first step in a conventional spectral analysis. After $D(z)$ is estimated, the original signal can be passed through the filter $D(z)$ and subjected to further spectral analysis by conventional methods. If the form assumed for the spectrum was appropriate the output will be nearly white, and this procedure will amount to an "automatic" prewhitening technique which can be used in conjunction with conventional spectral analysis.

Finally, it might be mentioned that the identification method can be used with the adaptive information processing method described by Chang.[31]

We have seen in this part how the concept of digital filtering can be applied to the problem of measuring the power-spectral-density of a digital signal. We first showed how the idea of bandpass filtering can be carried over from the analog case to the digital case to generate spectral windows that always give positive estimates of the spectrum. Furthermore, we have pointed out along the way how digital filters can be used to advantage as prefilters and postfilters much as analog filters are used for continuous signals. For these applications, the approximation techniques described in part 2 are especially useful. Lastly, we described a method of identifying unknown parameters in a power spectrum of an assumed form; a method which is promising as an identification program that can be incorporated into an adaptive loop.

SUMMARY

Our main goal has been to tie together the theories of filtering digital signals and analog signals. With the axiomatization of filtering and signal theory in terms of Hilbert space, we saw how an isomorphism could be constructed between the analog and digital signal spaces which allowed us to transfer many concepts from one domain to the other. The use of Hilbert space showed how the z-transform can be defined with much the same generality as the Fourier transform, and led to a definition of stable filters that can be used in both the analog and the digital cases. We then saw how any such filter, whether time-varying or not, could be represented by an infinite matrix of numbers. In particular, we saw that in the time-invariant case the digital and analog theories are essentially identical. Thus, many common optimum-filtering problems can be solved simultaneously for analog and digital signals, both in the deterministic and the random case. We also looked at data reduction filters and their interpretation in terms of frequency response.

In part 2 we showed that the approximation problem for time-invariant digital and analog filters were equivalent, and we discussed some methods that were particularly applicable to the design of digital filters for some common purposes, such as prefiltering prior to data reduction. We showed in particular how Fourier series can be used to design digital filters with prescribed magnitude characteristics that were polynomials in z^{-1}, and hence could be implemented economically.

Part 3 was devoted to the application of these ideas to the problem of measuring power-spectral-density from digital information. We saw in particular how bandpass digital filters could be used as spectral windows which always give positive estimates of the power-spectral-density. We then derived the optimum bandwidth and the optimum shape for such digital filters, following the results of Chang[23] for analog filters. Throughout this discussion we indicated how the approximation techniques of part 2 could be used effectively in the processing of digital information; prewhitening being an example. We then presented a method of identifying unknown parameters in a power spectrum. This method results in an analytical form for the spectrum, and is suitable for a systematic prewhitening program, or for use in an adaptive control loop.

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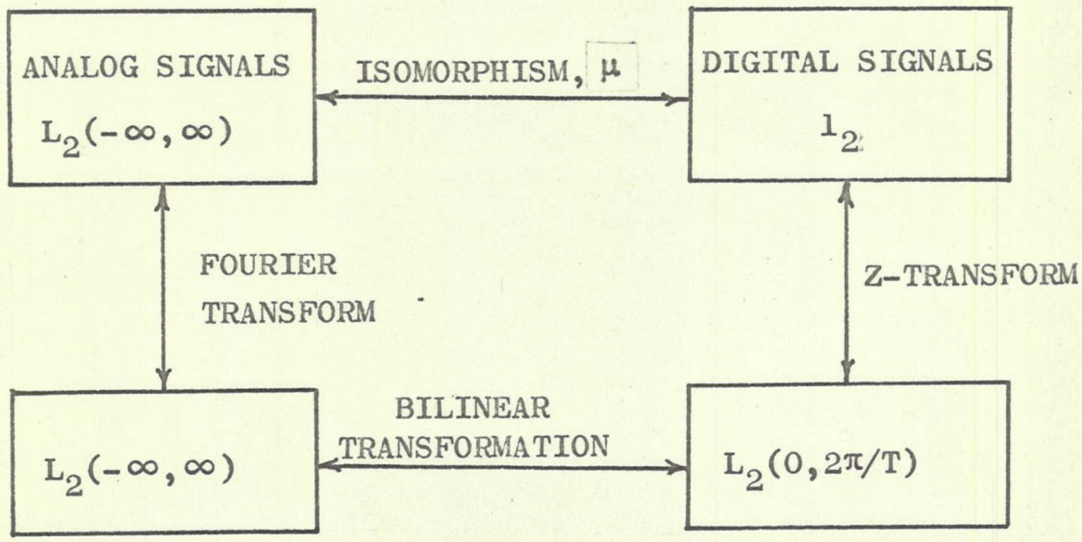

Fig. 1. A schematic representation of the mapping μ and its relations to the various signal spaces.

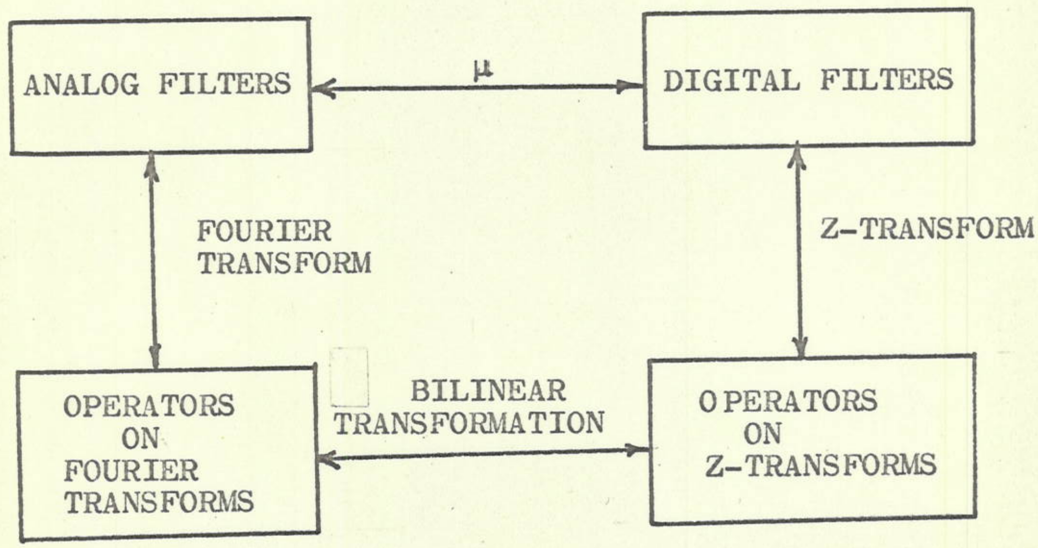

Fig. 2. A schematic representation of the mapping μ and its relations to the various spaces of operators.

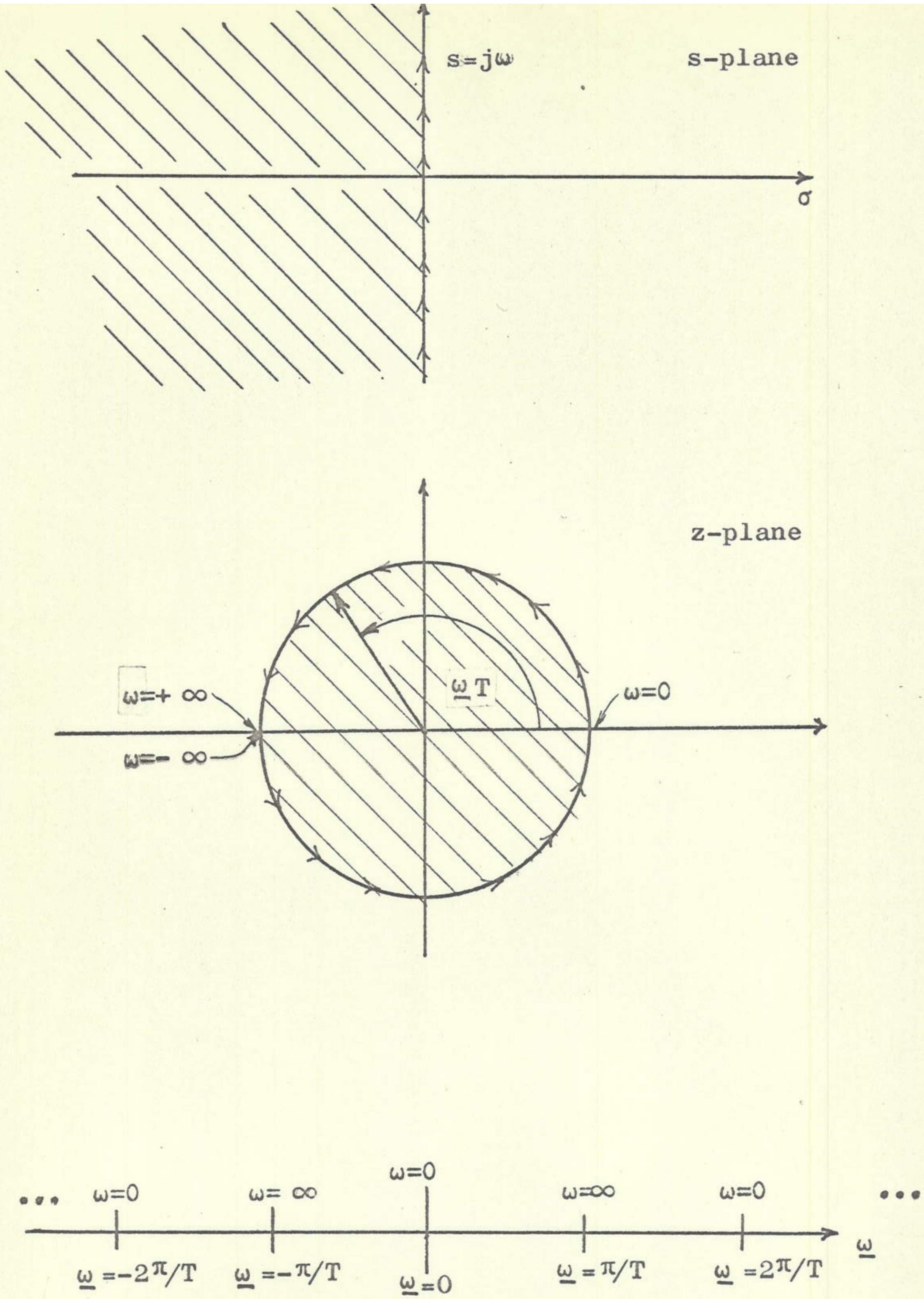

Fig. 3. The s-plane, the $z=e^{sT}$ plane, and the $\underline{\omega}$-axis; when $z = \frac{1+s}{1-s}$.

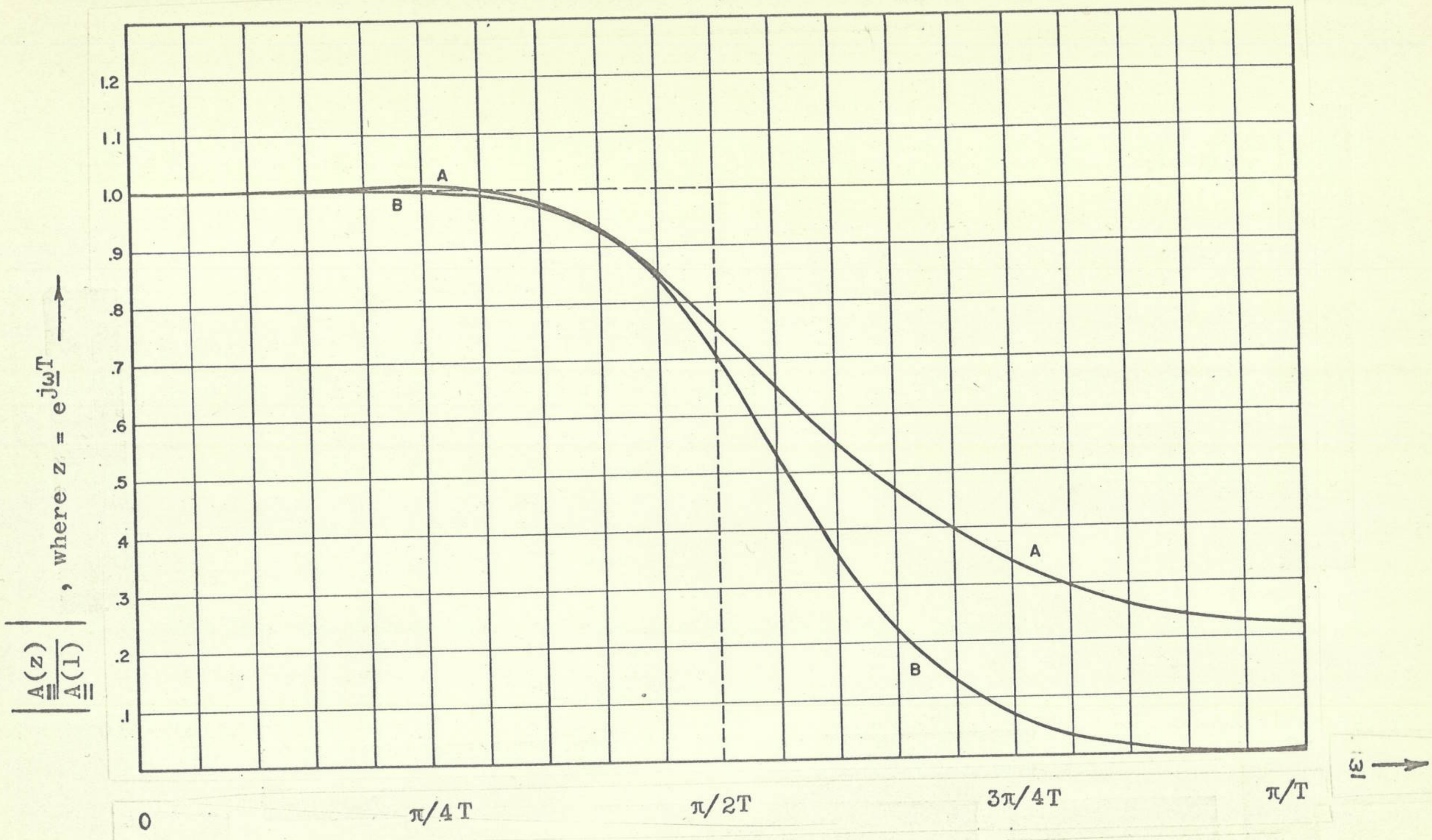

Fig. 4. Curve A is the normalized magnitude characteristic of the digital filter corresponding to the z-transform of a third-order Butterworth filter. Curve B is the normalized magnitude characteristic of the digital filter corresponding to the μ-transform of the same analog filter.

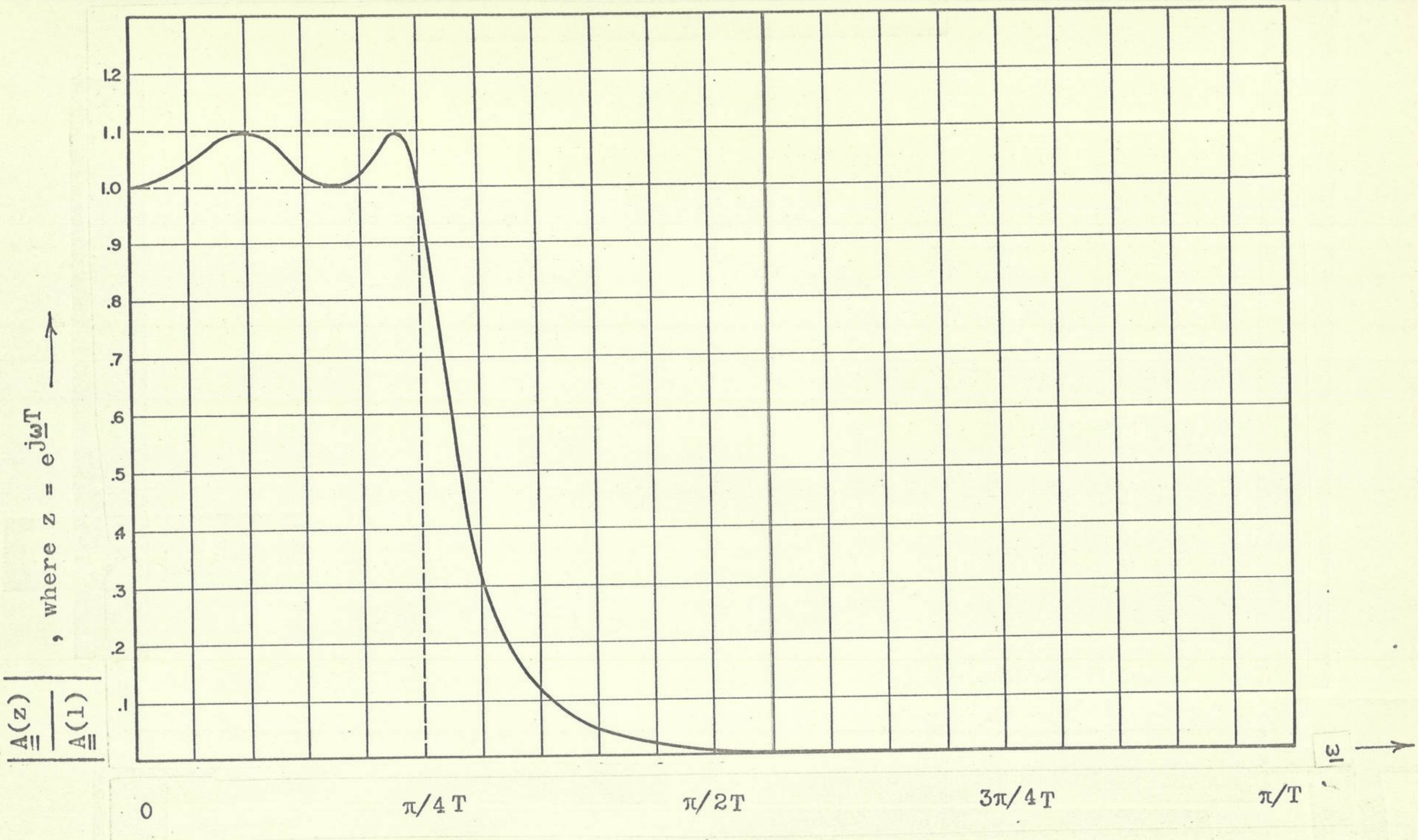

Fig. 5. The normalized magnitude characteristic of the digital filter corresponding to the μ-transform of a fourth-order Tchebycheff low-pass filter with 10% ripple.

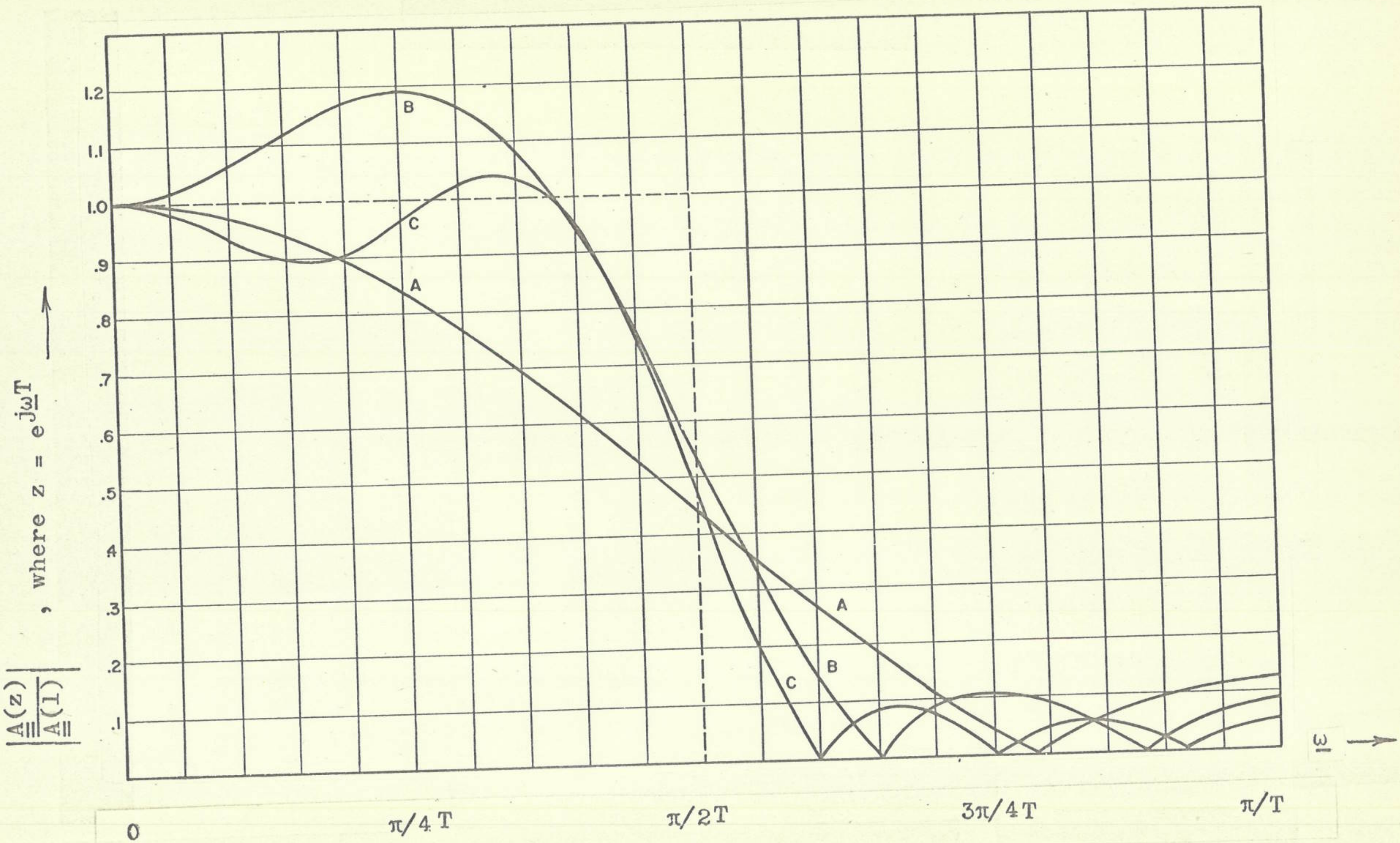

Fig. 6. Fourier series approximations to an ideal low-pass digital filter magnitude characteristic.

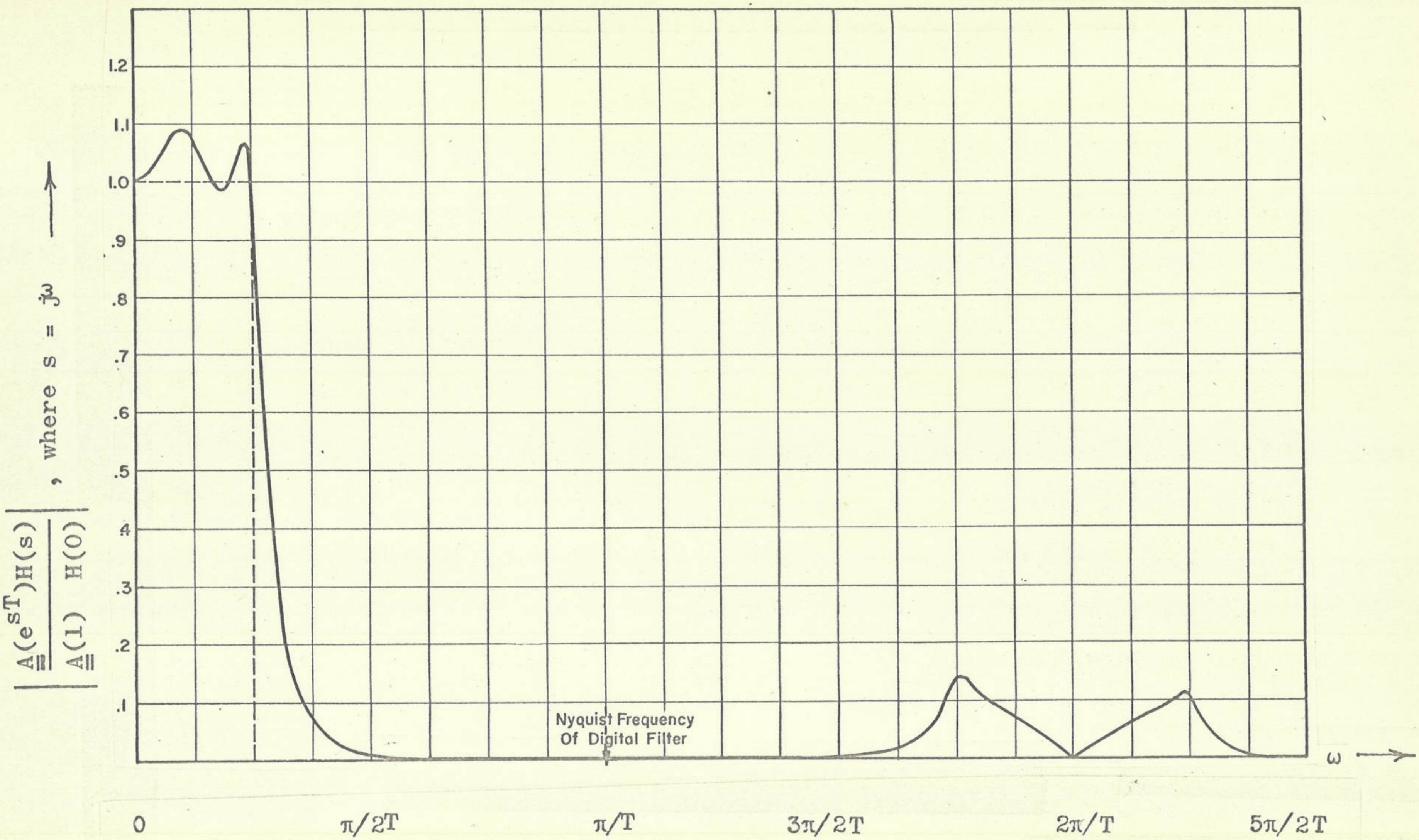

Fig. 7. The normalized magnitude characteristic of an analog low-pass filter constructed from a Tchebycheff digital filter and a zero-order hold circuit.

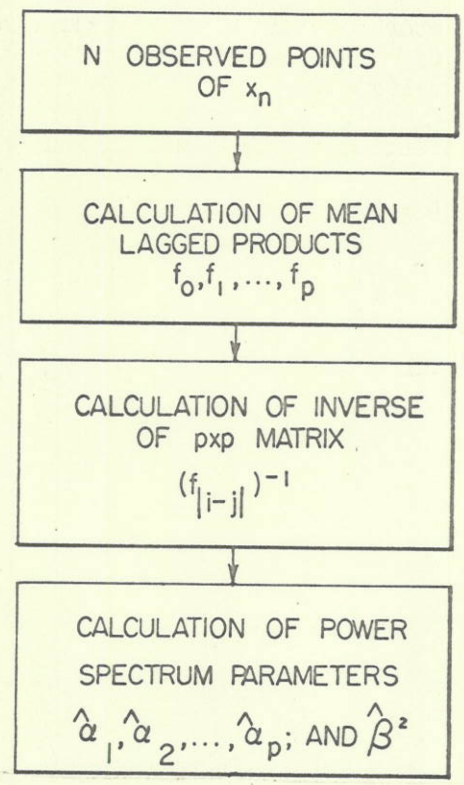

Fig. 8. Estimation of power spectrum parameters.

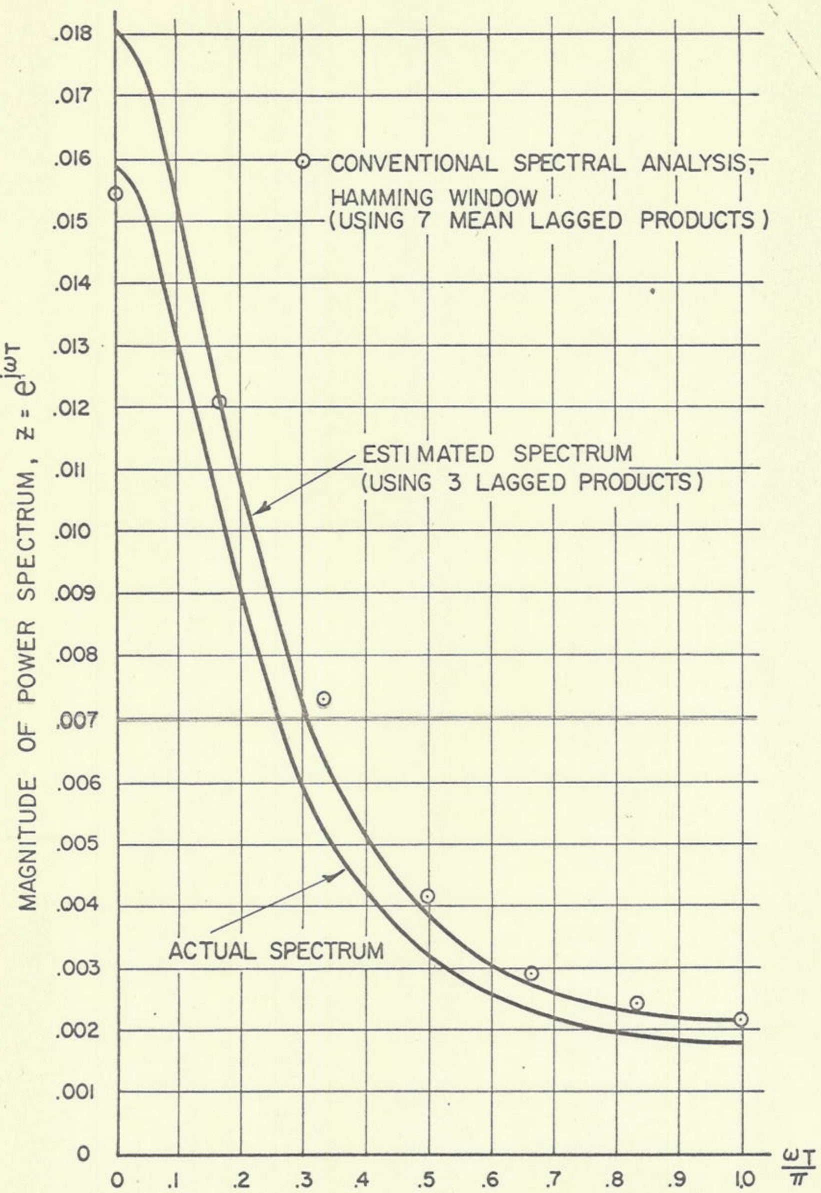

Fig. 9. Comparison of actual and estimated power spectra.