Note

I originally wrote this document in the spring of 2011 as part of an independent study on probability with Professor Steven Miller\(^1\) at Williams College. The piece was intended to be a chapter in a probability textbook Professor Miller was writing. Though I don’t believe it ever was included in the textbook, I think it serves as an example of my ability to explain complex topics in an easy-to-understand manner.

1 Introduction

Probability is a rich mathematical topic, and it is certainly worth studying from a purely theoretical standpoint. However, probability is also a vital component in a variety of models of real world phenomena; often, when we try to model actual events, we need to use probability. Models depending on probability can both guide our future decisions and influence our opinions about past events. In this chapter, we’ll use baseball as a lens to illustrate some practical applications of the topic, and the questions we ask and answer about baseball events will serve as a springboard to new mathematical techniques. Along the way, we’ll continue our focus on thinking about the right questions to ask and how changing our question just a little can dramatically affect the math that we use to solve the problem.

2 First Four Games

We’ll start with a mathematically fairly simple example, but we’ll use this question to illustrate some key concepts of models based on probability. The Boston Red Sox began the 2011 season by losing their first four games. (Actually, they lost six in a row before earning their first win, but we’ll examine the situation after four games.) What should we think about this losing streak? Is it likely that a team will start the season 0-4? Or is this a very unlikely event, and therefore we should worry that the team is flawed in some fundamental way? To come to any type of conclusion, we need to transform our general questions into precise mathematical terms so that we can create a model. Thus, what we really need to figure out is:

What is the probability that a given team loses its first four games?

Now we’re getting somewhere - the math involved in answering this question isn’t very hard if we make enough simplifying assumptions. In the real world, even this seemingly easy question is difficult to solve if we try to account for the complex interactions among different teams in the league. If we assume that we can assign one number to give the probability that the Red Sox win any game they play - that is, if we assume their chance of winning on a given day isn’t affected by the opponent, or the lineups both teams are using, or the weather, or any other variable - then this question is not mathematically complex. However, we will discuss in detail how to attack it, because this simple example will illustrate many common themes of probability. Already, we see that one of the most difficult aspects of using probability for real world phenomena is figuring out the right way to ask the question; we need to formulate our question such that the answer will be the information we are looking for, but also such that we are able to carry out the math necessary to get to an answer.

Let \( p \) be the probability that a team, in our case the Red Sox, wins any given game. We are assuming that they have the same probability of winning regardless of the team they are playing. This assumption is not completely realistic, but, when we create models, we need to strike a balance between capturing reality and working with equations that we can solve. In this case, for our first attempt at modeling the Sox’s situation, we will choose a very simplistic model. Therefore, we assume that the probability that the Red Sox win any

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\(^1\)https://web.williams.edu/Mathematics/sjmiller/public_html/index.htm
given game is $p$, and they lose with probability $1 - p$. We need the Sox to lose their first game, and then lose their second game, and then their third, and then their fourth. This gives us the equation:


It seems like we’re in great shape - we have an answer! The probability that the Red Sox lose their first four games is $(1 - p)^4$. But here is where modeling real world phenomena is more complicated than theoretical mathematics. We aren’t interested in an answer in terms of symbols - we want to know what we should think about the Sox’s poor start. An answer of $(1 - p)^4$ doesn’t help us with this at all. We would like an actual number as our answer, and to achieve this, we need a value for $p$.

But what do we use for $p$? We defined $p$ as the probability that the Red Sox win any given game. Since they haven’t won any games, should we take $p = 0$? This doesn’t seem to make sense, because then our answer would be that the probability that a team with no chance of winning loses its first four games is 1. This is certainly true, but it doesn’t help us with our question. By “the probability that the Red Sox win any given game,” what we really mean is the probability with which we can expect the team to win, which can be very different from the percentage of games the team actually has won. But how do we know the probability with which we can expect the Sox to win if they’ve only played four games and lost all of them? We need to use other models! We can model a team’s $p$ in a variety of ways, using individual players’ past performances and, once our sample size of games the team has played this season becomes reasonably large, past performance of the team overall. Now, we see that our mathematically straightforward problem has become much more complicated when we want to apply the results to real life - we need to use other models in order to gain any benefit from our model.

We are not going to try to create a model for the Red Sox’s $p$ in this hypothetical example. Instead, we’ll evaluate our equation for a few values of $p$, which will allow us to compare what the probability of this losing streak is for good, bad, and average teams.

### 2.1 A Few Different $p$

We’ll start by assuming the “true talent” level of the team is average. With $p = 0.5$, we have:

$$\text{Probability of losing first four games} = 0.5^4 = 0.0625.$$ 

Now we know that there is a 6.25 percent chance that an average team loses its first four games if facing average opponents. This certainly isn’t high, but it tells us that a losing streak like this to open the season isn’t unheard of.

Of course, not every team in the league has $p = 0.5$, which is what we assumed by giving the Red Sox $p = 0.5$ and asserting that their opponents are average. It will help us know how to think about the Red Sox’s first four games if we see the difference in likelihood that a true talent 0.600 winning percentage team loses the first four games compared to a team with $p = 0.4$. At the end of the season, most teams’ winning percentages fall somewhere between 0.4 and 0.6. Thus, plugging $p = 0.4$ into our model shows the probability that one of the truly worst teams in the league opens 0-4, and using $p = 0.6$ shows the probability that a team that is actually quite talented experiences a streak of bad luck at the beginning of the season.

For $p = 0.4$, we have:

$$\text{Probability of losing first four games} = (1 - 0.4)^4 = 0.1296.$$ 

For $p = 0.6$, we have:

$$\text{Probability of losing first four games} = (1 - 0.6)^4 = 0.0256.$$
Therefore, we expect that 12.96 percent of the time the worst teams in the league open the season 0-4, or about twice as often as a 0.500 team. The best teams in the league open the season 0-4 only 2.56 percent of the time, or about one-third as often as a 0.500 team. Interestingly, we see that, while a \( p = 0.4 \) team loses only one-third more games than a \( p = 0.6 \) team, the bad team is almost five times more likely to open the season with four consecutive losses. Figure 1 shows the probability of a team with any given \( p \) losing the first four games.

These results aren’t surprising, but they are somewhat discouraging. Attempting to predict a team’s true talent level from four games is a mistake, but if we were to do so, we would see that an 0-4 record is much more likely for a bad team than a good one, which suggests that the Sox might struggle all season.

Even though our answers are intuitive, it’s useful to discuss the implications of our simple model in order to get in the habit of stepping back to think about what our math is telling us. We would be very surprised if we obtained the opposite results, and we would need to think carefully about possible flaws in our model or with our calculations. On the other hand, just because our model gives us the answer we expect does not necessarily mean that we did everything right - if we were already sure about the answer, we wouldn’t have needed to create the model. This is just one example illustrating a key idea to keep in mind for math in general, and especially for applied math: we need to check that our answers make sense.

### 2.2 Different Opponents

Before we extend our investigation in a manner that leads us to a very different mathematical model, we consider one final possible improvement based on our current equation: What if we wanted to consider the Red Sox’s opponents having \( p \) something other than 0.5? Intuitively, we understand that, even if the Sox are a 0.500 team overall, if they happen to play a great team for the first four games, they have a higher chance of losing all the games than if they were playing a bad team. Maybe we can excuse the dismal start to the season by attributing it to the strength of the opponent! One way to address this question is to incorporate the expected winning percentage of the opponent, which we’ll call \( q \), into the \( p \) that we use to model the
Sox’s chance of winning. We need some model that takes the overall $p$ for the Red Sox and the overall $q$ for our opponent and combines them to give the chance that the Sox win against this specific team. Then, we can just use this new $p'$ in the model we discussed.

It turns out that baseball analysts have developed such a model. In 1981, Bill James, perhaps the most influential figure in revolutionizing the way people think about baseball statistics, presented the following equation to compute $p'$ from $p$ and $q$:

$$p' = \frac{p - pq}{p + q - 2pq}.$$  

We will not work out a full example here, because it is straightforward to choose numbers to plug in for $p$ and $q$ to find $p'$ and then use this $p'$ in the model we discussed earlier. James’s equation itself is a fascinating topic to investigate; for more information, see “A Justification of the log 5 Rule for Winning Percentages” by Steven J. Miller.

2.3 What Did We Find?

Stepping back for a minute before we leave the question about opening the season 0-4, we see that, in general, this is a very unlikely event. In addition, it is significantly more likely to occur to a true talent bad team. These facts make us worry that the Red Sox’s $p$ may be very low, which will make for a disappointing season. Nevertheless, we hold hope that the team’s $p'$ for these four games was strongly influenced by a very high $q$ for their opponents.

3 Any Four Games

Our first example has illustrated some basic aspects of using probability to answer questions arising from daily life. We had to articulate our question in such a way that we could write an expression to find the answer, and we had to make assumptions in order to evaluate this expression. Finally, we needed to think about our answer to ensure that it was reasonable.

Now, let’s use this question as a jumping off point for some questions involving more complicated math. What if we look at the question like this:

*What is the probability that a team loses four games in a row at any point in the season?*

Actually, this isn’t a good formulation of the question. As we saw in the Introduction with the Birthday Problem, we need to be careful with the specifics. Do we want to know the probability that a team loses exactly four games in a row, or the probability of losing at least four consecutive games? Both are interesting questions, but they have different answers. For now, we’ll work on the second:

*What is the probability that a given team loses at least four games in a row at any point in the season?*

This is a good, specific formulation of our question, but the wording still doesn’t lend itself to coming up with a mathematical equation that will let us answer it. Formally, what we want to know is:

3.1 At Least Four

*What is the probability that a given team has at least four consecutive losses in $n$ games?*

This is a question we can begin to attack. Let $a_n$ be the probability that the Red Sox have four consecutive losses in $n$ games, and let $b_n$ be the probability that this doesn’t happen (that is, the Sox’s longest losing streak in these $n$ games is no more than three). The answer to our question is $a_n$, so it might make sense to start by trying to figure out a way to break down $a_n$ into an expression we can evaluate. However, as is
often the case with math, we can save ourselves a lot of computational work later if we look at the situation slightly differently. We know that \(a_n = 1 - b_n\), so if we can solve for \(b_n\), then we can subtract our answer from 1 to find \(a_n\). In this case, solving for \(b_n\) turns out to be much easier than solving for \(a_n\), so we’ll use this strategy to simplify calculations. Later, we’ll explain why this decision makes sense.

3.2 An Expression for \(b_n\)

We’ll find our expression for \(b_n\) by considering all possible combinations of wins and losses within any string of four consecutive games within the \(n\) total. For each scenario (for example, lose the first game, lose the second game, win the third game), we’ll specify the probability that we lose no more than three consecutive games, if we have arrived at this scenario. To find the probability that this given scenario is our path to avoiding four losses in a row, we multiply the probability of getting to this situation by the probability of avoiding four consecutive losses once we’re there.

We can explain this mathematically with basic conditional probability. If we let \(B\) be the event of avoiding four consecutive losses in \(n\) games and \(A_i\) be the event of arriving at given win-loss scenario \(i\), we have:

\[
P(B|A_i) = \frac{P(A_i \cap B)}{P(A_i)}
\]

\[
P(A_i \cap B) = P(A_i)P(B|A_i)
\]

We have defined \(P(B)\) as \(b_n\). For each \(A_i\), we want to know \(P(A_i \cap B)\). We have \(P(B)\), and we will explain how to find \(P(B|A_i)\). Then \(P(A_i)P(B|A_i)\) gives the probability of avoiding four losses in a row for given game state \(A_i\). Since we have five win-loss scenarios to consider (see Figure 2 and Table 1), to find our overall \(b_n\), we need to sum \(P(A_i \cap B)\) for \(i = 1, 2, \cdots, 5\). We are allowed to sum these probabilities because all of our game states are unique; if \(A_i\) is the scenario in which we win the first game and \(A_j\) is the scenario in which we lose the first three games and then win the fourth, there is no overlap between \(A_i\) and \(A_j\). Since our five game states exhaust all the possible ways to avoid four consecutive losses, if we sum \(P(B|A_i)\) over all five values of \(i\), we are left with \(P(B)\), or \(b_n\). Overall, then, we are constructing an equation of the form:

\[
\sum_{i=1}^{5} P(A_i \cap B) = \sum_{i=1}^{5} P(A_i)P(B|A_i)
\]

\[
b_n = \sum_{i=1}^{5} P(A_i)P(B|A_i)
\]

Now that we’ve outlined our basic approach, we’ll show how to find \(P(B|A_i)\) for all of our \(A_i\). Let’s consider the first of our \(n\) games. We know that the Sox either win Game One with probability \(p\) or lose Game One with probability \(1 - p\). If the Red Sox win Game One, then they have successfully played one of these \(n\) games without making any progress toward four consecutive losses; they have only \(n - 1\) games left for which they need to avoid losing four in a row. That is, if the Red Sox win Game One, the probability that they do not have four consecutive losses in these \(n\) games is exactly the same as the probability that they do not have four consecutive losses in \(n - 1\) games, or \(b_{n-1}\). Thus, starting by winning Game One is our first possible way to avoid four consecutive losses in \(n\) games. It occurs with probability \(p\), and we lose no more than three in a row in this scenario with probability \(b_{n-1}\). In terms of our conditional probability approach, we have \(P(A_1) = p\), and \(P(B|A_1) = b_{n-1}\). Thus, \(pb_{n-1}\) is the first term in our expression for \(b_n\).

On the other hand, if the Sox lose Game One, then we need to look at Game Two. If they win Game Two, then their streak is reset, and they have \(n - 2\) games left to avoid losing four in a row. That is, if the Red Sox lose Game One and win Game Two, the probability that they do not have four consecutive losses in these \(n\) games is \(b_{n-2}\). We get to this scenario of lose Game One and then win Game Two with probability \(p(1-p)\), so our second term is \(p(1-p)b_{n-2}\). In terms of conditional probability, we have \(P(A_2) = p(1-p)\) and \(P(B|A_2) = b_{n-2}\).

If the Red Sox lose both Game One and Game Two, we move on to considering Game Three. Again, we have two possible outcomes: a win resets the streak and gives us a probability of \(b_{n-3}\) once in this game.
state, and a loss forces us to consider Game Four. Therefore, our third scenario in which we can lose no more than three in a row is lose-lose-win; this happens with probability \((1 - p)(1 - p)p\), and once we have these three outcomes, we still need to avoid losing four games in a row in the remaining \(n - 3\) games. Thus, our third term is \((1 - p)^2 pb_{n-3}\).

Game Four is the final game we need to explicitly include in our analysis, because we are trying to avoid four consecutive losses. We’re only looking at Game Four in the context of having lost Game One, Game Two, and Game Three, because we handled the cases in which the Red Sox win one of the earlier games by including the \(b_{n-i}\) factor in each term, with \(b_{n-i}\) defined as the probability that the Sox don’t lose four consecutive games within the remaining \(n - i\) games. We don’t need to explicitly state scenarios such as lose-win-win-lose, because once Sox win the second game, their losing streak is reset to zero games.

If the Sox win Game Four, then they need to avoid losing four in a row in the remaining \(n - 4\) games, and \(b_n\) once we are in this situation is \(b_{n-4}\). However, if they lose Game Four, \(b_n = 0\), because we are only looking at Game Four in the scenario of losses in the first three games. Since \(b_n\) is the probability that they don’t lose at least four in a row, if we’re in this situation, we know that the event for which \(b_n\) gives the probability cannot occur. Therefore, Game Four gives us \((1 - p)^3 pb_{n-4}\) as our fourth term in our expression for \(b_n\), and 0 as the fifth and final term. We’ve now considered how we can avoid four consecutive losses in all possible scenarios, and we simply sum the five terms we’ve discussed to find \(b_n\).

\[
b_n = pb_{n-1} + (1 - p)pb_{n-2} + (1 - p)^2 pb_{n-3} + (1 - p)^3 pb_{n-4} + 0.
\]

To review, we have just created an exhaustive breakdown of every possible different scenario of wins and losses within a sequence of \(n\) games, and our analysis holds for any \(n\). This is the power of the recurrence relation for \(b_n\). Figure 2 shows the complete breakdown graphically, and we list all relevant scenarios, and the term each contributes to our total \(b_n\), in Table 1 in order to further clarify our method.
3.3 Difference Equations: Solving for $b_n$

We see how modifying our question just a little can create a much more difficult problem! We’ve already done a lot of work to find this expression for $b_n$, and it’s still written in terms of other $b_i$, for $i = n - 1, \ldots, n - 4$. In addition, we’re again faced with the problem of needing to choose a $p$. Luckily, neither of these difficulties is insurmountable. For convenience, let’s take $p = 0.5$. Since we want to know in general how likely it is for a team to lose four consecutive games, it makes sense to evaluate our expression for an average team. Then, what we need to solve is:

$$b_n = 0.5b_{n-1} + (1 - 0.5)0.5b_{n-2} + (1 - 0.5)^20.5b_{n-3} + (1 - 0.5)^30.5b_{n-4}$$

$$= \frac{1}{2}b_{n-1} + \frac{1}{4}b_{n-2} + \frac{1}{8}b_{n-3} + \frac{1}{16}b_{n-4}. \tag{1}$$

Equation (1) is a difference equation, which we will introduce briefly here. See appendix for a more complete explanation of solving this type of problem.

Any equation of the form

$$\alpha_n = c_1\alpha_{n-1} + c_2\alpha_{n-2} + \ldots + c_k\alpha_{n-k}$$

for $k$ a fixed integer and $c_1, \ldots, c_k$ given real numbers, has a solution of the form

$$\alpha_{n-1} = \gamma_1^n + \ldots + \gamma_k^n \tag{2}$$

as long as the characteristic polynomial

$$r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0$$

has $k$ distinct roots. In our case, we have $k = 4$, and $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{4}$, $c_3 = \frac{1}{8}$, and $c_4 = \frac{1}{16}$. Therefore, we need to solve:

$$r^4 - \frac{1}{2}r^3 - \frac{1}{4}r^2 - \frac{1}{8}r - \frac{1}{16} = 0.$$

We can get an idea of where the roots of this quartic fall by graphing the function, and we can use a computer program to compute the roots for us. We find:

$$r_1 \approx -0.387$$

$$r_2 \approx -0.0382 - 0.407i$$

$$r_3 \approx -0.0382 + 0.407i$$

$$r_4 \approx 0.964.$$

We have four distinct roots, so we’re okay to proceed with solving this equation using the general form we gave above. When we first find these values for $r_1, \ldots, r_4$, we may be alarmed to see that we have complex numbers as solutions. We are looking for the probability that a team does not lose four games in a row out of $n$ total games, and probabilities must be real numbers between zero and one, inclusive. However, if we look more closely, we see that $r_2$ and $r_3$ are complex conjugates. When we use them to solve for $\gamma_1, \ldots, \gamma_4$ and then plug these values into our expression for $b_n$, the imaginary parts of each will cancel, and we’ll be left with a real probability.

From Equation (2) we know that the general form of our solution will be

$$b_n = \gamma_1r_1^n + \gamma_2r_2^n + \gamma_3r_3^n + \gamma_4r_4^n.$$

We have values for each of $r_1, \ldots, r_4$, but we need to solve for $\gamma_1, \ldots, \gamma_4$. To do so, we need to solve the system of equations:

$$b_0 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

$$b_1 = r_1\gamma_1 + r_2\gamma_2 + r_3\gamma_3 + r_4\gamma_4$$

$$b_2 = r_1^2\gamma_1 + r_2^2\gamma_2 + r_3^2\gamma_3 + r_4^2\gamma_4$$

$$b_3 = r_1^3\gamma_1 + r_2^3\gamma_2 + r_3^3\gamma_3 + r_4^3\gamma_4.$$
We know that $b_0 = b_1 = b_2 = b_3 = 1$; the probability that the Red Sox do not lose four consecutive games within any sequence of zero, one, two, or three games is 1. Then, we have this system to solve:

\[
\begin{align*}
1 &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \\
1 &= r_1 \gamma_1 + r_2 \gamma_2 + r_3 \gamma_3 + r_4 \gamma_4 \\
1 &= r_1^2 \gamma_1 + r_2^2 \gamma_2 + r_3^2 \gamma_3 + r_4^2 \gamma_4 \\
1 &= r_1^3 \gamma_1 + r_2^3 \gamma_2 + r_3^3 \gamma_3 + r_4^3 \gamma_4.
\end{align*}
\]

We have four equations and four unknowns (we’ve already solved for $r_1,...,r_4$, but the notation is cleaner using $r_i$ instead of long decimal approximations; our unknowns are $\gamma_1,...,\gamma_4$). From linear algebra, we know that we can rewrite this system of linear equations in matrix form as:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
r_1 & r_2 & r_3 & r_4 \\
r_1^2 & r_2^2 & r_3^2 & r_4^2 \\
r_1^3 & r_2^3 & r_3^3 & r_4^3
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4
\end{pmatrix}
=
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}.
\]

For any invertible matrix $A$ with inverse $A^{-1}$, the product $A^{-1}A$ is equal to the identity matrix. Therefore, multiplying each side of Equation (3) to the left by the inverse of our matrix, we obtain the following equation to solve for $\gamma_1,...,\gamma_4$:

\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4
\end{pmatrix}
=
\begin{pmatrix}
1 & 1 & 1 & 1 \\
r_1 & r_2 & r_3 & r_4 \\
r_1^2 & r_2^2 & r_3^2 & r_4^2 \\
r_1^3 & r_2^3 & r_3^3 & r_4^3
\end{pmatrix}^{-1}
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}.
\]

We could tediously solve this equation by hand, performing elementary row operations to compute the inverse matrix, and then multiplying this by $(1,1,1,1)^T$. However, this is not realistically practical, and instead we use a computer program to find:

\[
\begin{align*}
\gamma_1 &\approx -0.116 \\
\gamma_2 &\approx 0.0121 - 0.118i \\
\gamma_3 &\approx 0.0121 + 0.118i \\
\gamma_4 &\approx 1.092.
\end{align*}
\]

Again, it may be disconcerting to see complex numbers along the way toward computing a probability, but we’re about to see that our complex $r_2, r_3, \gamma_2,$ and $\gamma_3$ combine in our expression to give a valid probability.

Now, we essentially have an answer. We have

\[
b_n = \gamma_1 r_1^n + \gamma_2 r_2^n + \gamma_3 r_3^n + \gamma_4 r_4^n
\]

and we’ve solved for each of $r_1,...,r_4$ and $\gamma_1,...,\gamma_4$. We defined $n$ as the total number of games that we are examining to see if the Red Sox can avoid a losing streak of four games or longer within this sequence. Since a Major League Baseball season is 162 games long, we should take $n = 162$.

Lastly, we need to remember the simplifying modification that we made at the very beginning. Our question was about the probability of losing four consecutive games, but we defined $b_n$ as the probability of not losing four games in a row. We decided to solve for $b_n$ because doing so is easier than solving for $a_n$. Now that we’ve constructed our expression for $b_n$ and solved the recurrence relation, we can explain why.

### 3.4 An Aside: Why Not $a_n$?

Looking back at Figure 2, we can imagine what a breakdown would look like for $a_n$, as shown in Figure 3. We have the same five game states to consider. In addition, the recurrence relation for $a_n$ written in terms
of \( a_{n-i} \) is the same as the recurrence relation for \( b_n \) written in terms of \( b_{n-i} \) for each scenario. However, the case of lose-lose-lose-lose gives \( a_n = 1 \), because we have defined \( a_n \) as the probability that we have at least four consecutive losses in \( n \) games, while for \( b_n \) this state gave 0. Then, instead of having 0 as the fifth term in our recurrence relation, we have 1. This extra constant makes it more difficult to solve the difference equation.

To help further clarify how we would have constructed an expression for \( a_n \), we show an equivalent table to Table 1 for \( a_n \) in Table 2 and the corresponding equation:

\[
a_n = pa_{n-1} + (1-p)pa_{n-2} + (1-p)^2pa_{n-3} + (1-p)^3pa_{n-4} + 1.
\]

### 3.5 Back to \( b_n \)

Now that we see why we chose to take the seemingly indirect, but actually simpler, route to finding \( a_n \), we’re ready to convert back to find the answer we’re looking for. Putting everything together, we need to evaluate

\[
a_{162} = 1 - b_{162}.
\]

\[
= 1 - (\gamma_1 r_1^{162} + \gamma_2 r_2^{162} + \gamma_3 r_3^{162} + \gamma_4 r_4^{162})
\]

\[
\approx 1 - (-0.116)(-0.387)^{162} - (0.0121 - 0.118i)(-0.0382 - 0.407i)^{162} - (0.0121 + 0.118i)(-0.0382 + 0.407i)^{162} - 1.092(0.964)^{162}
\]

\[
\approx 0.997.
\]

It turns out that, even though there is only a 6.25 percent chance that an average team will lose the first four games of the season, there is a 99.7 percent chance that they’ll have a losing streak of at least four games at some point! If we think about the Red Sox’s poor performance in this way, we have no reason to worry at all!
Of course, if we think more carefully, we still might be worried. Even though we expect this four game losing streak at some point, it doesn’t seem likely that it starts at the first game of the season. This leads to an interesting question to consider:

After how many games is there at least a 50 percent chance that a given team has had at least one losing streak of four or more games?

This question is similar to a topic we investigated in relation to the Birthday Problem; there, we asked how many people we would need to have in the room in order to have at least a 50 percent chance of a common birthday. We will not address this question in detail here, but we can be fairly confident the answer is greater than four. That is, it’s not likely that this almost-guaranteed-to-exist four game losing streak occurs as early as it did for the Red Sox in 2011.

3.6 What About Different $p$?

Since we looked at the differences in the probability of a 0.400, 0.500, and 0.600 team losing the first four games of the season, it makes sense to run this model with $p = 0.4$ and $p = 0.6$ to compare the effects of varying true talent levels on the answers to our two questions. Remember, our original equation before we plugged in $p = 0.5$ was

$$b_n = pb_{n-1} + (1 - p)p_{n-2} + (1 - p)^2p_{n-3} + (1 - p)^3p_{n-4}.$$ 

Now, we can figure out the probability that a good team has a four game losing streak by following the same steps as above for

$$b_n = 0.6b_{n-1} + (1 - 0.6)0.6b_{n-2} + (1 - 0.6)^20.6b_{n-3} + (1 - 0.6)^30.6b_{n-4}.$$ 

We won’t go through the explanation of the mechanics again, but it turns out that the better teams in the league have approximately a 92.9 percent chance of a four game losing streak. For a bad team, we solve

$$b_n = 0.4b_{n-1} + (1 - 0.4)0.4b_{n-2} + (1 - 0.4)^20.4b_{n-3} + (1 - 0.4)^30.4b_{n-4}$$

to see that those teams lose four games in a row 99.9989 percent of the time. Interestingly, we find that the probability that a 0.500 team has a four game losing streak at some point in the season is far closer to the probability that a 0.400 team does the same, while earlier the average team was more similar to the good team.

Mathematically, this makes sense. All teams must have a probability of losing four or more in a row at some point between 0 and 1, and all teams worse than average should have a higher probability than an average team. This means that all winning percentage $p$ between 0.500 and 0 must give an answer between 99.7 percent and 100 percent, while all winning percentage $p$ between 0.500 and 1 can give an answer extending all the way from 99.7 percent to 0 percent. Intuitively, we see that the bad teams should be “squished” into the small 0.3 percentage points between 99.7 and 100, while the good teams need to extend to cover the large interval between 99.7 and 0. For our earlier question, we have this “squishing” on the opposite side. The probabilities of losing the first four games for teams with $p$ from 0.500 to 1 must fit between 0 and 0.0625, while bad teams have probabilities extending from 0.0625 to 1. Analyzing the recurrence relation in this way may not specifically help us know what to think about the Sox’s early season performance, but it is mathematically interesting and provides another opportunity to illustrate the importance of thinking about the reasonableness and implications of our answers.

3.7 Exactly Four

Now, let’s go back to the other question we suggested earlier: What is the probability that a team loses exactly four games in a row? Again, we need to state this question in a more precise way in order to be able to attack it mathematically:
What is the probability that a team has at least four consecutive losses in \( n \) games with no losing streaks of five games or longer?

In our above analysis, we repeatedly emphasized that we only needed to explicitly consider four games in order to find \( b_n \), because every win reset the process to a problem that we could solve recursively with \( b_{n-i} \), for \( i \) the number of elapsed games. Now, we care about the fifth game. This question seems similar to our earlier problem, but it also seems that our old approach is fundamentally flawed for finding a solution. Do we need to adopt an entirely different strategy?

Luckily, we do not. There is nothing special about four losses in a row; we could have computed \( c_n \), the probability of at least five consecutive losses, in a parallel manner, though we would have needed to add an extra level to our tree and an extra row to our table enumerating the possible win-loss scenarios. But how does it help us to be able to use our method to calculate \( c_n \)? In order to lose five games in a row, a team must also lose four. Therefore, we essentially want to find the difference between \( a_n \) and \( c_n \). The amount by which \( a_n \) is larger than \( c_n \) represents precisely those situations in which the team loses four in a row and then wins the fifth game.

Let \( d_n = 1 - c_n \). When we solve

\[
c_n = 1 - (pd_{n-1} + (1-p)pd_{n-2} + (1-p)^2pd_{n-3} + (1-p)^3pd_{n-4} + (1-p)^4pd_{n-5})
\]

for \( p = 0.5 \), we get \( c_{162} \approx 0.9345 \). For \( a_n \) with \( p = 0.5 \), we had \( a_{162} \approx 0.997 \). Therefore, the probability that a team with an expected true talent level \( p \) of 0.5 loses four games in a row but never five over the course of a 162 game season is

\[
a_{162} - c_{162} \approx 0.997 - 0.9345 = 0.0625.
\]

When we see this answer, we should have the opposite reaction to when we found complex roots in solving for \( r \) and \( \gamma \). A value of 0.0625 seems “normal.” Both \( a_{162} \) and \( c_{162} \) are long decimals, but when we subtract them our answer is very close to 0.54. It seems that we may have “ugly” decimals cancelling each other to give a precise answer to the four-not-five game losing streak question! Even though we have an answer to the question we started investigating, we should think about this interesting result more carefully. After all, this is how mathematicians, and scientists in other fields, make new discoveries - when they come to a puzzling scenario, they think of the right questions to ask to make sense of what they’re seeing, even if those questions lead them away from their original topic of study. As we’re learning to become mathematicians, we should diverge briefly from our original question to see if we can discover something new here.

### 3.8 A New Discovery?

Unfortunately, in our case, we may not have the mathematical breakthrough of the century; it seems that this is a coincidence. Had we subtracted \( c_{162} \) from \( a_{162} \) without rounding, our answer would have been, to six decimal places, 0.062799. This value is still similar to 0.54, but, looking at this number, we would no longer suspect that terms are cancelling to make this an exact answer. Further, we need to remember that, in the context of difference equations in general, choosing \( n = 162 \) is arbitrary. For very large \( n \), say 1000, we would expect \( a_{1000} - c_{1000} \approx 0 \), because we should almost be guaranteed to have a losing streak of at least four games, and a losing streak of at least five games, if we play 1000 times. Conversely, for very small \( n \), say 10, we should expect \( a_{10} - c_{10} \approx 1 \), because the value of each of \( a_{10} \) and \( c_{10} \) is extremely low. What we realize, looking at these two extremes, is that \( a_n - c_n \) depends entirely on the \( n \) we choose, and we have no specific reason to believe that \( n = 162 \) is “special” enough to have \( a_{162} - c_{162} = 0.54 \). We might be a little disappointed that our tangential inquiry didn’t lead to a fascinating discovery, but this, too, is part of being a mathematician. Not every investigation leads to important new results - that’s why we need to pursue many, many questions.
4 Conclusion

Now that we’ve taken the time to answer two different broad questions relating to the Sox’s early season struggles, each of which led us to several additional topics, let’s step back and both review the math that we’ve learned and think about our original goal. In terms of math, we investigated each of the following questions:

- What is the probability that a given team with an expected winning percentage of 0.500 loses its first four games, playing average opponents?

- What is the probability that a given 0.600 team loses its first four games, playing average opponents? A 0.400 team?

- How can we account for the strength of our team’s opponent to make our first four game model more accurate?

- What is the probability that a given average team loses at least four consecutive games at any point during the season?

- By what point in the season do we expect the first such losing streak to have begun?

- What is the probability that a given 0.600 or 0.400 team loses at least four consecutive games within 162 total played?

- What is the probability that a team has at least four consecutive losses during the season with no streaks of five or more?

We answered our first three questions using variations of a simple model based on finding the probability of a sequence of independent events by multiplying the probabilities of each individual event. In order to apply this model to our question, we made some key assumptions. First, baseball games aren’t truly independent events. If the Red Sox play a 15 inning game one night and the next day face a team that didn’t even play the day before, the Sox’s chances of winning the second game are significantly lower than they would be under normal circumstances. Their pitchers, and players in general, are fatigued, while their opponents have their best players well rested. Essentially, we assumed that we could assign one value of \( p \) to the Red Sox for every game, when this is not the case in reality. In addition, we didn’t account for interactions among different teams in the league. Nevertheless, our assumptions seem to be reasonable enough for the answers we found to be somewhat informative.

Our last four questions all depended on solving difference equations, a powerful mathematical technique. We were able to transform a question that seemingly involves an intractable number of different cases into a closed form that we can solve for any total number of games \( n \) for a team with any given probability of winning. Additionally, we used conditional probability identities to write our expression for \( b_n \). We made the same simplifying assumptions in constructing this model as in our first but, again, they seem reasonable enough to allow us to find value in our results.

Many of the questions we investigated along the way were not directly related to informing our opinion about the Sox’s start to the season. Nevertheless, each was a valuable diversion. By trying three values for \( p \) in both equations, we were able to examine the effect of our assumption about the team’s winning percentage on the results each model produced. We also noticed an interesting difference in the effects of taking \( p = 0.4 \), \( p = 0.5 \), or \( p = 0.6 \) in the two models, and we were able to justify mathematically why this is the case. By looking at the probability that a team loses no more than four games in a row at some point during the season, we learned how we can adapt our model to a new situation that, at first glance, seems to call for an entirely novel approach. We saw the importance of precisely wording questions in order to be able to translate them into mathematical equations, and we practiced diving into an investigation of interesting results in the hopes of discovering something new.
After our meandering educational journey, let’s go back to our original goal: we wanted to know if the outcome of the Red Sox’s first four games was unusual. In our first way of looking at the situation, in which we examined the scenario in which a team loses its first four games, we came to the conclusion that it is fairly uncommon, and considerably moreso if the Sox truly are as good a team as we hope they are. In our second broad formulation of the question, we looked at the probability that a team loses four games in a row at any point during the season. Then, we decided that we would expect this to happen to almost every team, even the best in the league. Unfortunately, we also acknowledged that we don’t expect this losing streak to start as early as it did, which brings us back to the more pessimistic conclusion from our first question.

Should we panic yet? If we need to give a simple “yes” or “no” answer, the evidence seems to point to “yes”; the probability is not high that a true talent good team opens the season this poorly. Nevertheless, if we’d like to take an optimistic approach to the season, we can calm ourselves with the knowledge that 92.9 percent of very good teams go through rough stretches like this at some point. Probabilities and models of real world phenomena are valuable for informing our thoughts, but they can’t tell us exactly what to think in any given situation. Ultimately, we need to synthesize the information we have and formulate our own opinions.

Epilogue

In January of 2019, I’m still not sure of the answer to our question “Should we panic yet”? Ultimately, the 2011 Red Sox season ended with one of the worst collapses in baseball history. But between the terrible start and terrible finish, the team was among the best in baseball. In some cases hindsight is 20-20. But, just knowing that the 2011 Red Sox finished with a 0.556 winning percentage doesn’t mean the team’s true talent probability of winning a game against an average opponent was 0.556. To determine the actual $p$ we should have used in these models, we would need to replay the season thousands or millions of times. Baseball research and technology have both come a long way since 2011, and we certainly could simulate the 2011 season repeatedly to get an idea of each team’s true talent $p$. Nevertheless, even now we don’t have the ability to actually replay seasons at our whim in the interest of research. But if we did, . . .