PRINCETON UNIV SPRING '26 COS 423: THEORY OF ALGORITHMS

Homework 1

Out: *Feb 2*

Problems:

§1 Consider a undirected weighted graph G, where the edge weights are NOT necessarily distinct.

(a) Give an example of such G, of which Minimum Spanning Trees are not unique.

(b) Prove that for some edge (u, v) if there exists a cut (X, Y) such that (u, v) is the *unique* minimum weight edge (no ties), then (u, v) must be in every MST of G.

(c) Prove if there exists a cycle that contains (u, v) and has (u, v) as the *unique* maximum weight edge, then (u, v) must not be in any MST of G.

(d) (Optional) Prove from first principles (without going through MST), that for an edge (u, v), there exists a cut (X, Y) such that (u, v) is one of the minimum weight edges if and only if for all cycles that contain (u, v), (u, v) is not the unique maximum weight edge.

§2 Let G be an undirected and weighted graph with distinct edge weights. Consider the *second-best minimum spanning trees* of G.

(a) Show that while the MST is unique, the second-best MST may not be unique.

(b) Show that a second-best MST can be obtained by replacing one edge from the MST.

(c) Design an $O(n^2)$-time algorithm for computing a second-best MST.

(d) (Optional) Design an $O(m+n \log n)$-time algorithm for computing a second-best MST.

§3 Let $G = (V, E)$ be a weighted graph with (possibly negative) edge weights $w(u, v)$.

(a) Suppose only edges leaving the source s may have negative weights. Prove that Dijkstra's algorithm computes shortest paths from s correctly.

(b) Let $f : V \longrightarrow \mathbb{R}$ be a weight function on the vertices. Let G' be another graph obtained by changing the edge weights of G to $w'(u, v) := w(u, v) + f(u) - f(v)$ for all edges (u, v). Prove that for any s, t, the shortest paths in G from s to t is the same as the shortest paths in G' from s to t.

(c) Prove that if one can compute in time $O(T)$, a weight function f such that $w'(u, v) \geq 0$ for all (u, v), then shortest paths from s in G can be computed in time $O(T+m \log n)$; if one can compute shortest paths from s in G in time $O(T)$, then one can compute a weight function f such that $w'(u, v) \geq 0$ for all (u, v) in time $O(T)$.