ROOT LOCI IN FEEDBACK SYSTEMS

A THESIS

by

Kenneth Steiglitz

B.E.E.

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APPROVAL OF READERS

This is to certify that I have read and approved the thesis entitled "Root Loci in Feedback Systems", submitted by Kenneth Steiglitz in partial fulfillment of the requirements for the degree of Master of Electrical Engineering, College of Engineering, New York University.

Date______________________________

(1)____ (Advisor)

(2)____

(3)____

(4)____

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1. INTRODUCTION

When Laplace Transform techniques are used to analyse a lumped linear system, the transfer function of the system can be expressed as a rational function of the complex frequency variable s. The poles of this rational transfer function represent roots of the characteristic equation of the system, and the location of these poles determine the natural modes and hence the dynamic response of the system. Thus, if the designer can control the position of the poles of the transfer function of a linear system, he can ensure that the system will be stable and will have certain prescribed dynamic characteristics.

The root-locus method enables the designer of linear feedback amplifiers and control systems to relate the pole positions of a closed-loop system to the poles and zeros of the open-loop system. While the root-locus method is usually thought of as a way of determining the poles of a single-loop feedback structure as a function of the midband loop gain, it may be used to study pole loci in multiple-loop systems and as functions of parameters other than loop gain.

The root-locus method was introduced in 1948 by W. R. Evans[5]. Quickly after this, those elementary properties which enable the designer or analyst to sketch the loci became well known, and engineers adopted graphical and semi-graphical techniques.

Generally, the fact that these loci are algebraic curves, often of low degree, has seldom been utilized; and the general properties of the curves have usually been studied only with a view towards their rapid construction. In the present work, the author has attempted to present in a unified manner the well known properties of root loci and any extensions of these that his research has uncovered. It is shown how the root locus for a higher order system is restrained by the loci of its lower order sub-systems. The algebraic equations of root loci have been determined and classified, with special attention given to those cases in which the curves are quadratic and cubic. A method is presented for determining algebraically those frequencies and gains for which a given system is on the threshold of instability. Finally, the loci for a class of phase shift oscillator are found. This is an example of a class of systems where the root-loci are quadratic and a completely analytical root-locus analysis is more fruitful than a graphical analysis.

2. DEFINITION OF THE ROOT LOCUS PROBLEM

We shall take the unity-feedback single-loop structure shown in Figure 1. as the motivating system for the root-locus problem and all that follows will be in terms of this basic system. Actually, little generality is lost by the adoption of this simple model, since most of the following results can be easily extended to any system where a parameter enters linearly into the characteristic equation.

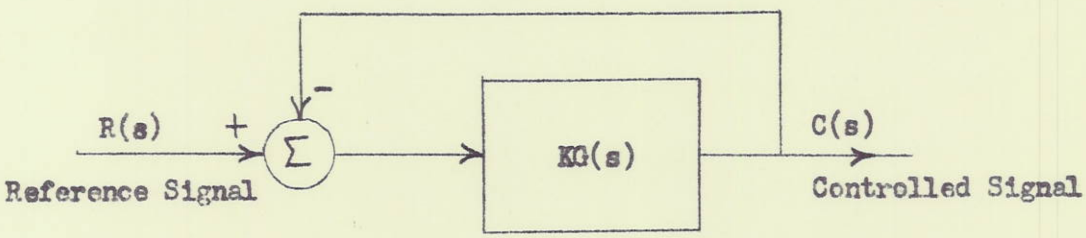

Figure 1. The Basic Single-Loop System.

We shall assume that the open-loop transfer function, KG(s), is a rational function of s with real coefficients, and that the polynomials N(s) and D(s) that are the numerator and denominator of G(s) are of degree n and d, respectively. Sometimes, it will be convenient to assume that the leading coefficients of N(s) and D(s) are unity. We may do this with no loss of generality, since any constant factor in G(s) can be absorbed in the gain constant K. Thus, KG(s) may be written:

$$KG(s) = K\frac{N(s)}{D(s)} = K\,\frac{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}{s^d + b_{d-1}s^{d-1} + b_{d-2}s^{d-2} + \cdots + b_0} \tag{1}$$

$$= K\,\frac{\sum_{k=0}^{n} a_k s^k}{\sum_{k=0}^{d} b_k s^k}$$

where

$$a_n = b_d = 1$$

We may also write the open-loop transfer function in factored form:

$$KG(s) = K \frac{\prod_{j=1}^{Z} (s - z_j)^{n_j}}{\prod_{j=1}^{P} (s - p_j)^{d_j}} \tag{2}$$

where

$$\sum_{j=1}^{Z} n_j = n$$

$$\sum_{j=1}^{P} d_j = d$$

Thus, $G(s)$ has Z distinct zeros z_j, each of degree n_j; and P distinct poles p_j, each of degree d_j.

Returning now to the system of Figure 1., we may write

$$KG(s)\left[R(s) - C(s)\right] = C(s) \tag{3}$$

This equation may be solved for the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)} \tag{4}$$

From this relation it can be seen that the zeros of the closed-loop system are the same as those of the open-loop system. The poles of the closed-loop system, however, differ from those of the open loop system, and are functions of the gain constant K. The closed-loop poles are solutions of the characteristic equation:

$$1 + KG(s) = 0 \tag{5}$$

or

$$G(s) = -\frac{1}{K} \tag{6}$$

We now define the locus of all solutions of Equation 6. for real K, $-\infty < K < +\infty$, as the <u>root locus</u> of the open-loop function G(s). In most of the literature, K is usually restricted to be positive, so that our definition is more general than usual.

From Equation 6. we can see that when K=0, the closed-loop poles coincide with the open-loop poles; and that as K becomes large in magnitude, the closed-loop poles approach the zeros of the open-loop system. Thus, the root locus starts at the poles of G(s) for K=0 and proceeds to the zeros of G(s) as $K \to \infty$.

We may equivalently define the root locus as all those values of s for which G(s) is real. Thus alternate conditions for a point to be on the root-locus is that it satisfy the equation:

$$\mathrm{Im}\left\{G(s)\right\} = 0 \tag{7}$$

Or $$\mathrm{ARG}\left\{G(s)\right\} = 0, \pm k\pi \tag{8}$$

$$k = 1, 2, 3, \cdots$$

From this it is clear that if we consider G(s) to be a mapping from the s-plane to the G-plane, then the root locus as defined above is the image in the s-plane of the entire real axis in the G-plane. This formulation gives the locus independently of the parameter K, and will be useful in our subsequent investigation of the algebraic equations of root loci.

3. INTERPRETATIONS OF THE ROOT LOCUS PROBLEM

It is well known that if F(s) is an analytic function of the complex variable $s = \sigma + j\omega$ in a region, and if

$$F(\sigma + j\omega) = F_r(\sigma,\omega) + j F_i(\sigma,\omega) \tag{9}$$

then the functions F_r and F_i are harmonic in that region and the curves (10)

$$\begin{cases} F_r(\sigma,\omega) = \text{const.} \\ F_i(\sigma,\omega) = \text{const.} \end{cases}$$

are a set of mutually orthogonal lines. In particular we may choose

$$F(s) = \ln G(s) \tag{11}$$

so that

$$\ln G(s) = \ln |G(s)| + j \, \mathrm{ARG}\left[G(s)\right] \tag{12}$$

The lines of constant phase and constant magnitude of G(s) in the s-plane are therefore a set of mutually orthogonal lines which represent the solution of Laplace's Equation in the plane with certain boundary conditions. Since the root loci in which we are interested are just the 0^0 and 180^0 phase lines in the solution of this potential problem, we may consider the root locus plot as having come about in the solution of some appropriate physical problem.

Evans[34] shows the analogy between flux plots and root locus plots. This analogy is made evident if we substitute the factored form of G(s) from Equation 2. into Equation 12. and decompose the logarithm of the product as follows:

$$\ln G(s) = \sum_{j=1}^{Z} n_j \ln|s-z_j| - \sum_{j=1}^{P} d_j \ln|s-p_j| + j\left[\sum_{j=1}^{Z} n_j \operatorname{ARG}(s-z_j) - \sum_{j=1}^{P} d_j \operatorname{ARG}(s-p_j)\right] \tag{13}$$

The condition that defines the root locus is

$$\ln G(s) = -\ln|K| \pm jk\pi \qquad k = 0, 1, 2, \cdots \tag{14}$$

Thus, on the locus we have

$$\begin{cases} \sum_{j=1}^{Z} n_j \ln|s-z_j| - \sum_{j=1}^{P} d_j \ln|s-p_j| = -\ln|K| \\ \sum_{j=1}^{Z} n_j \operatorname{ARG}(s-z_j) - \sum_{j=1}^{P} d_j \operatorname{ARG}(s-p_j) = \pm k\pi \end{cases} \tag{15}$$

$$k = 0, 1, 2, \cdots$$

In order to produce analogous equations, we have only to consider infinite line charges perpendicular to the s-plane at each of the points where G(s) has a pole or a zero. To each line charge we assign a charge $+n_j$ or $-d_j$ depending on whether it corresponds to a zero or pole. Then at any point s in the plane, the potential is given by

$$\text{Potential} = \sum_{j=1}^{Z} n_j \ln|s-z_j| - \sum_{j=1}^{P} d_j \ln|s-p_j| + \text{const.} \tag{16}$$

By a comparison of this equation with Equation 15. It is evident that lines of constant potential in this system are analogous to lines of constant magnitude of G(s). Similarly, the field lines in this system are perpendicular to the potential lines and are analogous to the lines of constant phase of G(s). Reza[14] suggests the possibility of pursuing the analogy with potential problems further, and provides references to earlier work on potential curves.

Tsien[20] suggests as another physical interpretation of Equation 15. the complex potential function of a two-dimensional irrotational flow of a perfectly incompressible fluid. In this analogy, sources of appropriate strengths are placed at the poles of G(s) and sinks of appropriate strengths are placed at the zeros of G(s).

Perhaps the most easily grasped and hence the most valuable interpretation of the root locus plot is that based on the contour map of G(s). If we construct a surface whose height above the s-plane is the magnitude of $G(s)$, then the level curves are curves of constant magnitude of $G(s)$, and the orthogonal trajectories of these curves are the curves of constant phase of $G(s)$. These latter curves represent streamlines, which are the paths that an inertialess fluid would take on the surface in a gravity field. Figure 2. shows such a surface for a $G(s)$ which has two poles. The streamline corresponding to the 180°- phase line of $G(s)$ is shown. This is then the part of the root locus corresponding to a positive gain constant.

Jawor[6] has actually constructed such contours with rubber membranes and found root loci by laying thin flexible chains on these surfaces. The potential analogy has also been followed through, and theoretically, any of the physical interpretations may be used to find root loci. We shall come back to some of these physical analogies when we interpret the critical points of $G(s)$.

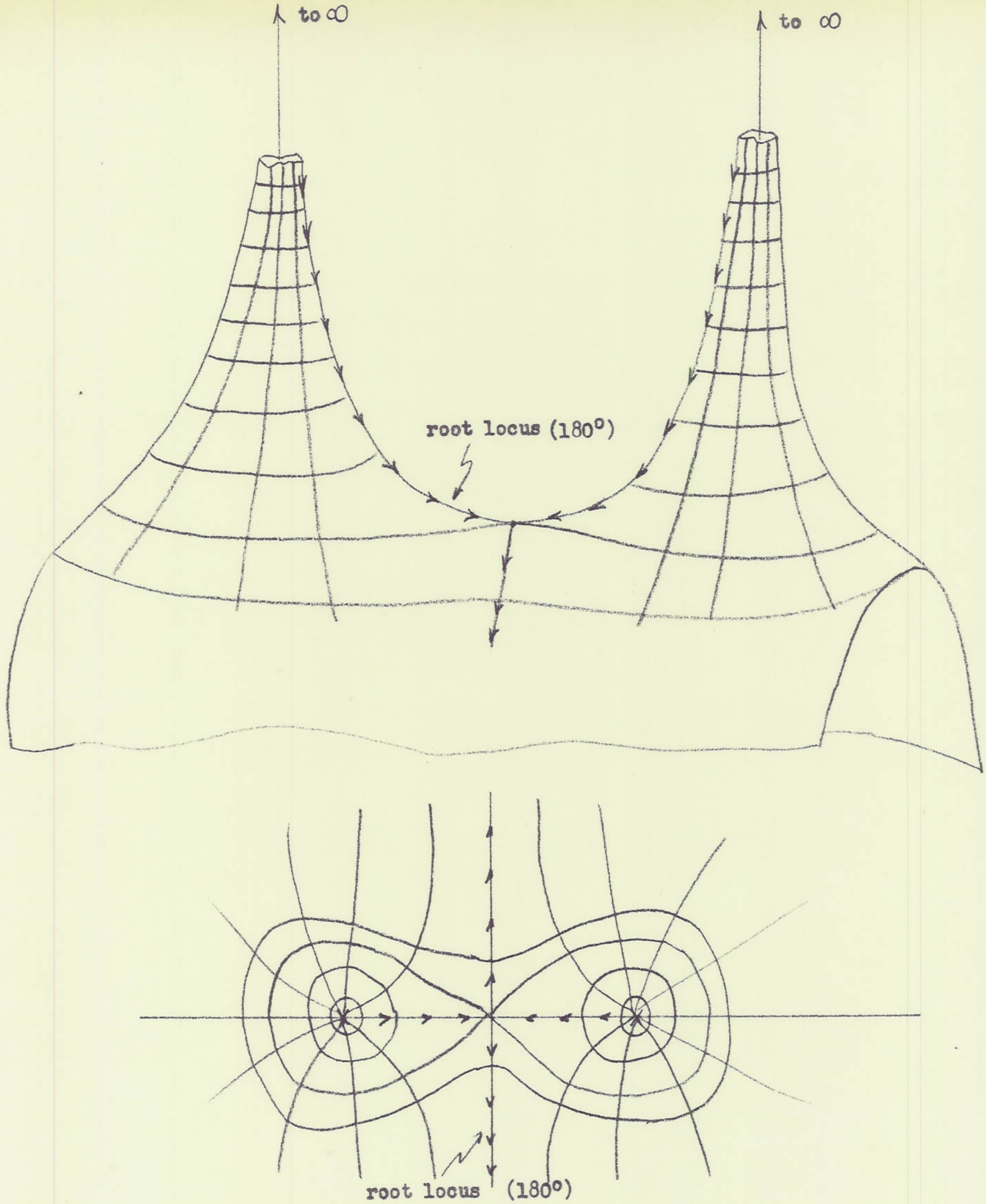

Figure 2. Perspective and top view of a contour map which shows the root locus for two poles as a streamline.

4. WELL KNOWN THEOREMS USEFUL IN CONSTRUCTING ROOT LOCI

In this section we shall state the various well known theorems which are commonly used to facilitate the sketching of root loci. Proofs of these can be found in textbooks such as Truxal[19], Savant[15], etc., and we shall dwell only upon those points which are not generally treated. Our inclusion of the 0^o-locus in the root locus, for instance, makes some of the material more general.

The general graphical procedure for constructing root loci is based on Equations 15. By trial and error a point is found which fulfills the condition on the angle of $G(s)$, the second line of Equations 15. The root locus may then be calibrated in terms of gain by using the first condition of Equations 15. A plastic device called the Spirule* facilitates the rapid addition of angles and multiplication of lengths. The designer has the following theorems available to form a preliminary sketch:

__Theorem 1.__ The real axis is always part of the root locus. Segments between simple zeros and poles on the real axis are alternately on the 0^o-locus and the 180^o-locus, with the segment that terminates at $+\infty$ on the 0^o-locus.

__Theorem 2.__ The root locus is symmetrical with respect to the real axis.

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* A patented device, available from The Spirule Company, 9728 El Venado, Whittier, California.

Theorem 3. The angle of departure ϕ_i of the root locus from the i^{th} pole is given by

$$d_i \phi_i = \sum_{j=1}^{Z} n_j \, ARG(s-z_j) - \sum_{\substack{j=1 \\ j \neq i}}^{P} d_j \, ARG(s-p_j) \pm k\pi \qquad (17)$$

$$k = \begin{cases} 1,3,5,\cdots & \text{FOR } 180^\circ\text{-LOCUS} \\ 0,2,4,\cdots & \text{FOR } 0^\circ\text{-LOCUS} \end{cases}$$

Similarly for the angle of arrival of the root locus at a zero.

Theorem 4. If N(s) and D(s) are not of the same degree; that is, if $n \neq d$, then the loci approach asymptotes for large s which are at angles

$$\phi_\infty = \pm \frac{k\,360^\circ}{|n-d|} \qquad \text{For } 0^\circ\text{-Locus} \qquad (18)$$

$$\phi_\infty = \pm \frac{180^\circ + k\,360^\circ}{|n-d|} \qquad \text{For } 180^\circ\text{-Locus}$$

$k=0,1,2,\ldots$

These asymptotes radiate from the asymptotic center:

$$\sigma_\infty = \frac{\sum_{j=1}^{P} d_j p_j - \sum_{j=1}^{Z} n_j z_j}{n-d} \qquad (19)$$

If N(s) and D(s) are of the same degree, then by Ur[21] the asymptotic center and the asymptotic angles are given by the above equations, but with D(s)-N(s) replacing D(s).

Theorem 5. If the difference in degree of N(s) and D(s) is greater than 1; that is if $d-n \geqslant 2$ then the center of gravity of the closed-loop poles, $\sigma^{\star}_{CL}$, is invariant with K and is equal to the center of gravity of the open-loop poles, $\sigma^{\star}_{OL}$. That is,

$$\sigma^{\star}_{CL} = \sigma^{\star}_{OL} \qquad d-n \geqslant 2 \tag{20}$$

If $d-n=1$, then the center of gravity changes with K and is given by:

$$\sigma^{\star}_{CL} = \sigma^{\star}_{OL} - \frac{K}{n} \tag{21}$$

These equations, and others that relate to the product of the loci, are useful in calibrating the locus with respect to K, and are discussed by Truxal[19] and Yeh.[24]

5. LOCI WHICH CAN BE CONSTRUCTED IMMEDIATELY

Although we shall investigate the general algebraic form of the root loci in a later section, it will be convenient now to find explicitly certain loci which can be constructed immediately. These will be used often in the application of Theorem 8. of the following section.

To find the equations of the circular loci; we shall use the method suggested by Equation 7.; that is, we shall set

$$\operatorname{Im}\left\{G(s)\right\} = 0 \tag{22}$$

This procedure was suggested in a note by Lass[7]. Consider, for example, the pole-zero configuration shown in Figure 3.

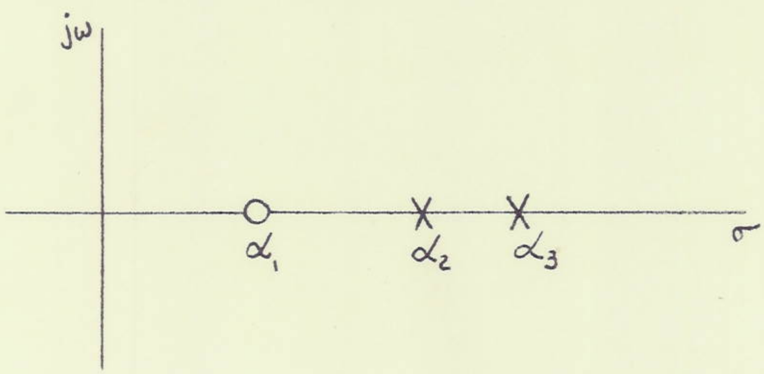


Figure 3. a pole-zero configuration that leads to a circular root locus.

for which

$$G(s) = \frac{(s-\alpha_1)}{(s-\alpha_2)(s-\alpha_3)} \tag{23}$$

Instead of setting the imaginary part of G(s) to zero, we may more conveniently set the imaginary part of its reciprocal to zero. Thus

$$\mathrm{Im}\left\{\frac{1}{G}\right\} = \mathrm{Im}\left\{\frac{(\sigma+j\omega-\alpha_2)(\sigma+j\omega-\alpha_3)}{(\sigma+j\omega-\alpha_1)}\right\} = 0 \qquad (24)$$

Rationalizing,

$$\mathrm{Im}\left\{\frac{(\sigma+j\omega-\alpha_2)(\sigma+j\omega-\alpha_3)(\sigma-j\omega-\alpha_1)}{(\sigma-\alpha_1)^2+\omega^2}\right\} = 0 \qquad (25)$$

After multiplying out, we have

$$\omega\left[\sigma^2 - 2\sigma\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_2\alpha_3 + \omega^2\right] = 0 \qquad (26)$$

The factor of ω gives the real axis as part of the locus. The rest of the locus is, completing the square,

$$(\sigma-\alpha_1)^2 + \omega^2 = (\alpha_3-\alpha_1)(\alpha_2-\alpha_1) \qquad (27)$$

This is a circle with center at α_1, and a radius which is the geometric mean of the distances from the zero to the poles. If the zero lies between the two poles, the right member of Equation 27. is negative and there is no root locus off the real axis.

For one pole, one zero, or one pole and one zero, there is no locus off the real axis. For two poles, the locus consists of the real axis and a line $\sigma =$ constant through the center of gravity of these poles. This locus can be considered as a circle with an infinite radius, produced by moving the zero in Figure 3. to infinity. In Figure 4. all the lower order loci that are straight lines or circles are shown. Enough information is provided so that each locus can be drawn with a straight edge and a compass.

Another locus which can be drawn immediately is that for a multiple pole. In this case the locus coincides with the asymptotes. Yeh[24], and Lorens and Titsworth[8] discuss this situation.

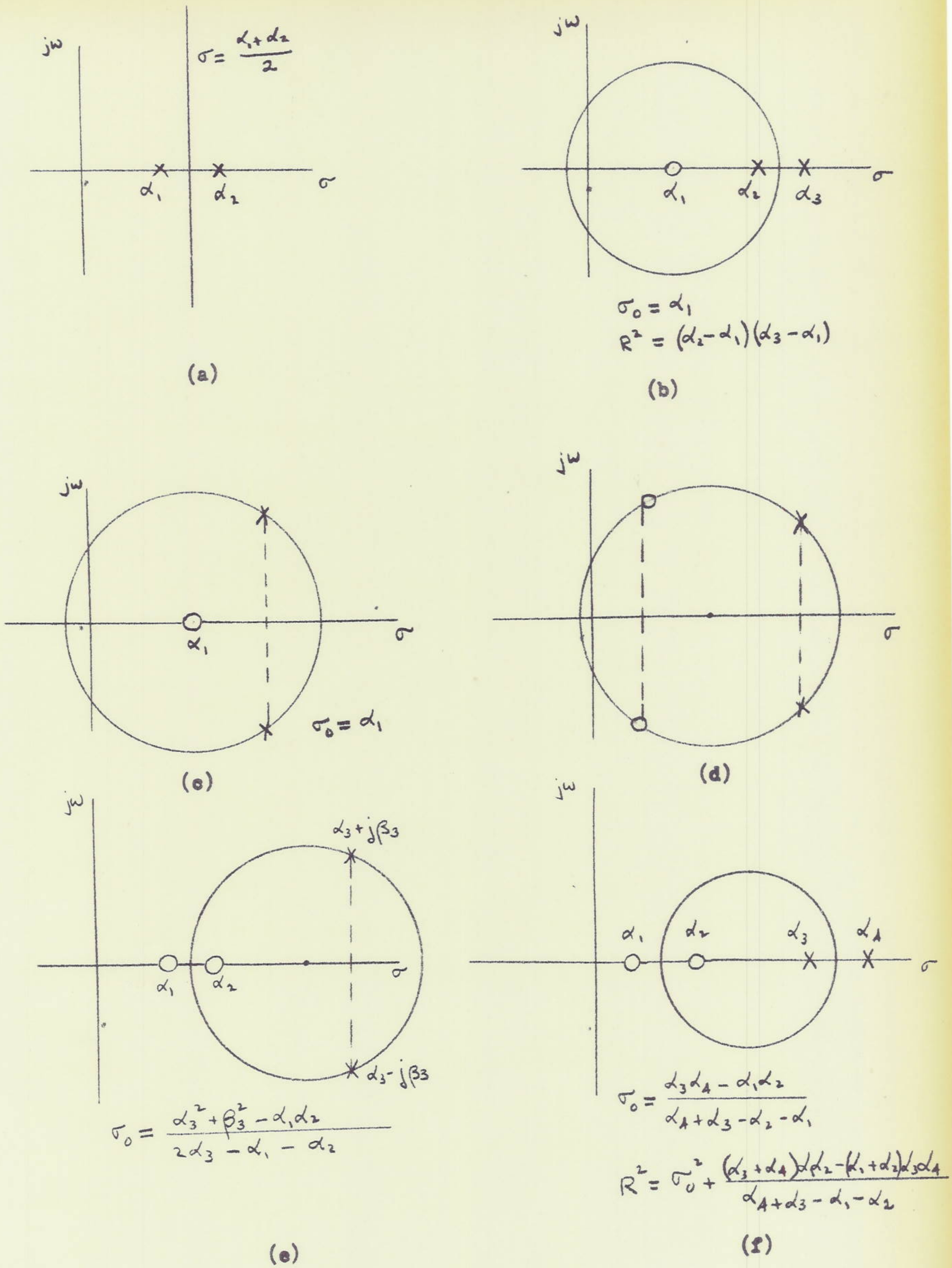


Figure 4. The Circular Loci: $(\sigma - \sigma_0)^2 + \omega^2 = R^2$

6. FURTHER WORK ON THE GENERAL PROPERTIES OF ROOT LOCI

We shall turn now to some less well known properties of root loci. Our notation will be that of Yeh[24] and the locus of any open-loop function with n zeros and d poles will be denoted by T(n,d).

The following theorem is stated by Ur[21]:

<u>Theorem 6.</u> T(n,d)=T(d,n). That is, the locus is unchanged by the interchange of the open-loop poles and zeros.

<u>Proof:</u> The locus is defined by

$$\mathrm{Im}\{G(s)\} = 0 \tag{28}$$

which is equivalent to

$$\mathrm{Im}\left\{\frac{1}{G(s)}\right\} = 0 \tag{29}$$

because G(s) is real when and only when $\frac{1}{G(s)}$ is real. At any point on the locus the open-loop function has a magnitude which is the reciprocal of its former value, so that after the interchange of poles and zeros K is replaced by $\frac{1}{K}$. This theorem is useful for classifying loci, since only half of the possible configurations need now be considered. Figure 4., then, gives all the loci T(0,2), T(2,0), T(1,2), T(2,1), and T(2,2). T(0,1), T(1,0), and T(1,1) are just the real axis, as mentioned above.

The next theorem is stated by Evans[4] and again by Ur[21]:

Theorem 7. If the open-loop poles are replaced by the closed-loop poles for some given value of gain, say $K = K_1$ then the root locus for this new system is the same as that for the original system, and the locus is recalibrated with the new gain

$$K' = K - K_1 \tag{30}$$

Proof: The root locus for the original system is given by

$$1 + KG = 0 \tag{31}$$

The new open-loop system function is

$$\frac{G}{1 + K_1 G} \tag{32}$$

and the root locus for this new system is given by

$$1 + K'\left[\frac{G}{1 + K_1 G}\right] = 0 \tag{33}$$

This can be rewritten as

$$1 + (K_1 + K')G = 0 \tag{34}$$

which, as K varies from $-\infty$ to $+\infty$, generates the same locus as the original open-loop function. It is evident from a comparison of Equations 31. and 34. that if a point on the original locus corresponded to a gain of K, this point on the new locus corresponds now to a gain $K' = K-K_1$.

Evans[4] introduces this idea to investigate the effect of gain variations on the closed-loop pole positions, but the theorem is also useful in classifying loci. If one has found the root locus for a particular configuration, for example, then one knows immediately other configurations that will have the same locus. For example, if K is increased in a positive direction in Figure 4. (b), the poles will move together, coalesce to a double pole at the point where the circle intersects the real axis; and then break away from the real axis, moving along the circle. If the gain is fixed at any of these values, and the resultant pole pattern taken as an open-loop function, the root locus will remain the same. From this may be deduced the locus in Figure 4. (c).

The next theorem is apparently original with the author, and enables the designer to use his knowledge of the simple loci like those in Figure 4. when sketching higher order loci.

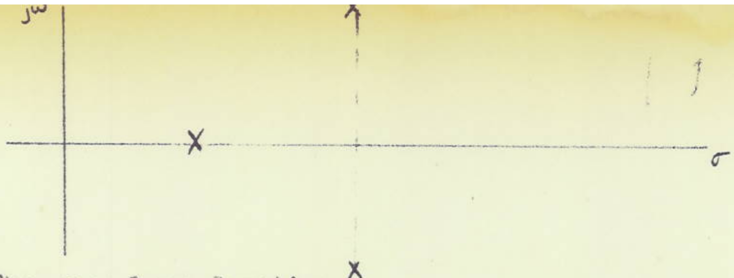
<u>Theorem 8</u>. Let T_1 be the root locus associated with G_1, and let T_2 be the locus associated with G_2. Then intersections of T_1 and T_2 are on the root locus associated with $G_1.G_2$. Furthermore, the locus for $G_1.G_2$ cannot cross the remaining parts of T_1 and T_2.

<u>Proof</u>: At any point which is on both T_1 and T_2, G_1 and G_2 are both

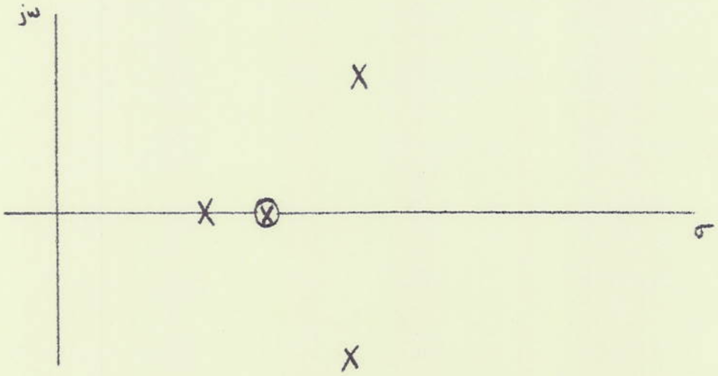
real, and hence so is $G_1 . G_2$. At a point on T_1 and not on T_2, G_1 is real and G_2 is not; so that $G_1.G_2$ is not real and this point is not on the root locus for $G_1.G_2$. Similarly for a point on T_2 and not on T_1.

This theorem is most useful when the total open-loop function can be broken up into the product of two other functions whose loci can be drawn immediately. Then the designer knows that the final loci can cross these loci only at their intersections.

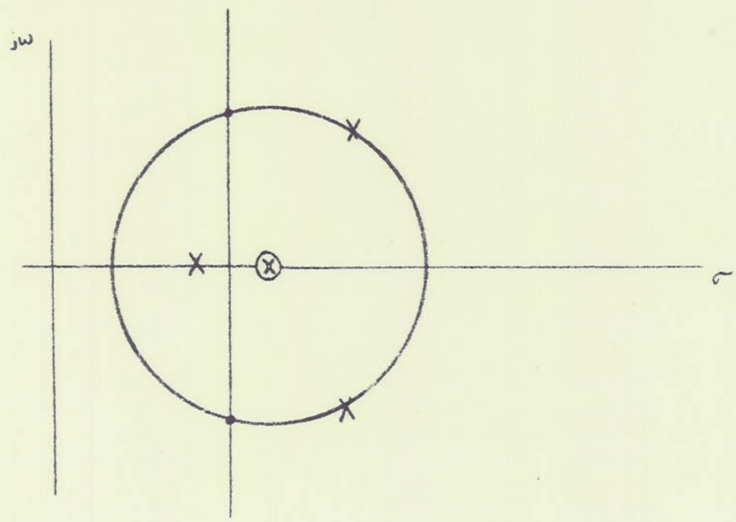
In Section 9. it is shown that the loci $T(0,3)$ and $T(3,0)$ are hyperbolas, and their equations may be found in a manner similar to those of the circular loci. However Theorem 8. now gives us a simple procedure for constructing these hyperbolas. Consider the three-pole open-loop function shown in Figure 5. (a). Now introduce a pole and zero which coincide, as in (b), so that the open-loop function and the locus is unchanged. We may now take the two real poles as the G_1 of Theorem 8., and the zero together with the imaginary pair of poles as G_2. As shown in (c), T_1 is a straight line and T_2 is a circle that is easily traced. The intersections of T_1 and T_2 give two points on the final locus $T(0,3)$. By introducing another pole-zero pair along the real axis, another pair of points may be found, and so on. In this way the loci $T(0,3)$ and $T(3,0)$ can be quickly sketched with a straight edge and a compass, as shown in Figure 5. (d). These loci may in turn be used to sketch higher order loci.



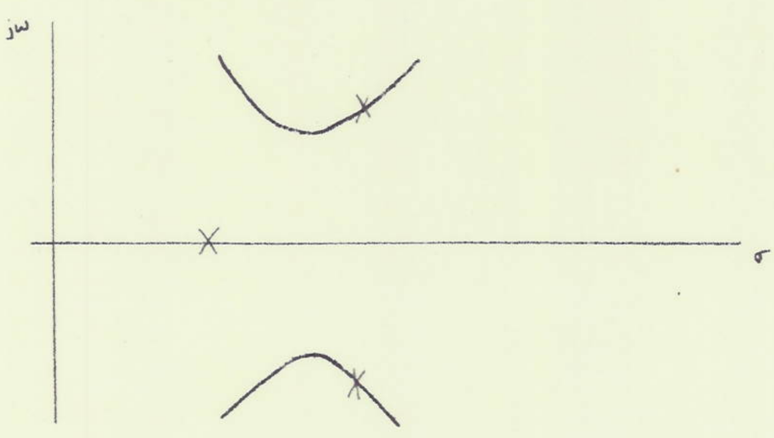
(a) The open-loop function



(b) The addition of a coincident pole and zero does not change the locus.



(c) The composite loci, T(0,2) and T(1,2).



(d) The final locus constructed as above.

Figure 5. A graphical procedure for constructing T(0,3) and T(3,0).

Figures 6. and 7. illustrate a situation where a few simple applications of Theorem 8. provide enough information to sketch the general shape of the locus. The open-loop function is one that might have been encountered in a compensated transistor amplifier. The asymptotes are drawn first; then the zeros and poles can be divided into various groups for which simple loci can be drawn. For instance, in Figure 6. the zero at -3 can be associated with the double pole at -5 and a circle drawn. The locus for the remaining two poles is a straight line through -3.5, and the intersections of these two loci give two points on the final locus. Moreover, this circle and line represent barriers for the final locus. Another circle-line combination is possible, and this gives two more points on the locus. When the zero is associated with other pairs of poles, it lies between them and produces no locus off the real axis, and the lines for the remaining two poles represent barriers to the locus. In Figure 7., the zero is at -4.5, the two circles do not intersect their corresponding lines, and the four lines and two circles all represent barriers to the locus. Here it can be quickly seen that the part of the locus leaving the double pole must come back to meet the real axis again.

It becomes apparent in this example that it is usually to the advantage of the designer to keep track of which parts of the root locus are the 0°-phase lines and which are the 180°-phase lines. The 0°-locus and the 180°-locus can meet only at a zero, pole, or a multiple point on the locus, and this restriction can be of considerable value in determining the behavior of some loci. For

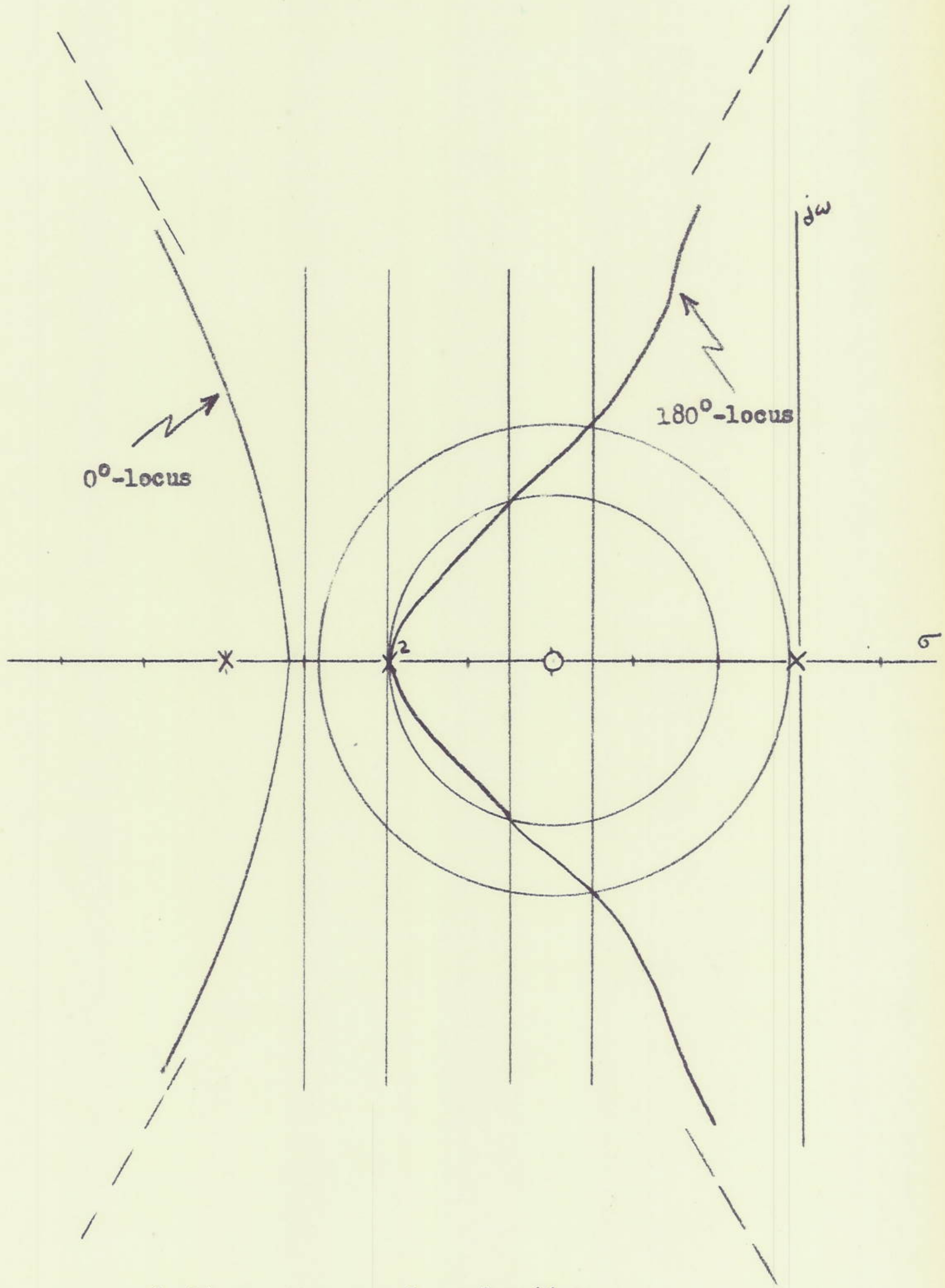


Figure 6. The locus T(1,4) for the open-loop function

$$G(s) = \frac{(s+3)}{s(s+5)^2(s+7)}$$

sketched with the aid of Theorem 8.

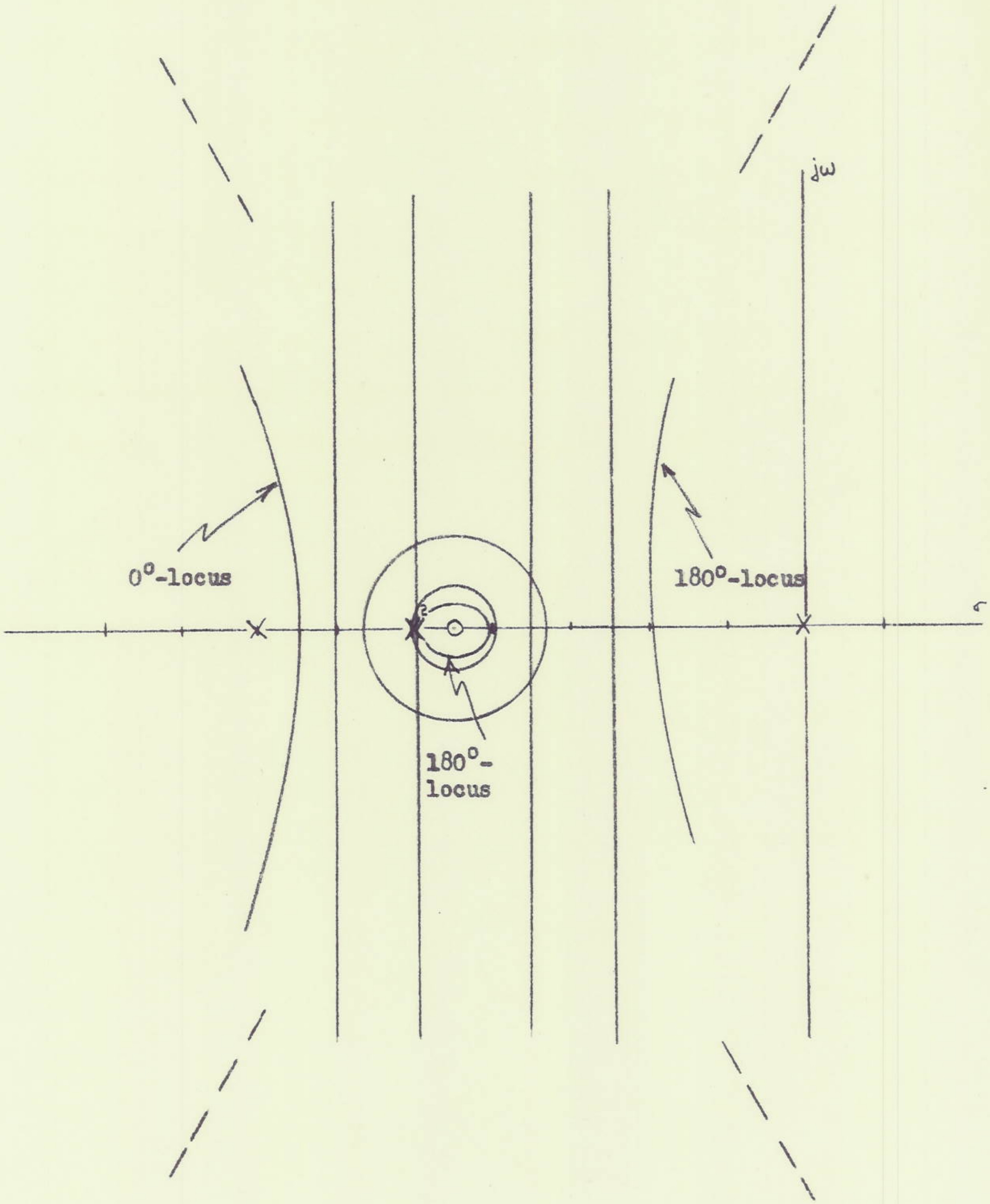


Figure 7. The locus T(1,4) for the open-loop function

$$G(s) = \frac{(s+4.5)}{s(s+5)^2(s+7)}$$

sketched with the aid of Theorem 8.

instance, in Figure 7., the part of the locus that leaves the double pole can come back to meet the real axis only on that part of the real axis that corresponds to the 180^o-locus.

When a point on the root locus is found by Theorem 8., the part of the locus that the point is on can be found by determining which parts of the T_1 and T_2 loci the point is on. If a point is on the 0^o-locus of T_1 and the 180^o-locus of T_2, for instance, the point must be on the 180^o-locus of $G_1 \cdot G_2$. On the other hand, if the point is on the 180^o-locus of both T_1 and T_2, it is on the 0^o-locus of $G_1 \cdot G_2$; and so on.

7. MULTIPLE POINTS ON THE ROOT LOCUS

In order that we may further investigate the general structure of root loci, we shall now concern ourselves with multiple points on the locus; — points on the locus where the closed-loop system has repeated poles. The following argument is similar to that presented by Bower and Schultheiss[3].

Consider that the point $G_0 = -1/K_0$ on the real axis of the G-plane is mapped into the point s_0 on the root locus in the s-plane, as shown in Figure 8.

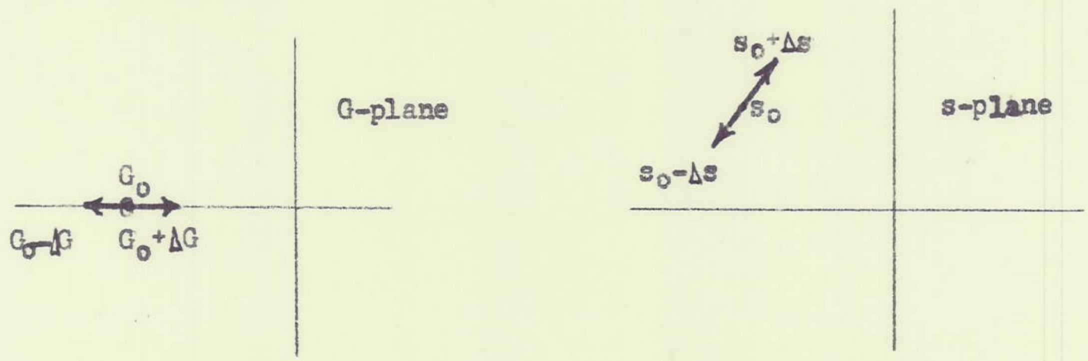


Figure 8. An infinitesmal part of the root locus plot in the G-plane and the s-plane.

If we assume that G(s) is analytic in the vicinity of the point in question, we may represent G(s) by a power series as follows:

$$\Delta G = \left.\frac{dG}{ds}\right|_0 \Delta s + \frac{1}{2!}\left.\frac{d^2G}{ds}\right|_0 \overline{\Delta s}^2 + \cdots \tag{35}$$

where $\Delta G = G - G_0$ corresponds to $\Delta s = s - s_0$ in the vicinity of the point 0. If we further assume that

$$\left.\frac{dG}{ds}\right|_0 \neq 0 \tag{36}$$

then we may by taking Δs small enough, make ΔG as close to the first term of Equation 35. as we wish. We shall write this as

$$\Delta G \doteq \left.\frac{dG}{ds}\right|_0 \Delta s \tag{37}$$

In order for the point s to remain on the root locus, G must remain real; but $\Delta G = -\frac{1}{K}$ can be positive or negative. Thus Δs is

$$\Delta s \doteq \pm \frac{\Delta G}{dG/ds|_0} \tag{38}$$

This corresponds to an ordinary point on the root locus; the infinitesmal line segment $(G_0 - \Delta G, G_0 + \Delta G)$ maps into the infinitesmal segment on the root locus $(s_0 - \Delta s, s_0 + \Delta s)$, where the angle of Δs is determined by $\left.\frac{dG}{ds}\right|_0$.

If $\left.\frac{dG}{ds}\right|_0 = 0$, however, the situation is different. By Equation 35., we can see that for small Δs, ΔG is now given by

$$\Delta G \doteq \frac{1}{2!} \left.\frac{d^2G}{ds^2}\right|_0 \overline{\Delta s}^2 \tag{39}$$

If K is now varied about its center value K_0, we have

$$\overline{\Delta s}^2 \doteq \pm \frac{\Delta G}{\frac{1}{2!} \left.\frac{d^2G}{ds^2}\right|_0} \tag{40}$$

$$\Delta s \doteq \pm \sqrt{\frac{|\Delta G|}{\frac{1}{2!}\left.\frac{d^2G}{ds^2}\right|_0}} \qquad \text{when } \Delta G > 0$$

or

$$\Delta s \doteq \pm j\sqrt{\frac{|\Delta G|}{\frac{1}{2!}\left.\frac{d^2G}{ds^2}\right|_0}} \qquad \text{when } \Delta G < 0 \tag{41}$$

Thus, the locus enters the point from two opposite directions and leaves perpendicular to these directions as shown in Figure 9. This behavior

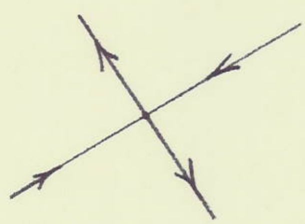


Figure 9. The behavior of the root locus in the vicinity of a first order multiple point.

can be observed at the intersection of the circular loci with the real axis. In general, we have:

<u>Theorem 9</u>. At a point s_0 on the root locus for which the first r derivatives of G(s) vanish and for which G(s) not infinite, the closed-loop function has a pole of order r+1 . In the vicinity of s_0, the root locus consists of r+1 straight lines which meet at equal angles. Furthermore, the branches of the locus alternately enter and leave.

The restriction that $G(s_0)$ be different from infinity is necessitated by the assumption that $G(s)$ be analytic at the point in question. The result is valid at a zero, however, and since the locus for $\frac{1}{G(s)}$ is the same as that for G, the above result is valid at a pole of G after all.*

If $G(s)$ has a zero of order r at s_0, it may be written

$$G(s) = (s-s_0)^r F(s) \tag{42}$$

where

$$F(s_0) \neq 0, \infty$$

then

$$\begin{aligned} G'(s) &= r(s-s_0)^{r-1} F(s) + (s-s_0)^r F'(s) \\ &= (s-s_0)^{r-1}\left[rF(s) + (s-s_0)F'(s)\right] \end{aligned} \tag{43}$$

so that $G'(s)$ has a zero at s_0 of degree one lower than that of $G(s)$. Thus, if $G(s)$ has a zero of order r at s_0, the first $r-1$ derivatives of $G(s)$ vanish at s_0, and the locus in the vicinity of s_0 will consist of r concurrent straight lines. The locus will behave similarly at a multiple pole of $G(s)$, but of course, no derivatives will exist at such a point.

---

*The behavior of the root locus in the vicinity of a pole or a zero of $G(s)$ is also given by Theorem 3.

The zeros of $G'(s)$ can be divided into two classes: those due to repeated zeros of $G(s)$ and those for which $G(s)$ is different from zero. This can be seen by writing

$$G'(s) = G(s)\frac{G'(s)}{G(s)} = G(s)\left[\frac{d}{ds}\ln G(s)\right] \tag{44}$$

From this we can see that the zeros of $G'(s)$ which are not zeros of $G(s)$ must satisfy the equation

$$\frac{d}{ds}\left[\ln G(s)\right] = 0 \tag{45}$$

Such points are called <u>critical points</u> or <u>saddle points</u> of $G(s)$. With the substitution of the product form for $G(s)$, Equation 45. becomes

$$\frac{d}{ds}\ln G(s) = \sum_{j=1}^{z}\frac{n_j}{s-z_j} - \sum_{j=1}^{p}\frac{d_j}{s-p_j} = 0 \tag{46}$$

This equation may be used to find possible multiple points of the root locus. A critical point of $G(s)$, however, does not necessarily lie on the root locus of $G(s)$.

From Equation 46. we may deduce the following theorems:

<u>Theorem 10</u>. If $G(s)$ has a pole or a zero at infinity; that is, if $n \neq d$, then the maximum number of saddle points that can lie on the root locus for $G(s)$ is one less than the number of distinct finite poles plus the number of distinct finite zeros of $G(s)$. That is, if S is the number of saddle points (counting multiplicities) on the

root locus,

$$S \leq P+Z-1 \tag{47}$$

Proof: The coefficient of s^{Z+P-1} when fractions are cleared in Equation 46. will be

$$\sum_{j=1}^{Z} n_j - \sum_{j=1}^{P} d_j = n - d \neq 0 \tag{48}$$

Therefore, Equation 45. will be of degree P+Z-1, and G(s) will have P+Z-1 finite critical points, any number of which may be saddle points on the root locus. Note that Equation 46. may have multiple roots, in which case the root locus may have a multiple saddle point. The behavior of the root locus at a multiple saddle point is given by Theorem 9., since a saddle point of order **r** corresponds to the first **r** derivatives of G vanishing.

Theorem 11. If G(s) has neither a pole nor a zero at infinity, but if the center of gravity of the poles does not coincide with the center of gravity of the zeros; that is, if $n = d$, but

$$\sum_{j=1}^{Z} n_j z_j \neq \sum_{j=1}^{P} d_j p_j \tag{49}$$

then

$$S \leq P+Z-2 \tag{50}$$

Proof: By the above proof, the coefficient of s^{P+Z-1} will vanish in the equation whose solutions give the critical points of G(s), and the degree of this equation will be of degree P+Z-2 or less. Moreover, if the coefficient of s^{P+Z-2} in Equation 46. is computed, it is found to be

$$\sum_{j=1}^{Z} k_j z_j - \sum_{j=1}^{P} d_j p_j \tag{51}$$

Thus, if this coefficient does not vanish, G(s) has P+Z-2 finite critical points, and the proof is complete. In the case of Theorem 11. G(s) can be considered to have a saddle point at infinity. If the center of gravity of the poles and zeros do coincide, then G(s) has at least two saddle points at infinity, and a more complicated condition determines the number of finite critical points of G(s)

We shall now take up the interpretation of the preceding theorems in terms of the contour map of G(s). At a single order saddle point of G(s), the streamline enters the point from two opposite directions and leaves perpendicular to these directions. This is illustrated in Figure 2. Thus, the contour map in the vicinity of such a point will be saddle-shaped, whence the name "saddle point". The following argument, in slightly different terms, is presented by James C. Maxwell[10] in an article called "On Hills and Dales".

Consider that we have a contour map with P distinct finite poles, Z distinct finite zeros, and a pole or a zero at infinity. Let us start with a plane resting on the $|G| = 0$ level and lift it slightly. We shall now have a small level curve surrounding each zero (including the zero at infinity, if there is one). As the plane is lifted gradually there are two ways in which a saddle point will be formed, leading to two types of saddle points:

Type 1. (Bar) A bar will be formed when two of the above regions of depression merge. Such a situation is illustrated in Figure 10.

Figure 10. (a) The level curves at the formation of a bar.
(b) The high and low areas around the bar.

Type 2. (Pass) A pass will be formed when two arms from one region of depression meet to isolate a region of high ground. For a rational function this region of high ground must lead to a pole. Figure 11. shows this situation.

(a) (b)

Figure 11. (a) The level curves at the formation of a pass.
(b) The high and low areas around the pass.

Suppose first that we have a pole at infinity. Then as the plane is lifted to infinity, there will be formed Z-1 bars and P passes. If there is a zero at infinity, there will be formed Z bars and P-1 passes. Thus, there will be in all P+Z-1 finite saddle points; in agreement with Theorem 10.

The distinction between a bar and a pass is not always clear. Consider, for example the function:

$$G(s) = \frac{(s-\alpha)(s+\alpha)}{s} \tag{52}$$

This function has saddle points given by

$$\frac{d}{ds} \ln G(s) = \frac{1}{s-\alpha} + \frac{1}{s+\alpha} - \frac{1}{s} = 0 \tag{53}$$

or

$$s^2 + \alpha^2 = 0 \tag{54}$$

Thus, G(s) has saddle points at $\pm j\alpha$, and the contour map is similar to that shown in Figure 12. If one of these two saddle points is considered to have been formed slightly before the other as the plane is lifted, the first becomes a bar and the second a pass. Note that in this case the saddle points do not lie on the root locus, but lie rather on the 90^0-locus and are at the intersections of the circle and the line that are this locus.

Another interpetation of saddle points is that of Gauss as presented in Walsh[22] and Marden[9]. Here the "force field" of a rational function is constructed by placing a point mass $+n_j$ at each zero and a point mass $-d_j$ at each pole. Then if the force at any point due to a mass is inversely proportional to the distance to the mass and proportional to the size of the mass, the force at any point is given by

$$\text{FORCE} \propto \sum_{j=1}^{Z} \frac{n_j}{\bar{s}-\bar{z}_j} - \sum_{j=1}^{P} \frac{d_j}{\bar{s}-\bar{p}_j} \tag{55}$$

This is just the conjugate of Equation 46., so that the condition that a point be a saddle point is equivalent to the condition that it

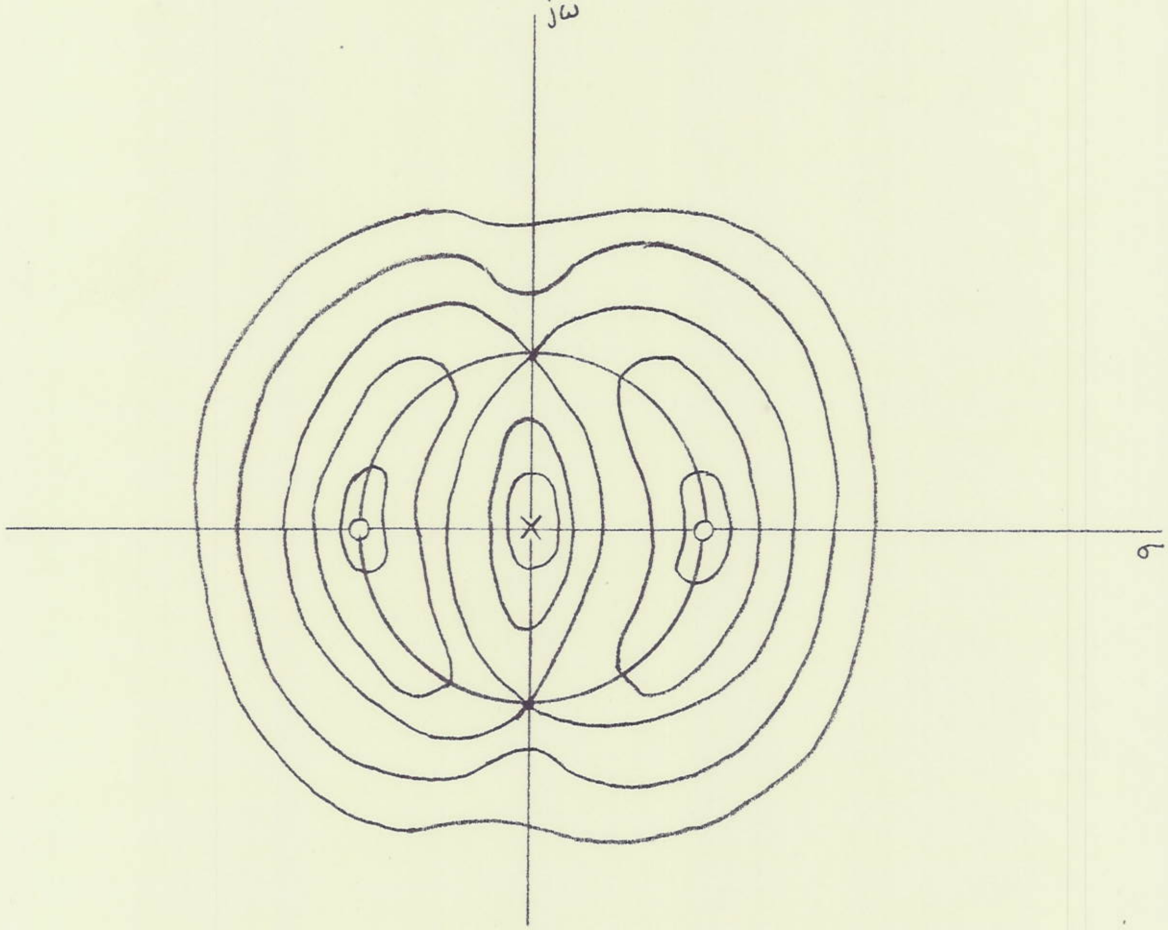


Figure 12. The contour map for the closed-loop function

$$G(s) = \frac{(s-\alpha)(s+\alpha)}{s}$$

with the 90°-locus shown.

be a point of equilibrium in the above force field. This fact is useful in visualizing the possible locations for critical points.

The force field interpretation is the usual point of departure for the study of locations of critical points. The author has found, however, that in so far as the root-locus problem is concerned, the contour map interpretation is also useful. The preceding discussion, in fact, leads naturally to the following pair of theorems, which seem to be original with the author.

Theorem 12. If the portion of the root locus between two consecutive zeros (poles) is traced, there must be traversed an odd number of saddle points, a saddle point being counted only if traversed so as to subtend an angle which is an integral multiple of $360^\circ/(r+1)$ at an r^{th} order saddle point. Besides this condition the saddle points are counted without regard to their multiplicities.

Proof: Consider a section of a root locus between two zeros. On this $G(s) = \pm |G(s)|$ is a real continuous function of distance along the locus, that does not change sign or become infinite. Therefore on the locus between two zeros, $G(s)$ must have an odd number of peaks and troughs at which $dG(s)/ds$ changes sign. These peaks and troughs correspond to saddle points traversed in the manner stated in the theorem. It is also apparent that if we travel along a line of steepest descent (or ascent) and encounter a minimum (or maximum) height, then we must be at a saddle point. To prove the theorem for the locus between two poles, we need just consider $1/G(s)$, which has the same locus as $G(s)$.

The proof of the following theorem is parallel to the above proof.

<u>Theorem 13</u>. If the portion of the root locus between a zero and a pole is traced, there must be traversed an even (0 included) number of saddle points, a saddle point being counted only if traversed so as to subtend an angle which is an integral multiple of $360^\circ/(r+1)$ at an r^{th} order saddle point. As before, the saddle points are counted without regard to their multiplicities.

Another way of determining when saddle points are counted is to note the direction of increasing gain on the branches of the root locus. We count a saddle point only if the direction of increasing gain on our path changes as we traverse the saddle point. Consider, for example, the root locus at a second order saddle point as shown in Figure 13.

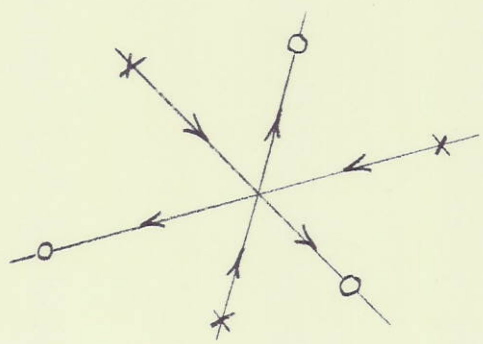


Figure 13. A second order saddle point.

The locus consists of three lines meeting with 60° between them. The poles and zeros on the branches must alternate if there are no other saddle points on the portions of the branches shown.

8. THE EQUATIONS OF ROOT LOCI IN POLAR COORDINATES

We turn now to finding the general equations of root loci, in polar coordinates in this section and in Cartesian coordinates in the next section. We might be tempted to start with the factored form of the open-loop function. Experience has shown, however, that since the terms in the equations will be symmetric functions of the open-loop roots, it will more convenient to write the root loci equations in terms of the coefficients in N(s) and D(s). These coefficients are themselves symmetric functions of the roots[16], and the resultant equations will be more concise and easier to use.

G(s) can be written in polar form by replacing s by $Re^{j\theta}$. Thus, we have

$$G(s) = \frac{\sum_{k=0}^{n} a_k s^k}{\sum_{l=0}^{d} b_l s^l} = \frac{\sum_{k=0}^{n} a_k R^k \epsilon^{jk\theta}}{\sum_{l=0}^{d} b_l R^l \epsilon^{jl\theta}} \tag{56}$$

In order that the numbering scheme be apparent in the following expressions, we shall write a_n and b_d instead of unity. Rationalizing Equation 56. by multiplying by the conjugate of the denominator:

$$G(s) = \frac{\left(\sum_{k=0}^{n} a_k R^k \epsilon^{jk\theta}\right)\left(\sum_{l=0}^{d} b_l R^l \epsilon^{-jl\theta}\right)}{\left(\sum_{l=0}^{d} b_l R^l \epsilon^{jl\theta}\right)\left(\sum_{l=0}^{d} b_l R^l \epsilon^{-jl\theta}\right)} = \frac{\sum_{k=0}^{n}\sum_{l=0}^{d} a_k b_l R^{k+l} \epsilon^{j(k-l)\theta}}{\left|\sum_{l=0}^{d} b_l R^l \epsilon^{jl\theta}\right|^2} \tag{57}$$

We may now set the imaginary part of G(s) to zero as in Equation 6. to obtain the equation of the root locus:

$$\sum_{k=0}^{n} \sum_{l=0}^{d} a_k b_l R^{k+l} \sin(k-l)\theta = 0 \tag{58}$$

The terms in the left member of this equation may be arranged in a matrix with n+1 rows and d+1 columns as shown in Figure 14. (a).

Since the real axis will always be part of the locus, $R\sin\theta$ will always be a factor of Equation 58. The equation for the root locus may then be rewritten:

$$\sum_{k=0}^{n} \sum_{l=0}^{d} a_k b_l R^{k+l-1} \left[\frac{\sin(k-l)\theta}{\sin\theta} \right] = 0 \tag{59}$$

Since the locus will have conjugate symmetry, this expression must be even in θ. Therefore, there can be no terms involving $\sin\theta$ to an odd power, and the equation must be algebraic in $\cos\theta$ and R. We may in fact recognize the trigonometric functions in Equation 59. as Chebyshev polynomials of the second kind as defined in the tables of Chebyshev polynomials issued by the National Bureau of Standards[13]:

$$U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta} \tag{60}$$

$$
\left.
\overbrace{
\begin{array}{cccccc}
0 & -a_0 b_1 R \sin\theta & -a_0 b_2 R^2 \sin 2\theta & -a_0 b_3 R^3 \sin 3\theta & -a_0 b_4 R^4 \sin 4\theta & \cdots \\
+a_1 b_0 R \sin\theta & 0 & -a_1 b_2 R^3 \sin\theta & -a_1 b_3 R^4 \sin 2\theta & -a_1 b_4 R^5 \sin 3\theta & \cdots \\
+a_2 b_0 R^2 \sin 2\theta & +a_2 b_1 R^3 \sin\theta & 0 & -a_2 b_3 R^5 \sin\theta & -a_2 b_4 R^6 \sin 2\theta & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
}^{d+1 \text{ columns}}
\right\} n+1 \text{ rows}
$$

(a)

$$
\left.
\overbrace{
\begin{array}{cccccc}
0 & -a_0 b_1 & -a_0 b_2 R U_1(\cos\theta) & -a_0 b_3 R^2 U_2(\cos\theta) & -a_0 b_4 R^3 U_3(\cos\theta) & \cdots \\
+a_1 b_0 & 0 & -a_1 b_2 R^2 & -a_1 b_3 R^3 U_1(\cos\theta) & -a_1 b_4 R^4 U_2(\cos\theta) & \cdots \\
+a_2 b_0 R U_1(\cos\theta) & +a_2 b_1 R^2 & 0 & -a_2 b_3 R^4 & -a_2 b_4 R^5 U_1(\cos\theta) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
}^{d+1 \text{ columns}}
\right\} n+1 \text{ rows}
$$

(b)

Figure 14. (a) A matrix, the sum of whose terms set equal to zero gives the root locus.

(b) The same matrix with $R\sin\theta$ factored out and in terms of Chebyshev polynomials of the first kind.

Thus the locus may be written in terms of these polynomials as

$$\sum_{k=0}^{n} \sum_{l=0}^{d} a_k b_l R^{k+l-1} U_{k-l-1}(\cos\theta) = 0 \tag{61}$$

where

$$U_{-1} = 0$$

The corresponding matrix is shown if Figure 14. (b). The above reference gives explicitly the first twelve polynomials $U_n(x)$, and will be convenient to use for writing the explicit equations of the root loci for higher order systems. $U_n(x)$ is of degree n in x and contains only even powers of x if n is even, and only odd powers of x if n is odd.

These tables of the National Bureau of Standards[13] might also be useful in numerical computations. For instance, if cosθ is prescribed, the U_n can be looked up in tables, and Equation 61. will become, in general, a polynomial in R of degree n+d-1. The solutions of this equation locate points on the locus along the radius vector at the prescribed angle. Such a technique might be employed in situations where the damping ratio, which is directly dependent on the angle θ, is a design criterion[11,12].

We shall now illustrate the preceding by finding the locus for the function

$$G(s) = \frac{(s-\epsilon^{j60^\circ})(s-\epsilon^{-j60^\circ})(s-\epsilon^{j30^\circ})(s-\epsilon^{-j30^\circ})}{s^2} \tag{62}$$

Which has a double pole at the origin and four zeros on the unit circle at $\pm 30^\circ$ and $\pm 60^\circ$. We note first that the unit circle is the root locus for the functions

$$\left.\begin{aligned} G_1(s) &= \frac{(s-\epsilon^{j60^\circ})(s-\epsilon^{-j60^\circ})}{s} \\ \text{and} \quad G_2(s) &= \frac{(s-\epsilon^{j30^\circ})(s-\epsilon^{-j30^\circ})}{s} \end{aligned}\right\} \tag{63}$$

whose product is G(s). Therefore, by Theorem 8., the unit circle must also be part of the locus for G(s). By Theorem 4., the asymptotic center will be at

$$\sigma_\infty = \frac{1}{2}\left(\sqrt{3}+1\right) = 1.366 \tag{64}$$

and the asymptote will be perpendicular to the real axis. There will be two saddle points at ± 1, corresponding to the intersection of the real axis with the unit circle. To find the other saddle points, we may employ Equation 46.:

$$\frac{1}{s-\epsilon^{j60^\circ}} + \frac{1}{s-\epsilon^{-j60^\circ}} + \frac{1}{s-\epsilon^{j30^\circ}} + \frac{1}{s-\epsilon^{-j30^\circ}} - \frac{2}{s} = 0 \qquad (65)$$

Multiplying out and factoring, we get

$$(s^2-1)\left(s^2 - \frac{1+\sqrt{3}}{2}s + 1\right) = 0 \qquad (66)$$

By Theorem 10. there can be at most four saddle points on the locus, so that by Theorem 12. there must be one saddle point between each of the two complex zeros on the root locus. Thus is verified by Equation 66., which gives these saddle points as lying on the unit circle with projections on the real axis midway between the projections of the zeros. Finally, G(s) may be expanded to find the a_i and b_i :

$$G(s) = \frac{s^4 - (1+\sqrt{3})s^3 + 2(1+\frac{\sqrt{3}}{2})s^2 - (1+\sqrt{3})s + 1}{s^2} \qquad (67)$$

and the equation of the locus written from the matrix in Figure 14. (b). The only non-zero coefficient in the denominator is $b_2=1$, and the locus is

$$2a_0 R\cos\theta + a_1 R^2 - a_3 R^4 - 2a_4 R^5 \cos\theta = 0 \qquad (68)$$

When the a_i are substituted into this equation, and the unit circle factored out, we have

$$R(R^2-1)\left(R^2\cos\theta - R\frac{1+\sqrt{3}}{2} + \cos\theta\right) = 0 \tag{69}$$

The factor of R is due to the fact that besides the real axis there is another branch of the locus passing through the origin, because of the double pole.

Thus, besides the real axis, and the unit circle, we have the locus:

$$R^2\cos\theta - R\frac{1+\sqrt{3}}{2} + \cos\theta = 0 \tag{70}$$

It is interesting to note that if R is replaced by $1/R$, and θ by $-\theta$, this locus is unchanged, so that the root locus for G(s) is invariant under inversion with respect to the unit circle. This might have been expected, since the inversion of the original open loop function G(s) is in this case just $1/G(s)$, which has the same root locus as G(s). The locus of Equation 70. is most conveniently plotted by solving for $R\cos\theta = \sigma$ as a function of R:

$$\sigma = R\cos\theta = \frac{R^2}{1+R^2}\left(\frac{1+\sqrt{3}}{2}\right) = \frac{R^2}{1+R^2}\,\sigma_\infty \tag{71}$$

The entire root locus is shown in Figure 15.

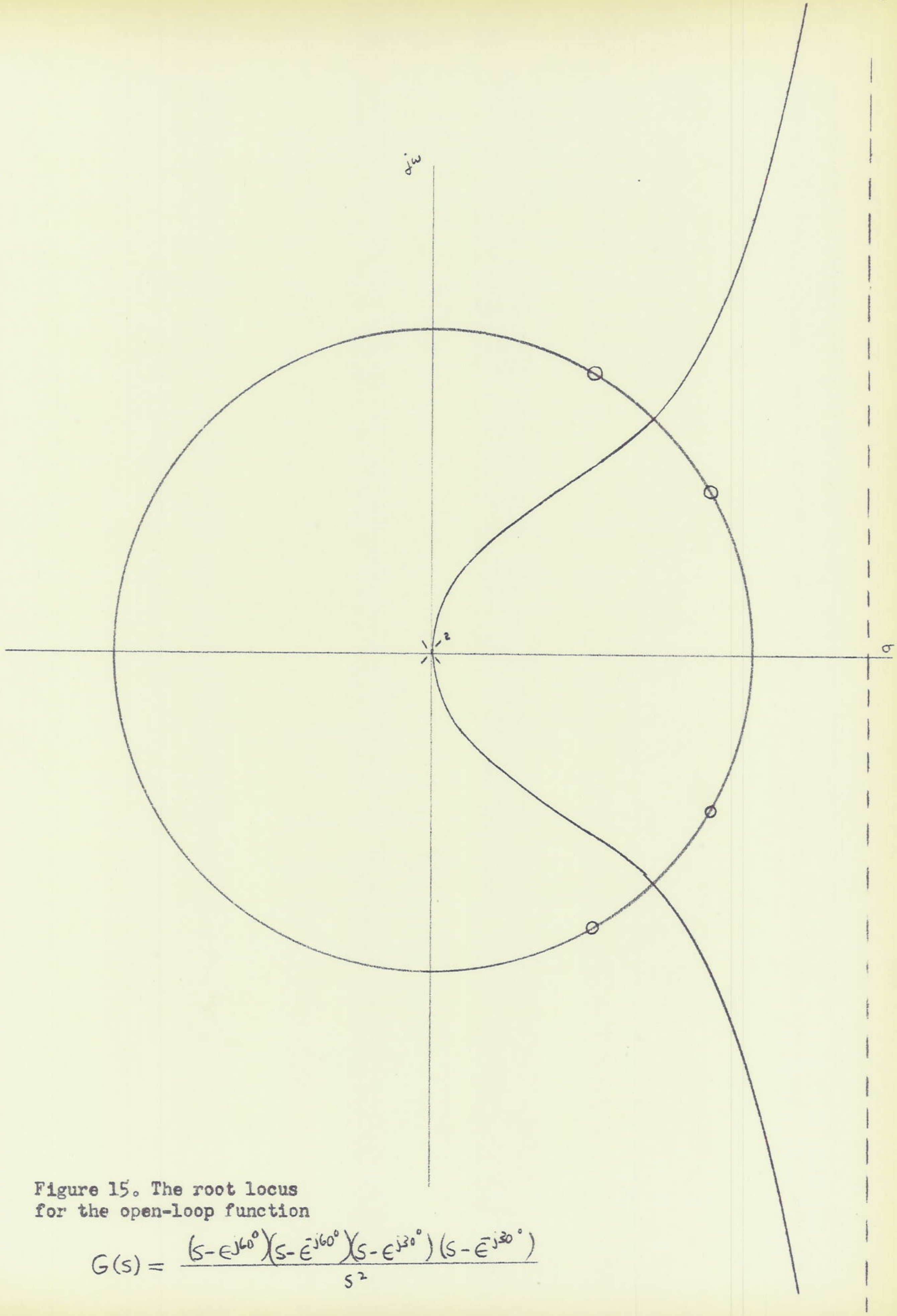


Figure 15. The root locus
for the open-loop function

$$G(s) = \frac{(s-\epsilon^{j60^\circ})(s-\epsilon^{-j60^\circ})(s-\epsilon^{j30^\circ})(s-\epsilon^{-j30^\circ})}{s^2}$$

We have thus given a procedure for finding the equation of any root locus in polar coordinates. Thus far the problem of calibrating this curve in terms of K has been ignored, and we shall postpone this problem further, to the time when we have also found the Cartesian equations for the loci.

9. THE EQUATIONS OF ROOT LOCI IN CARTESIAN COORDINATES

To find the root loci equations in terms of σ and ω, we shall use an idea that was suggested by Bendrikov and Teodorchik[2] in a Russian journal, but which seems not to have been carried out. The idea is that of expanding the N(s) and D(s) in a power series in $j\omega$.

N(s) is analytic everywhere in the s-plane, and can indeed be expanded in a Taylor series about any σ with an infinite radius of convergence to obtain the identity:

$$N(\sigma + j\omega) = N(\sigma) + j\omega \frac{N'(\sigma)}{1!} + (j\omega)^2 \frac{N''(\sigma)}{2!} + \cdots + (j\omega)^n \frac{N^{(n)}(\sigma)}{n!} \tag{72}$$

Grouping the real and imaginary components of this expression, we have

$$N(\sigma + j\omega) = \left[N(\sigma) - \omega^2 \frac{N''(\sigma)}{2!} + \cdots\right] + j\omega\left[\frac{N'(\sigma)}{1!} - \omega^2 \frac{N'''(\sigma)}{3!} + \cdots\right] \tag{73}$$

When this is also done for D(s), G(s) may be written:

$$G(s) = \frac{\left[N(\sigma) - \omega^2 \frac{N''(\sigma)}{2!} + \cdots\right] + j\omega\left[\frac{N'(\sigma)}{1!} - \omega^2 \frac{N'''(\sigma)}{3!} + \cdots\right]}{\left[D(\sigma) - \omega^2 \frac{D''(\sigma)}{2!} + \cdots\right] + j\omega\left[\frac{D'(\sigma)}{1!} - \omega^2 \frac{D'''(\sigma)}{3!} + \cdots\right]} \tag{74}$$

Multiplying by the conjugate of the denominator, and setting the imaginary part equal to zero, as before, we have as the equation of the root locus:

$$\left[N(\sigma) - \omega^2 \frac{N''(\sigma)}{2!} + \cdots\right]\left[\frac{D'(\sigma)}{1!} - \omega^2 \frac{D'''(\sigma)}{3!} + \cdots\right] - \left[\frac{N'(\sigma)}{1!} - \omega^2 \frac{N'''(\sigma)}{3!} + \cdots\right]\left[D(\sigma) - \omega^2 \frac{D''(\sigma)}{2!} + \cdots\right] = 0 \tag{75}$$

We shall now collect terms in like powers of ω^2:

$$\left[\frac{N(\sigma)}{0!}\frac{D'(\sigma)}{1!} - \frac{N'(\sigma)}{1!}\frac{D(\sigma)}{0!}\right] - \omega^2\left[\frac{N(\sigma)}{0!}\frac{D'''(\sigma)}{3!} - \frac{N'(\sigma)}{1!}\frac{D''(\sigma)}{2!} + \frac{N''(\sigma)}{2!}\frac{D'(\sigma)}{1!} - \frac{N'''(\sigma)}{3!}\frac{D(\sigma)}{0!}\right]$$

$$+\omega^4\left[\frac{N(\sigma)}{0!}\frac{D^{v}(\sigma)}{5!} - \frac{N'(\sigma)}{1!}\frac{D^{iv}(\sigma)}{4!} + \frac{N''(\sigma)}{2!}\frac{D'''(\sigma)}{3!} - \frac{N'''(\sigma)}{3!}\frac{D''(\sigma)}{2!} + \frac{N^{iv}(\sigma)}{4!}\frac{D'(\sigma)}{1!} - \frac{N^{v}(\sigma)}{5!}\frac{D(\sigma)}{0!}\right] \tag{76}$$

$$-\omega^6\left[\quad \cdots \right. \qquad = 0$$

The coefficients of powers of ω in this equation will be polynomials in σ, and it will be convenient to introduce the following notation for these polynomials:

$$Q_R = \sum_{\nu=0}^{R} (-1)^{\nu} \frac{N^{(\nu)}(\sigma)}{\nu!} \frac{D^{(R-\nu)}(\sigma)}{(R-\nu)!} \tag{77}$$

The root locus is then

$$Q_1(\sigma) - \omega^2 Q_3(\sigma) + \omega^4 Q_5(\sigma) - \cdots = 0 \tag{78}$$

The equation of the root locus for any rational function may be written in the form of Equation 78., so that it will be desirable to learn as much about the σ-polynomials as possible.

Let us first consider $Q_1(\sigma)$. Writing the first two terms of $Q_1(\sigma)$, we have

$$Q_1(\sigma) = N(\sigma)D'(\sigma) - N'(\sigma)D(\sigma)$$
$$= (d-n)\sigma^{n+d-1} + \left[(d-n)(b_{d-1}+a_{n-1}) + (a_{n-1}-b_{d-1})\right]\sigma^{n+d-2} \tag{79}$$
$$+ \ldots$$

If $d \neq n$, $Q_1(\sigma)$ is of degree $(d+n-1)$ with a leading coefficient of $(d-n)$. Similarly, if $d=n$, but if $a_{n-1} \neq b_{d-1}$, $Q_1(\sigma)$ is of degree $d+n-2$, with a leading coefficient of $(a_{n-1}-b_{d-1})$. It can be seen from the first part of Equation 79. that if $N(s)$ or $D(s)$ has a multiple zero, then $Q_1(\sigma)$ will have a zero of one lower order at that point. This accounts for $(n-Z) + (d-P)$ zeros of $Q_1(\sigma)$. To find the other roots we need only consider the logarithmic derivative of $G(s)$, whose zeros are the finite saddle points of $G(s)$:

$$\frac{d}{ds}\left[\ln \frac{N(s)}{D(s)}\right] = \frac{N'(s)}{N(s)} - \frac{D'(s)}{D(s)} = 0 \tag{80}$$

Every root of this equation also satisfies the equation

$$N(s)\,D'(s) - N'(s)\,D(s) = 0 \tag{81}$$

and the finite saddle points are therefore roots of $Q_1(\sigma)$. These account for all the zeros of $Q_1(\sigma)$, because if $n \neq d$, we have $P+Z-1$ finite saddle points and $(n-Z) + (d-P)$ zeros due to repeated roots of $G(s)$, making a total of $n+d-1$, as above. If $n=d$, but $a_{n-1} \neq b_{d-1}$, there are P+Z-2 finite saddle points which means that $Q_1(\sigma)$ is of degree n+d-2, as above. In general, it can be seen that the degree of $Q_1(\sigma)$ and the number of finite saddle points are determined by the same conditions, so the saddle points and multiple zeros and poles of G(s) will always account for all the zeros of $Q_1(\sigma)$. Thus, $Q_1(\sigma)$ has zeros of appropriate orders at all the finite saddle points of G(s), whether they are on the root locus or not; and at the multiple poles and zeros of G(s). We can now write $Q_1(\sigma)$ in factored form. If $d \neq n$,

$$Q_1(\sigma) = (d-n)(\sigma - s_{k1})(\sigma - s_{k2}) \cdots (\sigma - s_{k(n+d-1)}) \tag{82}$$

where the s_k may be complex and are the zeros of the logarithmic derivative (saddle points), and the multiple zeros and poles of G(s). If d=n, but $a_{n-1} \neq b_{d-1}$,

$$Q_1(\sigma) = (a_{n-1} - b_{d-1})(\sigma - s_{k1}) \cdots (\sigma - s_{k(n+d-2)}) \tag{83}$$

Higher order saddle points are, of course, represented in Equations 82. and 83. by higher order factors.

In general, $Q_r(\sigma)$ will be of degree d+n-~~1~~r. It is evident from Equation 77. that

$$Q_{n+d}(\sigma) = (-1)^n \tag{84}$$

$Q_{n+d-1}(\sigma)$ will have two terms:

$$Q_{n+d-1}(\sigma) = (-1)^{n-1}\frac{\overset{(n-1)}{N}(\sigma)}{(n-1)!}\frac{\overset{(d)}{D}(\sigma)}{d!} + (-1)^{n}\frac{\overset{(n)}{N}(\sigma)}{n!}\frac{\overset{(d-1)}{D}(\sigma)}{(d-1)!} \tag{85}$$

or

$$Q_{n+d-1}(\sigma) = (-1)^{n}\left[d\sigma + b_{d-1} - n\sigma - a_{n-1}\right] \tag{86}$$

or, if $d \neq n$:

$$Q_{n+d-1}(\sigma) = (-1)^{n}(d-n)\left[\sigma - \frac{a_{n-1} - b_{d-1}}{d-n}\right] \tag{87}$$

a_{n-1} is the negative of the sum of the zeros of the numerator, and b_{d-1} is the negative of the sum of the zeros of denominator; all the roots being weighted according to their multiplicity. Thus, by Equation 19. we may write that

$$Q_{n+d-1}(\sigma) = (-1)^{n}(d-n)(\sigma - \sigma_{\infty}) \qquad \text{if } d \neq n \tag{88}$$

and

$$Q_{n+d-1}(\sigma) = (-1)^{n}(b_{d-1} - a_{n-1}) \qquad \begin{array}{l}\text{if } d = n\\ \text{but } a_{n-1} \neq b_{d-1}\end{array}$$

Of course, only one of Q_{n+d} and Q_{n+d-1} will appear in the equation of the locus, depending on whether n+d or n+d-1 is odd.

The lower order loci may now be written in terms of the critical points with little more effort. The circular loci, T(1,2) and T(2,2) reduce to

$$\omega^2 + (\sigma - s_{k1})(\sigma - s_{k2}) = 0 \tag{89}$$

which is obviously the equation of a circle which intersects the real axis at s_{k1} and s_{k2}. The locus T(0,3) is

$$Q_1(\sigma) - \omega^2 Q_3(\sigma) = 0 \tag{90}$$

and

$$\begin{aligned} Q_1(\sigma) &= 3(\sigma - s_{k1})(\sigma - s_{k2}) \\ Q_3(\sigma) &= 1 \end{aligned} \tag{91}$$

so that the locus T(0,3) is

$$3(\sigma - s_{k1})(\sigma - s_{k2}) - \omega^2 = 0 \tag{92}$$

which is the equation of a hyperbola, as was mentioned in Section 6.

The loci T(1,3) and T(0,4) are the only cubic loci, and represent the next order of complexity after the quadratic loci. For these loci, Q_5 is zero, and the highest power of ω is ω^2. Thus, any line $\sigma = \text{const.}$ can intersect the locus in at most a pair of points. Let us consider these cubic loci in detail.

By Equation 78, the T(1,3) locus is of the form

$$Q_1(\sigma) - \omega^2 Q_3(\sigma) = 0 \tag{93}$$

Furthermore,

$$
\begin{aligned}
Q_1(\sigma) &= 2(\sigma - s_{k1})(\sigma - s_{k2})(\sigma - s_{k3}) \\
Q_3(\sigma) &= -2(\sigma - \sigma_\infty)
\end{aligned}
\tag{94}
$$

Thus, the locus is

$$
\omega^2(\sigma - \sigma_\infty) + (\sigma - s_{k1})(\sigma - s_{k2})(\sigma - s_{k3}) = 0 \tag{95}
$$

or

$$
\omega^2 = -\frac{(\sigma - s_{k1})(\sigma - s_{k2})(\sigma - s_{k3})}{(\sigma - \sigma_\infty)} \tag{96}
$$

The right member of this equation must be positive for the locus to exist; so that is the s_k are all real, the non-real locus exists for alternate intervals of the real axis, these intervals having as end points the s_k and the asymptotic center.

Equation 96. suggests a semi-graphical technique for finding this cubic locus. First the critical points and asymptotic center are located, and portions of the σ-axis for which the locus exists are determined. A point on the real axis is then chosen and the three distances from this point to the three critical points are measured. The product of these three distances is then divided by the distance to the asymptotic center to give ω^2 corresponding to the chosen value of σ.

If all the poles and zeros are real, the loci T(1,3) can be broken down in two cases: the case when the zero lies outside of the three poles; and the case when the zero lies between the poles.

In the first case, by Theorem 12. there are three real critical points: two between the poles and one between the finite zero and the zero at infinity. The locus must then be shown in Figure 16. (a), with an oval on the 0°-branch. If the zero lies between two poles, on the other hand, the locus may have an oval or may not. The two possibilities and the limiting case are sketched in Figure 16. (b), (c), and (d). Here Theorem 8. is useful; a circle and three lines can be drawn which all represent barriers to the locus. Examples of these loci and some other higher order loci are sketched in Yeh[24], and in chapter 21 of <u>The Handbook of Automatation, Computation and Control</u>[18].

The other cubic locus, T(0,4), may be found as above, with the only difference in the result being the sign of one term:

$$\omega^2(\sigma-\sigma_\infty) - (\sigma - S_{k1})(\sigma - S_{k2})(\sigma - S_{k3}) = 0 \tag{97}$$

This curve is susceptible to the same analysis as that given the T(1,3) loci. In fact, given the critical points and the asymptotic center, the curve can be found with the same semi-graphical technique, except that the locus exists for the complementary segments of the real axis. Some examples of these loci can be found in Yeh[24], Evans[4], <u>et. al.</u>, and will not be given here.

Figure 17. shows a tabulation of the first fifteen loci, with the polynomials in σ of degree r represented by the symbol P_r. The

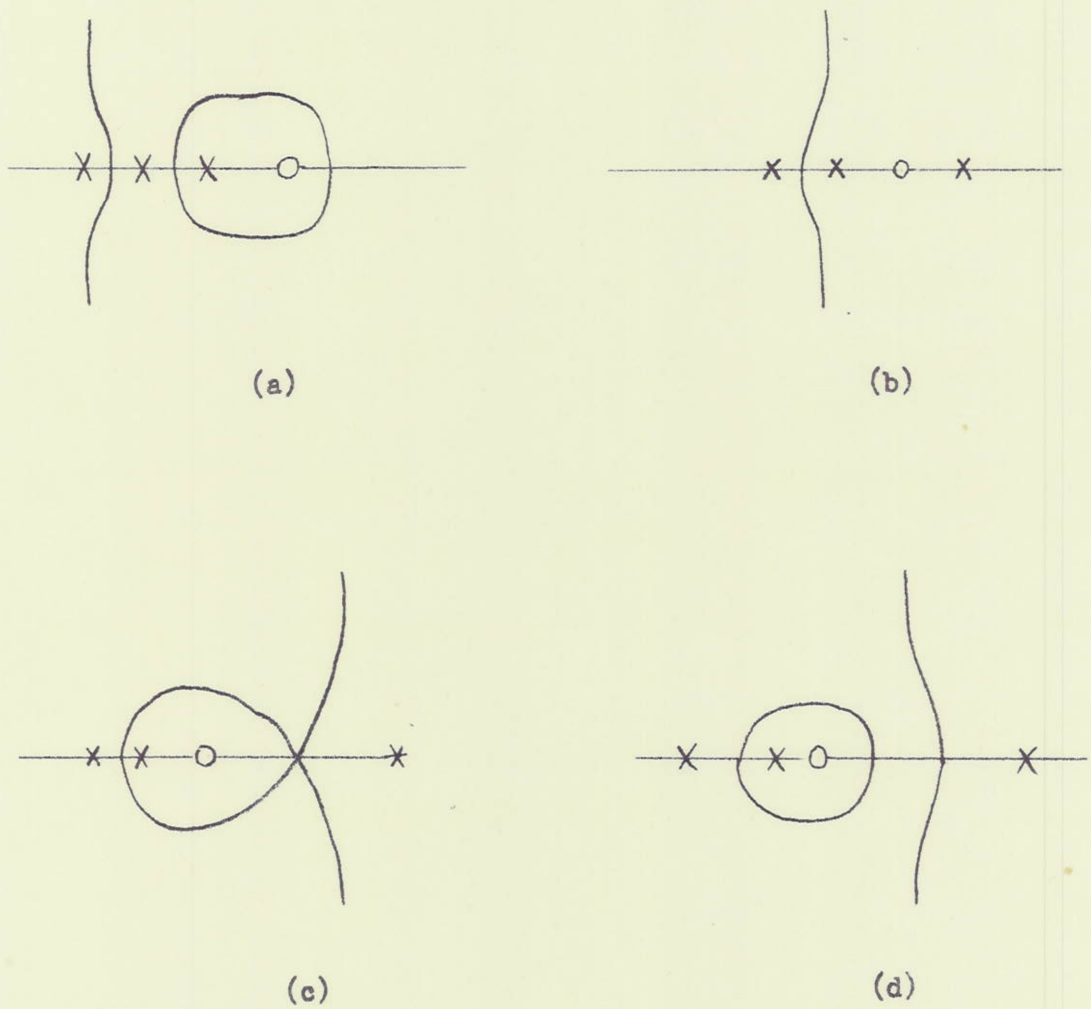


Figure 16. The four possible T(1,3) loci with real poles and zeros.
(Complex pole-zero patterns with these loci can be found
by applying Theorem 7.)

$T(0,1)$	only real axis	
$T(1,1)$	only real axis	
$T(0,2)$	$\sigma = S_K$	line
$T(1,2)$	$\omega^2 + (\sigma - S_{K_1})(\sigma - S_{K_2}) = 0$	circle
$T(2,2)$	$\omega^2 + (\sigma - S_{K_1})(\sigma - S_{K_2}) = 0$	circle
$T(0,3)$	$\omega^2 - 3(\sigma - S_{K_1})(\sigma - S_{K_2}) = 0$	hyperbola
$T(1,3)$	$\omega^2(\sigma - \sigma_\infty) + (\sigma - S_{K_1})(\sigma - S_{K_2})(\sigma - S_{K_3}) = 0$	cubic
$T(2,3)$	$P_4 + P_2\omega^2 + \omega^4 = 0$	quartic
$T(3,3)$	$P_4 + P_2\omega^2 + \omega^4 = 0$	quartic
$T(0,4)$	$\omega^2(\sigma - \sigma_\infty) - (\sigma - S_{K_1})(\sigma - S_{K_2})(\sigma - S_{K_3}) = 0$	cubic
$T(1,4)$	$P_4 + P_2\omega^2 + \omega^4 = 0$	quartic
$T(2,4)$	$P_5 + P_3\omega^2 + P_1\omega^4 = 0$	quintic
$T(3,4)$	$P_6 + P_4\omega^2 + P_2\omega^4 + \omega^6 = 0$	sextic
$T(4,4)$	$P_6 + P_4\omega^2 + P_2\omega^4 + \omega^6 = 0$	sextic
$T(0,5)$	$P_4 + P_2\omega^2 + \omega^4 = 0$	quartic

Figure 17. The first fifteen root loci; P_r is a polynomial in σ of degree r.

equations of the lower order loci as found above are also given in this table. Yeh[24] gives the explicit equations of the lower order loci, but to the author's knowledge these equations have not before been expressed in terms of the critical points, as in the equations in Figure 17.

It may happen that, under some conditions of symmetry, a locus decomposes into the product of lower order curves. Consider, for example, a pole-zero configuration that is symmetric in a line $\sigma = \sigma'$. (We require also that there be an even number of poles and an even number of zeros exactly on the line $\sigma = \sigma'$). Then it is clear that at any point on the line the net phase of the open-loop function must be some multiple of 180^0. Alternatively, we may argue as in Theorem 8. that G(s) is the product of real factors, and hence must be real. The line $\sigma = \sigma'$ is then part of the locus, and $(\sigma - \sigma')$ must be a factor of the equation of the locus. In the next section, we shall encounter another example of the factoring of higher order loci into low order factors.

It is often of interest to find the intersection of the root locus with the $j\omega$-axis. Since poles in the right half plane will lead to unstable behavior of a system, these points represent critical values of gain, for which the system is on the threshold of stability. To find these points, we shall let $\sigma = 0$ in Equation 76. recognizing that

$$\left.\begin{aligned} \frac{N^{(\nu)}(0)}{\nu!} &= a_\nu \\ \text{and} \qquad \frac{D^{\nu}(0)}{\nu!} &= b_\nu \end{aligned}\right\} \qquad (98)$$

we have:

$$(a_0 b_1 - a_1 b_0) - \omega^2 (a_0 b_3 - a_1 b_2 + a_2 b_1 - a_3 b_0) + \omega^4 (a_0 b_5 - a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1 - a_5 b_0) - \cdots = 0 \tag{99}$$

The real solutions of this equation will give the values of ω at which the locus crosses the $j\omega$-axis.

We shall now take up the question of determining the gain constant K at a point on the locus. We may of course calculate the value of G(s) at the point, and recognize that

$$K = -\frac{1}{G(s)} \tag{100}$$

The following simplification, however, has been pointed out by Sollecito and Reque[18], and by Andeyick[1]. As in Equation 73. we may write N(s) as $N_R + j\omega N_I$ where N_R and N_I are real polynomials in σ and ω. We may write D(s) similarly, and Equation 100. becomes

$$K = -\frac{D_R + j\omega D_I}{N_R + j\omega N_I} \tag{101}$$

K is real, so that the real and the imaginary parts of these expressions must both be in the proportion-K. Thus

$$K = -\frac{D_R}{N_R} = -\frac{D_I}{N_I} \tag{102}$$

We may now write K in terms of the coefficients in the open-loop transfer function. First, in terms of σ and ω :

$$K = -\frac{\frac{D(\sigma)}{0!} - \omega^2 \frac{D''(\sigma)}{2!} + \omega^4 \frac{D^{iv}(\sigma)}{4!} - \cdots}{\frac{N(\sigma)}{0!} - \omega^2 \frac{N''(\sigma)}{2!} + \omega^4 \frac{N^{iv}(\sigma)}{4!} - \cdots}$$

$$= -\frac{\frac{D'(\sigma)}{1!} - \omega^2 \frac{D'''(\sigma)}{3!} + \omega^4 \frac{D^{v}(\sigma)}{5!} - \cdots}{\frac{N'(\sigma)}{1!} - \omega^2 \frac{N'''(\sigma)}{3!} + \omega^4 \frac{N^{v}(\sigma)}{5!} - \cdots} \tag{103}$$

or in terms of the polar coordinates R and θ:

$$K = -\frac{\sum_{k=0}^{d} b_k R^k \cos k\theta}{\sum_{k=0}^{n} a_k R^k \cos k\theta} = -\frac{\sum_{k=1}^{d} b_k R^{k-1} \frac{\sin k\theta}{\sin\theta}}{\sum_{k=1}^{n} a_k R^{k-1} \frac{\sin k\theta}{\sin\theta}} \tag{104}$$

This may be rewritten in terms of the Chebyshev polynomials of the first and second kind[13] :

$$\left.\begin{aligned} T_n(\cos\theta) &= \cos n\theta \\ U_{n-1}(\cos\theta) &= \frac{\sin n\theta}{\sin\theta} \end{aligned}\right\} \tag{105}$$

so that

$$K = -\frac{\sum_{k=0}^{d} b_k R^k T_k(\cos\theta)}{\sum_{k=0}^{n} a_k R^k T_k(\cos\theta)} = -\frac{\sum_{k=1}^{d} b_k R^{k-1} U_{k-1}(\cos\theta)}{\sum_{k=1}^{n} a_k R^{k-1} U_{k-1}(\cos\theta)} \tag{106}$$

Again, tables can facilitate computation for higher order systems.

To find the values of K for points on the intersection of the locus with the $j\omega$ -axis, we may substitute $\sigma = 0$ in Equation 103., or $\cos\theta = 0$ in Equation 104. to obtain

$$K = -\frac{b_0 - b_2\omega^2 + b_4\omega^4 - \cdots}{a_0 - a_2\omega^2 + a_4\omega^4 - \cdots} = -\frac{b_1 - b_3\omega^2 + b_5\omega^4 - \cdots}{a_1 - a_3\omega^2 + a_5\omega^4 - \cdots} \tag{107}$$

where ω in this equation is an appropriate crossover point, as in Equation 99. Note that if either N(s) or D(s) is a constant, that both the numerator and the denominator of the rightmost fraction in Equation 107. vanish, and the first fraction must be used for calculations.

It might also be mentioned here that terms such as $\dfrac{N^{(\nu)}(\sigma)}{\nu!}$ can be conveniently evaluated numerically by an iterative synthetic division process.

10. THE ROOT LOCI FOR A CLASS OF PHASE SHIFT OSCILLATORS

We have thus treated the general properties of the root loci for rational transfer functions, and have investigated the general form of their equations. We shall now be concerned with the more specialized problem of finding the loci for transfer functions derived from certain iterative structures. Putting aside the practical importance of such an investigation, we shall be interested in studying how the symmetry of the network is reflected in the root loci.

In Figure 18. there is shown the generalized model that we shall here consider. The open-loop network consists of N identical cascaded symmetrical T-sections, each with a series impedance $Z_1(s)/2$ and a shunt impedance $Z_2(s)$. This artificial line is terminated in an amplifier with a gain of $A(-\infty < A < +\infty)$, which will be assumed to have an infinite input impedance and a zero output impedance. The output of the amplifier in series with some hypothetical input voltage, Vin, is then connected to the input of the network, forming a closed-loop system, the locus of whose pole positions as a function of A we wish to find.

It is well known[23] that the open-loop transfer function V_n/V_o is given by

$$\frac{V_N}{V_0} = \frac{1}{\cosh N\gamma} \tag{108}$$

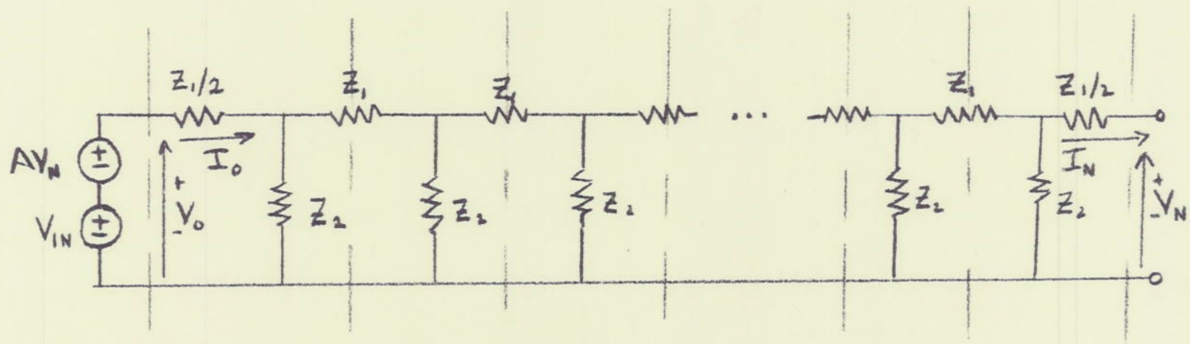


Figure 18. A generalized phase shift oscillator, consisting of N identical cascaded symmetrical T-sections and a non-reciprocal coupling A between the N^{th} mesh and the 0^{th} mesh.

where the propagation constant $\gamma=\alpha+j\beta$ is a function of s and is given by

$$\cosh \gamma = 1 + \frac{Z_1}{2Z_2} = u = x + jy \tag{109}$$

We have also introduced the complex variable $u = x+jy$. For a low-pass R-C structure, which we shall come to consider as basic, $Z_1 = R$ and $Z_2 = 1/_{sC}$ so that the variable u is simply related to s as follows:

$$u = 1 + \frac{RCs}{2} \tag{110}$$

or

$$\left.\begin{aligned} x &= 1 + \frac{RC\sigma}{2} \\ y &= \frac{RC\omega}{2} \end{aligned}\right\} \tag{111}$$

We shall find the root loci in the u-plane, and then consider the transformations necessary to go to the s-plane for different structures.

As Weber[23] points out, the structure we are dealing with is a finite lumped constant system for which we can write Kirchoff's equations in the usual manner. We should expect, therefore, that the transfer function is a rational function of s. Indeed this is the case, since the transfer function can be written

$$\frac{V_N}{V_0} = \frac{1}{\cosh\left[N \cosh^{-1}\left(1+\frac{Z_1}{2Z_2}\right)\right]} = \frac{1}{T_N\left(1+\frac{Z_1}{2Z_2}\right)} \tag{112}$$

so that V_n/V_o is just the reciprocal of a Chebyshev polynomial in $1 + \frac{Z_1}{2Z_2}$. The N poles of the open-loop system will occur when

$$\cosh N\gamma = 0 \tag{113}$$

or when

$$N\gamma = \pm j\,(2m+1)\frac{\pi}{2} \qquad m = 0, 1, 2, \cdots \tag{114}$$

thus

$$\cosh \gamma = x + jy = \cosh j\left(\frac{2m+1}{N} \cdot \frac{\pi}{2}\right) = \cos\left(\frac{2m+1}{N} \cdot \frac{\pi}{2}\right) \tag{115}$$

$$m = 0, 1, 2, \cdots$$

so that the poles lie on the x-axis and are the projections on the x-axis of unit vectors from the origin at the angles $\frac{2m+1}{N} \cdot \frac{\pi}{2}$.

We know by Theorems 10. and 12. that there must be exactly N-1 saddle points on the real axis. To find these, we set

$$\frac{d}{du}\left(\ln \frac{V_N}{V_0}\right) = \frac{d}{du}\ln\left(\frac{1}{\cosh N\gamma}\right) = 0 \tag{116}$$

or

$$\frac{d}{du}\ln(\cosh N\gamma) = \frac{d}{d\gamma}[\cosh N\gamma]\cdot\frac{d\gamma}{d\cosh\gamma}\cdot\frac{d\cosh\gamma}{du}\cdot\frac{1}{\cosh N\gamma}$$

$$= \frac{N\sinh N\gamma}{\cosh N\gamma\,\sinh\gamma} = 0 \tag{117}$$

The solutions of this equation occur when both

$$\left.\begin{array}{l}\sinh N\gamma = 0\\ \text{and} \quad \sinh\gamma \neq 0\end{array}\right\} \tag{118}$$

or when

$$N\gamma = \pm jk\pi \qquad \left\{\begin{array}{l}k = 1, 2, \cdots, N-1, N+1, \cdots\\ k \neq 0, N, 2N, \cdots\end{array}\right. \tag{119}$$

Thus

$$\cosh\gamma = x + jy = \cos k\frac{\pi}{N} \qquad k = 1, 2, \cdots, N-1 \tag{120}$$

The critical points, x_k, lie then on the x-axis at the projections of the unit vectors from the origin at the angles $k\frac{\pi}{N}$. The situation for N=4 is shown in Figure 19.

The asymptotes for the loci are just the extensions of the vectors whose projections locate the critical points. It is evident also, that when N is even, the poles are evenly symmetric about x=0; and as discussed before, the line x=0 will be part of the root locus. It is clear, then, that the loci must take the general form shown in Figure 19. For the low-pass R-C structure, the image of the $j\omega$ -axis in the u-plane is the line x=1, so that when the rightmost branch of the locus in Figure 19. crosses the line x=1, the phase shift oscillator with a low pass RC phase shifter will oscillate.

We shall now find the algebraic formulas for these loci. The closed-loop transfer function is

$$\frac{V_N}{V_{in}} = \frac{1}{\cosh N\gamma - A} \tag{121}$$

so that the root locus is given by

$$\cosh N\gamma = A \tag{122}$$

An equation similar to this is encountered in the derivation of the pole distribution of a Chebyshev filter, and our method of solution

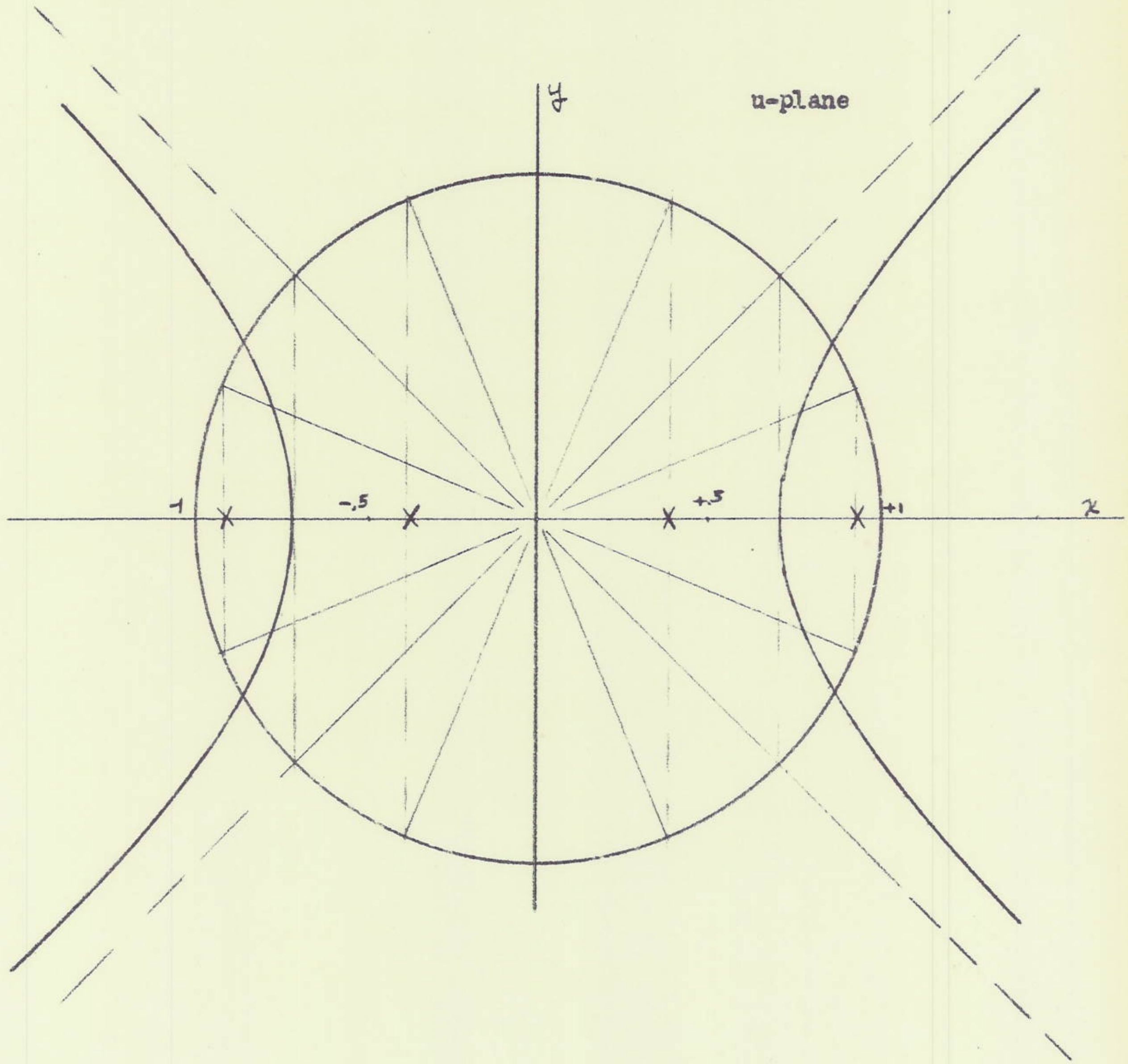


Figure 19. The root locus for a 4 section phase shift oscillator in the u=coshγ plane.

will be similar to that of Seshu and Balabanian[17].

Let us write $\gamma = \alpha + j\beta$ and expand the $\cosh(N\gamma)$ in Equation 122.

$$\begin{aligned} \cosh(N\alpha + jN\beta) &= \cosh(N\alpha)\cos(N\beta) + j\sinh(N\alpha)\sin(N\beta) \\ &= A + j0 \end{aligned} \tag{123}$$

The solutions of this equation depend on the magnitude of A, and are

$$\left.\begin{aligned} \cosh N\alpha &= 1 \\ \cos N\beta &= A \end{aligned}\right\} \text{ if } |A| < 1 \tag{124}$$

and

$$\left.\begin{aligned} \cosh N\alpha &= |A| \\ \cos N\beta &= \text{sign}(A) \end{aligned}\right\} \text{ if } |A| > 1 \tag{125}$$

Consider first the case $|A| < 1$. From Equation 124.,

$$\begin{aligned} \alpha &= 0 \\ \beta &= \frac{1}{N}\cos^{-1} A \end{aligned} \tag{126}$$

u is then given by:

$$u = \cosh\gamma = \cos\left(\frac{1}{N}\cos^{-1}A\right) \tag{127}$$

whereupon

$$\left.\begin{aligned} x &= \cos\left(\tfrac{1}{N}\cos^{-1}A\right) \\ y &= 0 \end{aligned}\right\} \tag{128}$$

This part of the root locus corresponds to the real axis. When A=0, the closed-loop poles are at the critical points.

When $|A|>1$, certain branches leave the real axis. Under this condition, we have from Equation 125.

$$\begin{aligned} \alpha &= \frac{1}{N}\cosh^{-1}|A| \\ &\left\{\begin{aligned} \beta &= \frac{1}{N}\cos^{-1}(\operatorname{sign} A) \\ &= \frac{1}{N}k\pi \\ \\ k &= 0, 2, 4, \cdots \qquad A>0 \\ &= 1, 3, 5, \cdots \qquad A<0 \end{aligned}\right. \end{aligned} \tag{129}$$

k even corresponds to the branches that leave the real axis for positive A; and k odd corresponds to the branches that leave the real axis for negative A. We may then find u as above:

$$u = \cosh(\alpha + j\beta) = \cosh\alpha \cos\beta + j \sinh\alpha \sin\beta \tag{130}$$

substituting for α and β:

$$u = \cosh\left(\frac{1}{N}\cosh^{-1}|A|\right)\cos\frac{k}{N}\pi + j\sinh\left(\frac{1}{N}\cosh^{-1}|A|\right)\sin\frac{k}{N}\pi \tag{131}$$

$$k = \begin{cases} 0, 2, 4, \cdots & A > 0 \\ 1, 3, 5, \cdots & A < 0 \end{cases}$$

or

$$\begin{cases} x = \cosh\left(\frac{1}{N}\cosh^{-1}|A|\right)\cos\frac{k}{N}\pi \\ y = \sinh\left(\frac{1}{N}\cosh^{-1}|A|\right)\sin\frac{k}{N}\pi \end{cases} \tag{132}$$

$$k = \begin{cases} 0, 2, 4, \cdots & A > 0 \\ 1, 3, 5, \cdots & A < 0 \end{cases}$$

k is a discrete variable, representing the branch of the root locus that we are on. If we consider k fixed, we may eliminate the continuous variable A from these equations to get the equation of the locus. We can do this by dividing each equation by the circular function on the right, squaring, and subtracting equations, obtaining:

$$\frac{x^2}{\cos^2 k\frac{\pi}{N}} - \frac{y^2}{\sin^2 k\frac{\pi}{N}} = 1 \tag{133}$$

$$k = \begin{cases} 2, 4, \cdots & A > 0 \\ 1, 3, 5, \cdots & A < 0 \end{cases}$$

Note that $k \neq 0, N, 2N \ldots$ in this equation, because we divided by $\sin k\frac{\pi}{N}$. Thus the root locus consists of N-1 hyperbolas intersecting the real axis at the critical points given by Equation 120. We may write the loci in terms of these critical points as

$$\frac{x^2}{x_k^2} - \frac{y^2}{1 - x_k^2} = 1 \qquad \begin{array}{l} k = 1, 2, \cdots, N-1 \\ x_k = \cos\frac{k\pi}{N} \end{array} \tag{134}$$

If we eliminate k from Equations 132, by dividing by the hyperbolic functions and adding the equations, we obtain

$$\frac{x^2}{\cosh^2\left(\frac{1}{N}\cosh^{-1}|A|\right)} + \frac{y^2}{\sinh^2\left(\frac{1}{N}\cosh^{-1}|A|\right)} = 1 \tag{135}$$

This equation represents intersections of the root locus with the contours of constant A, and is valid only for $|A| > 1$, and only for values of (x,y) on the root locus. Thus the points on the root locus corresponding to a value of $|A|>1$ lie on an ellipse given by Equation 135.

We have found that for any number of sections in the iterative structure under consideration, the root locus in the $u=1+Z_1(s)/2Z_2(s)$ plane consists of the hyperbolas of Equation 134. Thus, the symmetry in the structures has allowed us to decompose the root locus into quadratic factors. To find the loci in the s-plane, we must map the curves from the u-plane to the s-plane by means of the transformation

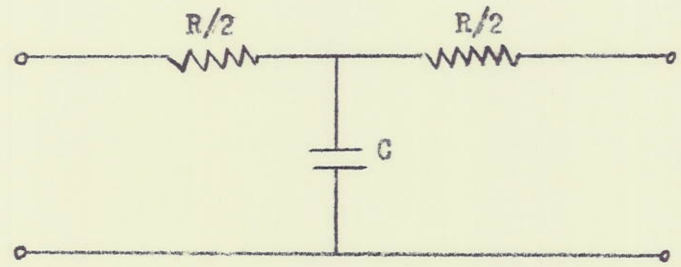
$$u = 1 + Z_1(s)/2Z_2(s) \tag{136}$$

In the more complicated cases, this may be impractical; but we may find points where the locus crosses the $j\omega$-axis by transforming the $j\omega$ -axis into the u-plane, and finding the intersections in the u-plane of the hyperbolas with this image of the $j\omega$-axis.

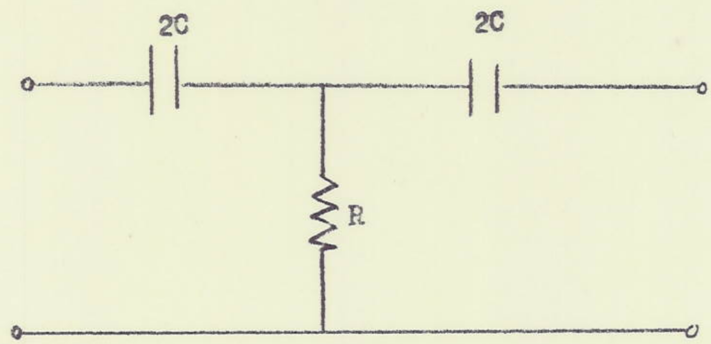
As noted before, for the low-pass R-C structure whose T-section is shown in Figure 20. (a), we have

$$\left.\begin{aligned} x &= 1 + \frac{RC\sigma}{2} \\ y &= \frac{RC\omega}{2} \end{aligned}\right\} \tag{137}$$

so that we can obtain the loci in the s-plane by just shifting the



(a)



(b)

Figure 20. (a) A low-pass R-C T-section that may be used in a phase shift oscillator.

(b) A high-pass R-C T-section.

hyperbolas to the left one unit and adjusting the scale. In terms of the normalized variable RCs, the loci for this case are given by:

$$\frac{\left(1+\frac{RC\sigma}{2}\right)^2}{\cos^2 k\pi/N} - \frac{\left(RC\omega/2\right)^2}{\sin^2 k\pi/N} = 1 \qquad (138)$$

$$k = \begin{cases} 2, 4, \cdots & A > 0 \\ 1, 3, 5, \cdots & A < 0 \end{cases}$$

To find the intersections of this locus with the $j\omega$-axis, set $\sigma = 0$ to obtain

$$\left(\frac{RC\omega}{2}\right)^2 = \frac{\sin^4 k\frac{\pi}{N}}{\cos^2 k\pi/N} \qquad (139)$$

or

$$RC\omega = \pm 2\left(\frac{1}{\cos\frac{k\pi}{N}} - \cos k\frac{\pi}{N}\right) \qquad (140)$$

To find the gain corresponding to these intersections, we may substitute for y in Equations 132 to obtain:

$$|A| = \cosh\left[N\sinh^{-1}\left(\tan k\frac{\pi}{N}\right)\right] \qquad (141)$$

It is clear from Figure 19. that when $A>0$ a pole moves along the σ-axis to the right and will eventually cross over into the right half plane. From Equation 132. we can see that this corresponds to k=0, and that the crossover point corresponds to A=+1. If $A<0$, the first crossover point is given by Equation 132. for k=1, so that the gain required for oscillation in a phase shift oscillator with a low-pass iterative phase shifting network is

$$A_{critical} = -\cosh\left[N\sinh^{-1}\left(\tan\frac{\pi}{N}\right)\right] \tag{142}$$

and the frequency of oscillation is

$$\omega_{critical} = \pm\frac{2}{RC}\left(\frac{1}{\cos\left(\frac{\pi}{N}\right)} - \cos\left(\frac{\pi}{N}\right)\right) \tag{143}$$

$$= \pm\frac{2}{RC}\sin\frac{\pi}{N}\tan\frac{\pi}{N}$$

In a high-pass R-C structure whose T-section is shown if Figure 20. (b):

$$u = 1 + \frac{Z_1(s)}{2Z_2(s)} = 1 + \frac{1}{2RCs} \tag{144}$$

This is the transformation for the low-pass case with RCs replaced by 1/RCs; so that the root locus in the RCs-plane for the high-pass R-C structure is the inversion with respect to the unit circle of the root locus for the low-pass structure. The high-pass locus for

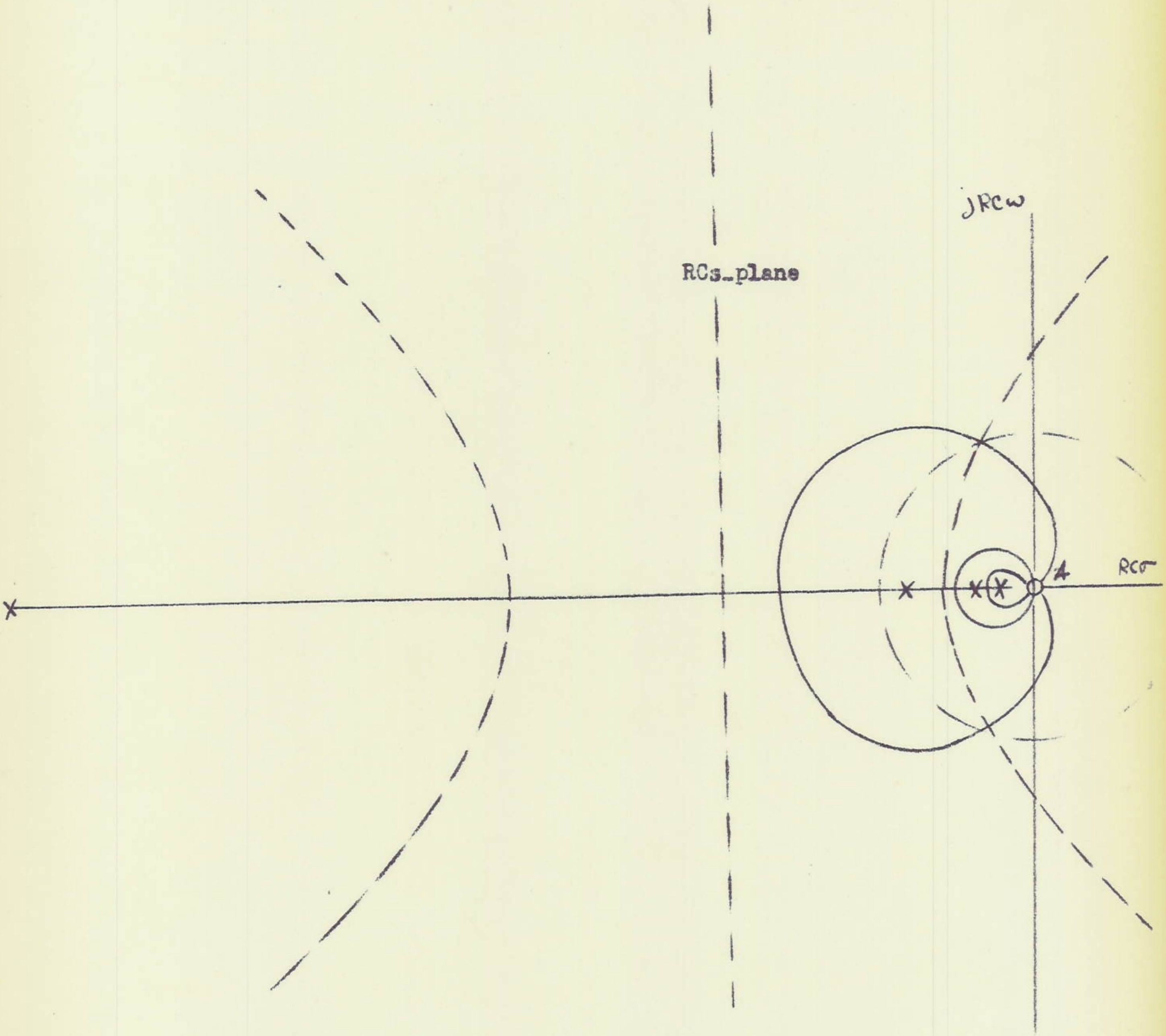


Figure 21. The root locus for a 4 section high-pass R-C phase shift oscillator shown as the inversion with respect to a unit circle in the RCs plane of the hyperbolic locus for the low-pass case. (The low-pass locus is shown dotted)

N=4 is shown in Figure 21. This idea of turning the root locus plot "inside out" is mentioned by Evans.[4]

Consider now the arrangement shown in Figure 22. Here the input terminals of the amplifier with gain A' are connected between the input and output terminals of the ladder network. To find the transfer function V_B/V_{in}, we may write

$$\frac{V_N}{V_0} = \frac{1}{\cosh N\gamma} = \frac{V_B - A'V_B - V_{IN}}{-A'V_B - V_{IN}} \tag{145}$$

or

$$\frac{V_B}{V_{IN}} = \frac{1}{\dfrac{\cosh N\gamma}{\cosh N\gamma - 1} - A'} \tag{146}$$

The closed-loop poles are given by

$$\cosh N\gamma = \frac{A'}{A'-1} = A \tag{147}$$

or

$$A' = \frac{A}{A-1} \tag{148}$$

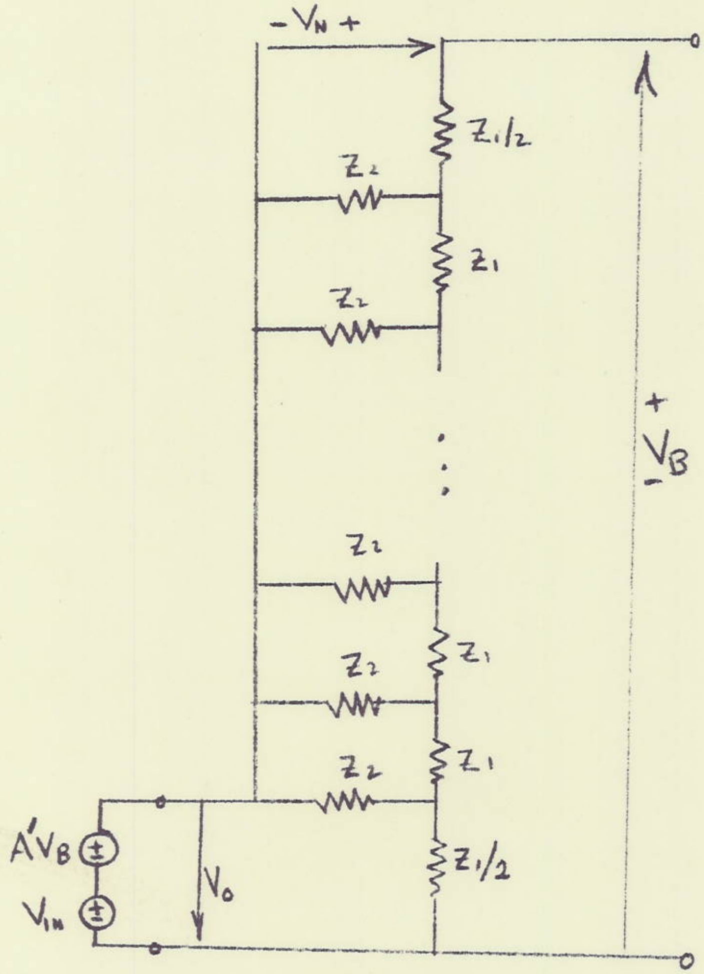


Figure 22. Another kind of phase shift oscillator, which has the same root locus as that shown in Figure 18., except for the calibration with respect to gain.

which is the same locus as the one found above, except that A is replaced by $A'/(A'-1)$. The new open-loop function is

$$\frac{V_B}{Y_{IN}} = \frac{\cosh N\gamma - 1}{\cosh N\gamma} \tag{149}$$

so that the new open-loop poles are the same as before, but there are now zeros at every point where the gain A was +1 on the old locus. On the other hand, it can be seen from Equation 148. that where there were zeros on the old locus, there are now points on the new locus where $A'=1$. Figure 23. shows the locus for a low-pass R-C structure for $N=4$ and for the new connection of the amplifier. This locus is, of course, the same as that for the old connection, except that it is recalibrated in terms of gain according to Equation 148. with zeros where $A = 1$. If the new connection were used for the network whose locus is shown in Figure 21., there would be a 3^{rd} order saddle point at the origin, corresponding to a gain of +1. Thus the connection shown in Figure 22. leads to systems where a high order saddle point appears on the root locus.

When L-C T-sections are used, or when resonating arms are used in the T-sections, the transformation represented by Equation 136. becomes complicated. The author has not attempted to analyse how the hyperbolas in the u-plane behave under these transformations. Suffice it to say that we have indicated a way of treating analytically the root-loci that arise when certain iterative structures are used in a feedback arrangement.

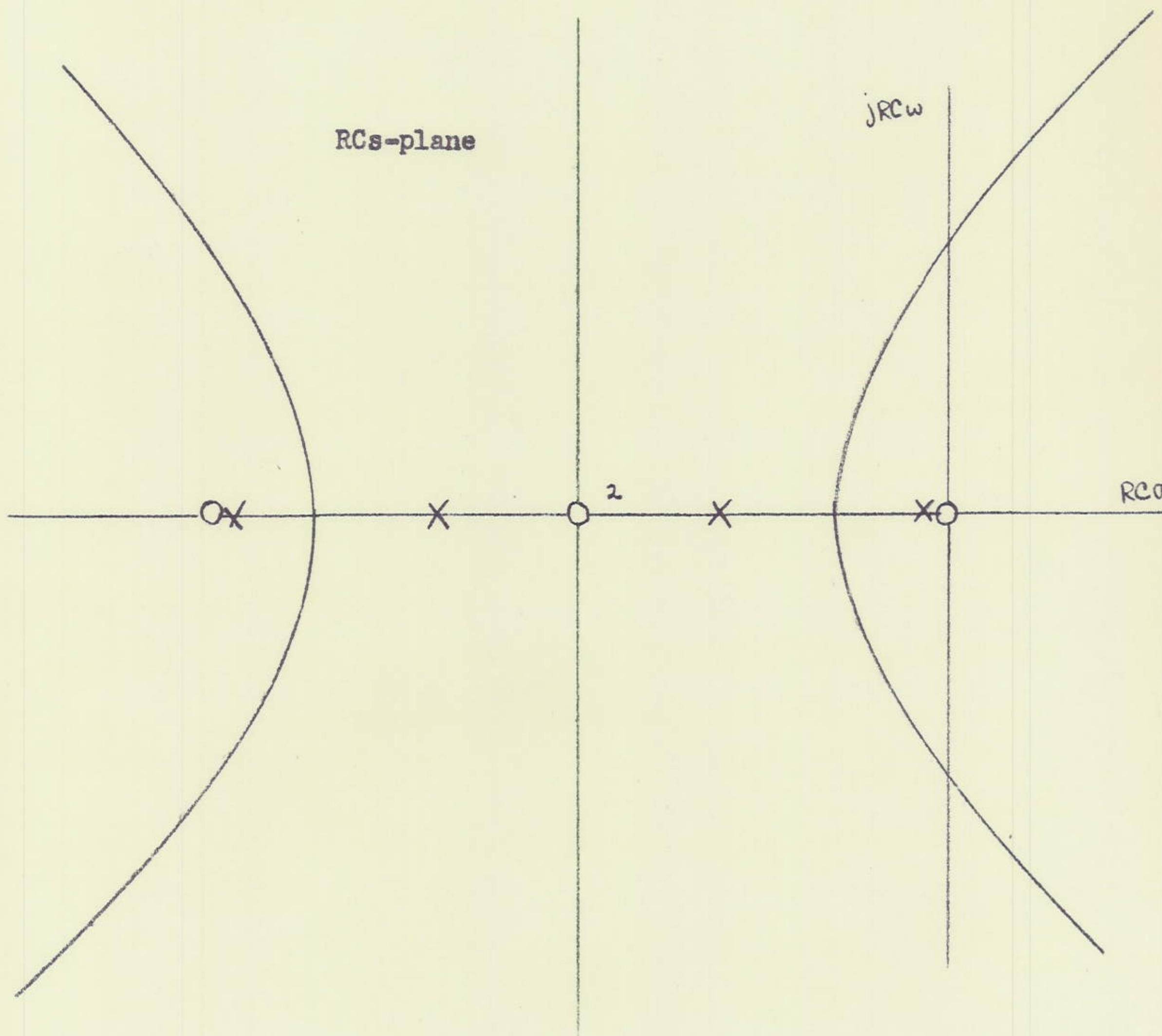


Figure 23. The locus for the amplifier connection of Figure 22. when the ladder network consists of 4 low-pass R-C sections.

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