Max flow

Given a graph $G=(V,E)$ with edge capacities $c(e)$, an s-t flow is

$f: E \to \mathbb{R}^{\geq 0}$, s.t.

- $\forall e \in E$, $f(e) \leq c(e)$
- $\forall u \neq s,t$, $\sum_{v:(u,v)\in E} f(u,v) = \sum_{v:(v,u)\in E} f(v,u)$

An s-t <u>max flow</u> is an s-t flow that maximizes <u>the value</u>

$$\sum_{u:(s,u)\in E} f(s,u) = \sum_{u:(u,t)\in E} f(u,t).$$

Problem: Given G, compute max flow

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residual network $G_f = (V, E_f)$: given an s-t flow f,

$\forall e=(u,v) \in G$,

$(u,v) \in E_f$ with capacity $c_f(u,v) = c(u,v) - f(u,v)$
if $f(u,v) \neq c(u,v)$,

$(v,u) \in E_f$ - - - - - $c_f(v,u) = f(u,v)$
if $f(u,v) \neq 0$

Ford-Fulkerson: repeatedly find augmenting paths from s to t in G_f. $O(m \cdot F)$ value of max flow

Edmonds-Karp: shortest augmenting paths
$O(n \cdot m^2)$

Dinic: find a maximal set of disjoint shortest augmenting paths $O(n^2 m)$. (Hopcroft-Karp)

Push-Relabel algorithm

Maintain a "preflow" f:

$$\forall u \neq s,t, \quad \sum_{v:(v,u)\in E} f(v,u) \geq \sum_{v:(u,v)\in E} f(u,v)$$

$$0 \leq f(u,v) \leq c(u,v).$$

(allow more incoming flow than outgoing flow)

The excess of a vertex $u (\neq s, t)$ is

$$e_f(u) = \sum_v f(v,u) - \sum_v f(u,v).$$

Residual network for a preflow is defined similarly.

Maintain a height $h(u)$ for each vertex u, s.t.

$$h(s) = n, \quad h(t) = 0, \quad \text{and}$$

$$\text{if } (u,v) \in E_f, \text{ then } h(u) \leq h(v) + 1. (*)$$

Lemma 1: $(*)$ implies that there is no path from s to t in the residual network.

<u>Proof</u>. O.w. let P be a path $s \to v_1 \to v_2 \to \cdots \to t$ in G_f. Then by $(*)$, $h(v_i) \leq h(v_{i+1}) + 1$.

P contains $\leq n-1$ edges. $h(s) \leq h(t) + (n-1)$.

Contradiction. $\square$

Goal: eliminate all excess $e_f(u)$.

Then f would become a flow, and Lemma 1 implies that there's no augmenting paths.

Push: let u be a vertex with excess $e_f(u) > 0$ $u \neq s, t$

let $(u,v) \in E_f$ and $h(u) = h(v) + 1$.

"push" $\delta = \min\{c_f(u,v), e_f(u)\}$ amount of flow through edge (u,v):

- $f(u,v) \leftarrow f(u,v) + \delta$.

Relabel: let u be a vertex with excess $e_f(u) > 0$ $u \neq s, t$

but $\forall v$ s.t. $(u,v) \in E_f$, $h(u) \leq h(v)$.

"increase" the height of u:

- $h(u) \leftarrow 1 + \min\{h(v) : (u,v) \in E_f\}$.

General algo:

set $h(s) \leftarrow n$, $h(u) \leftarrow 0$ for all $u \neq s$.

set $f(s,u) \leftarrow c(s,u)$ for all $(s,u) \in E$

$f(u,v) \leftarrow 0$ o.w.

while $\exists u$ s.t. $e_f(u) > 0$, $u \neq s, t$

if $\exists (u,v) \in E_f$

push(u,v)

else

relabel(u).

Lemma 2: Push / Relabel doesn't break (*).
(Easy to verify)

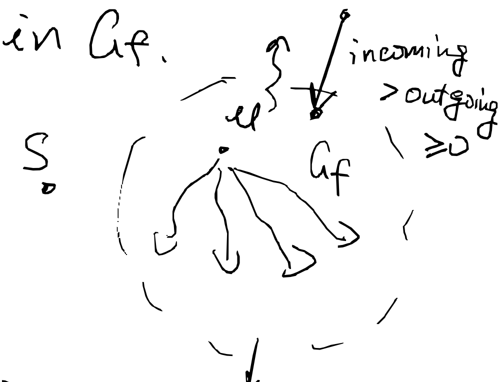
When the algo terminates, f is a flow.

Lemma 1+2 implies that f is max flow.

Analysis

Lemma 3: If u has $e_f(u) > 0$, then $\exists$ a path from u to s in G_f.

(omit proof)



Lemma 4: # relabel $\le O(n^2)$.

Proof: When u is relabeled, $e_f(u) > 0$.

Lemma 3 $\Rightarrow$ $\exists$ a path $u \to v_1 \to v_2 \to \cdots \to s$ in G_f.

(*) $\Rightarrow$ $h(u) \le h(s) + (n-1)$.

$\le 2n-1$.

The height only increases.

relabels $\le n \cdot (2n-1) = O(n^2)$. $\square$

Push(u,v): saturated if $e_f(u) \geq c_f(u,v)$
unsaturated o.w.

Lemma: # saturated push $\leq O(n \cdot m)$.

<u>Proof</u>: Consider each edge (u,v).

A saturated push removes (u,v) from G_f.

At the time of the push, $h(u) = h(v)+1$.

v must be relabeled before flow can be pushed back along (v,u), and restore (u,v) in G_f.

Relabeling v only happens $O(n)$ times.

saturated push on $(u,v) \leq O(n)$. $\square$

Lemma: # unsaturated push $\leq O(n^2 m)$.

<u>Proof</u>. An unsaturated push "moves" excess from u to v with $h(u) = h(v)+1$.

Consider $\Phi(f) = \sum_{u: e_f(u) > 0} h(u)$. Each unsat push decreases Φ by ≥ 1.

Each relabel increases Φ by $\le O(n)$

Each saturated push increases Φ by $\le O(n)$.

Φ starts at $\le O(n^2)$, total increase

$\le O(n^2 m)$.

$\Phi \ge 0$. Thus, # unsat push $\le O(n^2 m)$. $\square$