A Functional Space Model

COS 326
David Walker
Princeton University

Last Time

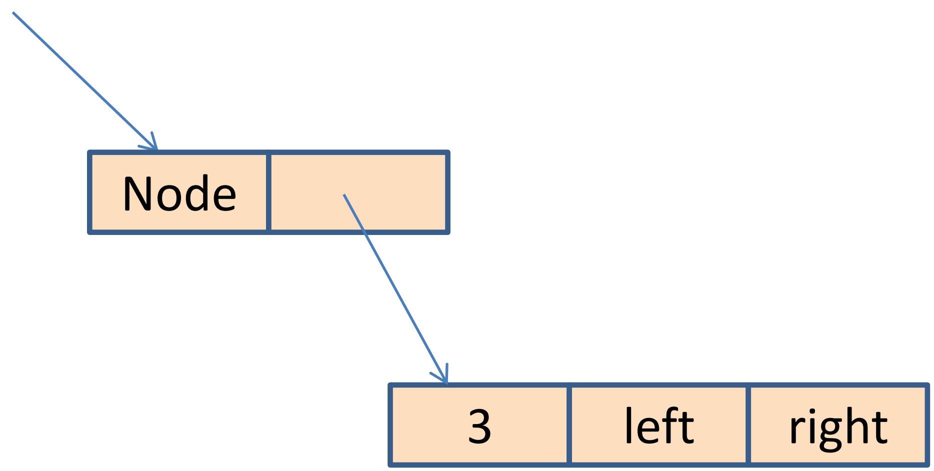
Data type representations:

```
type tree = Leaf | Node of int * tree * tree
```

Leaf:

0

Node(i, left, right):



This Time

Understanding the space complexity of functional programs

- At least two interesting components:
  - the amount of *live space* at any instant in time
  - the *rate of allocation*
    - a function call may not change the amount of live space by much but may allocate at a substantial rate
    - because functional programs act by generating new data structures and discarding old ones, they often allocate at a great rate
      - » OCaml garbage collector is optimized with this in mind
      - » interesting fact: at the assembly level, the number of writes made by a function program is typically roughly the same as the number of writes by an imperative program

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    - because functional programs act by generating new data structures and discarding old ones, they often allocate at a great rate
      - » OCaml garbage collector is optimized with this in mind
      - » interesting fact: at the assembly level, the number of writes made by a function program is typically roughly the same as the number of writes by an imperative program
- *What takes up space*?
  - conventional first-order data: tuples, lists, strings, datatypes
  - function representations (closures)
  - the call stack

CONVENTIONAL DATA

Allocating space

Whenever you use a constructor, space is allocated:

```
let rec insert (t:tree) (i:int) =
  match t with
    Leaf -> Node (i, Leaf, Leaf)
  | Node (j, left, right) ->
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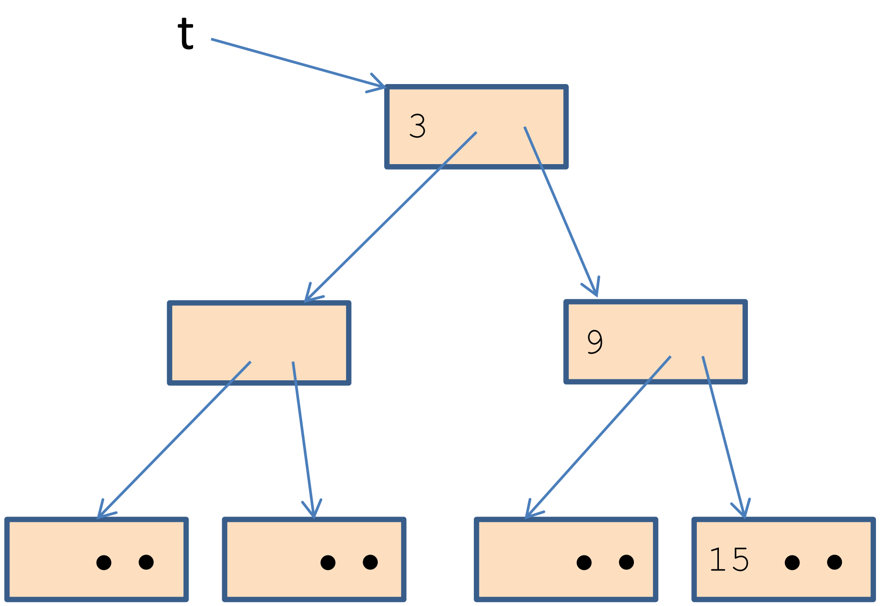

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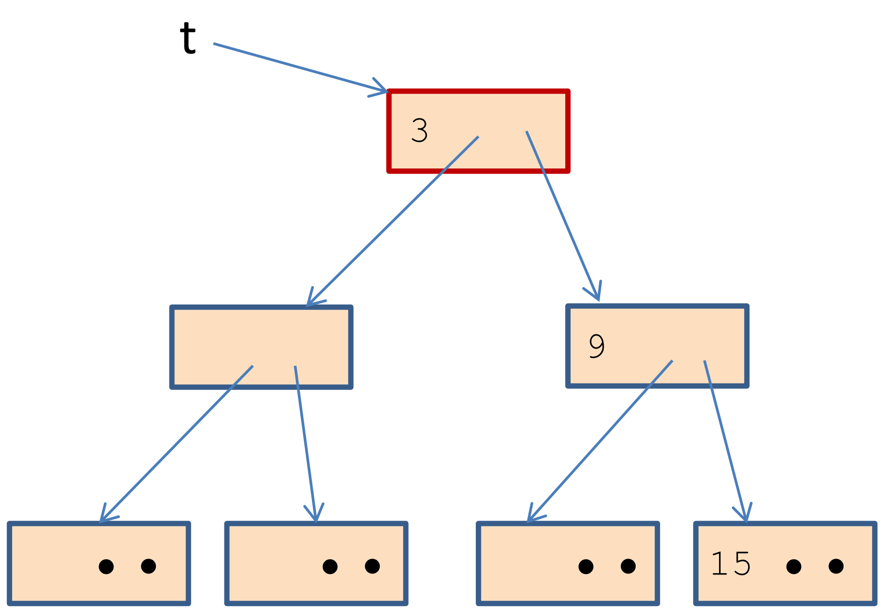

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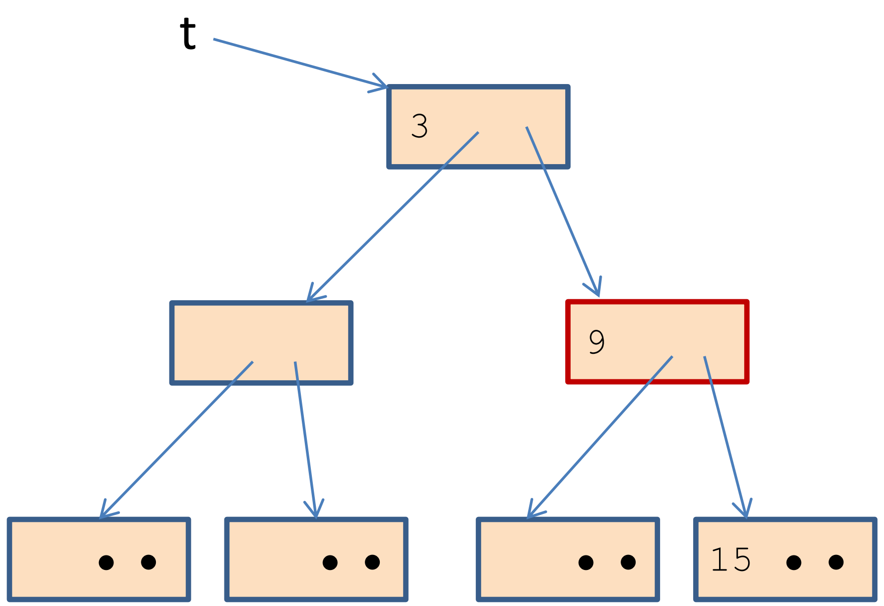

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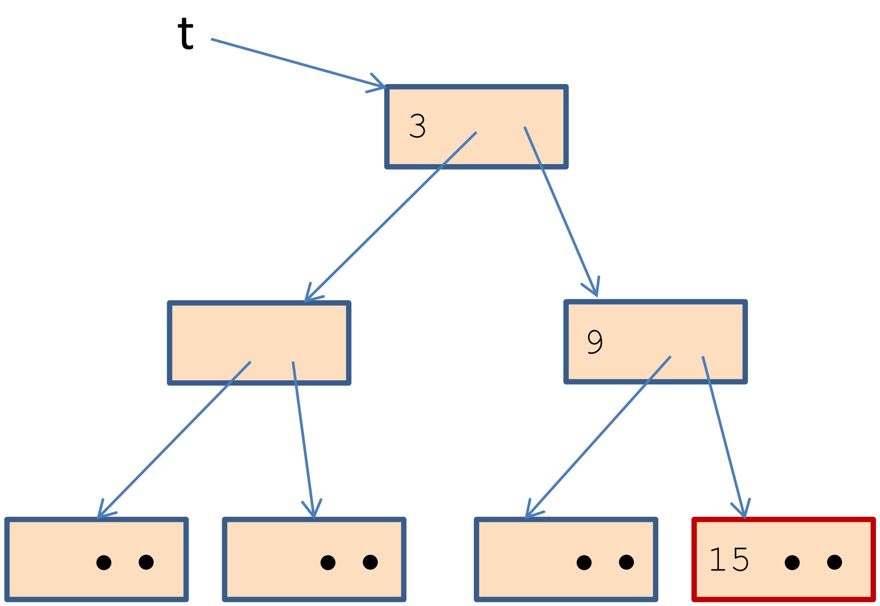

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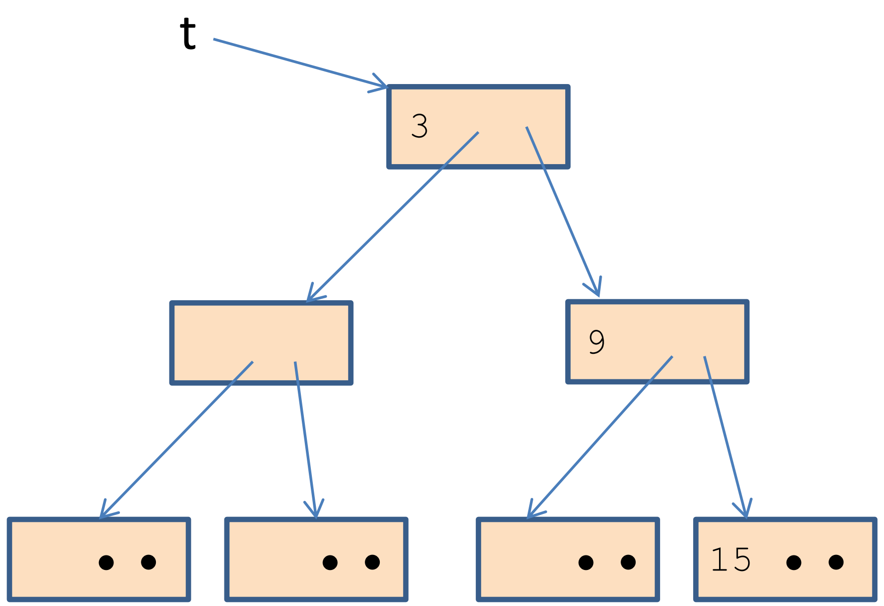

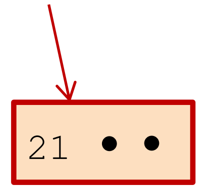

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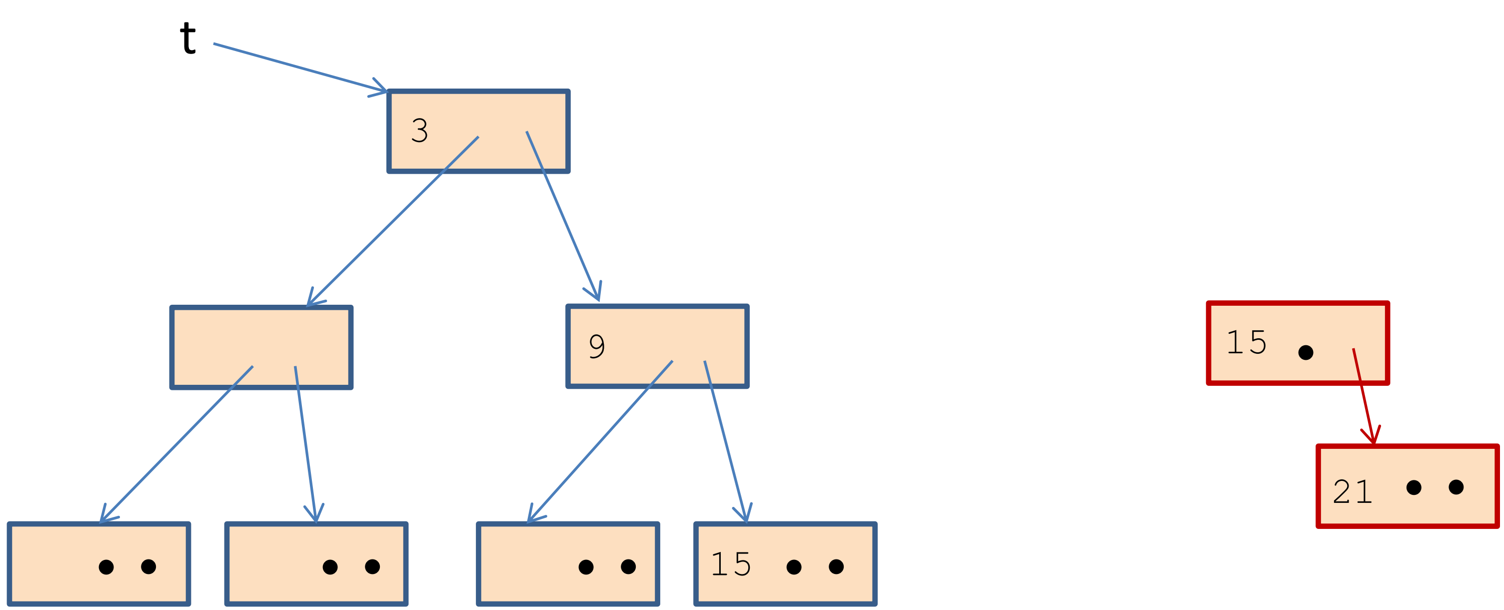

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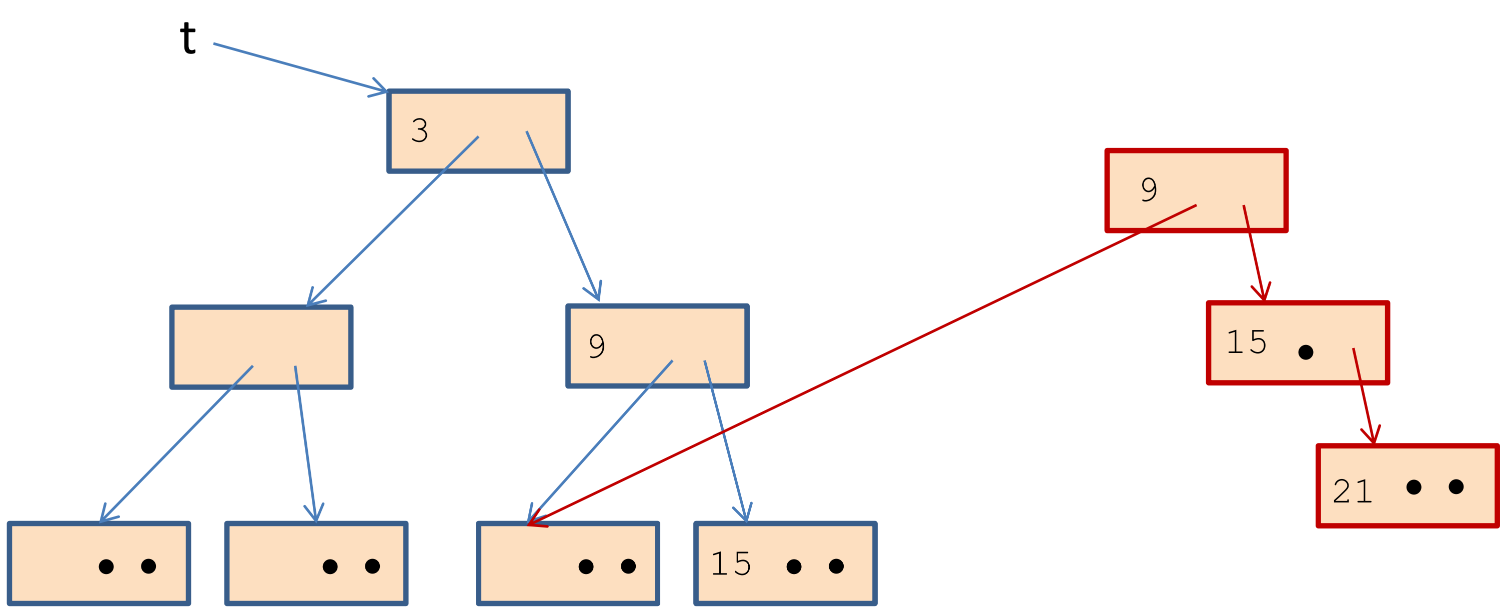

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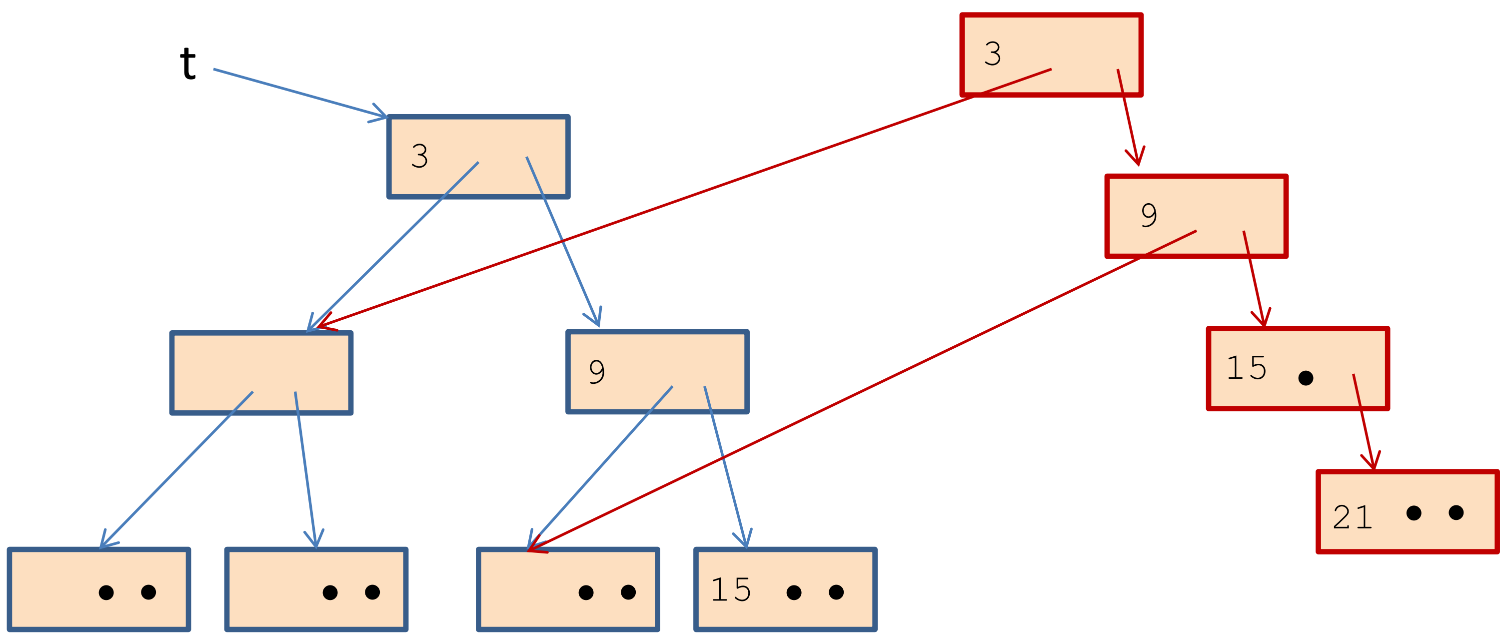

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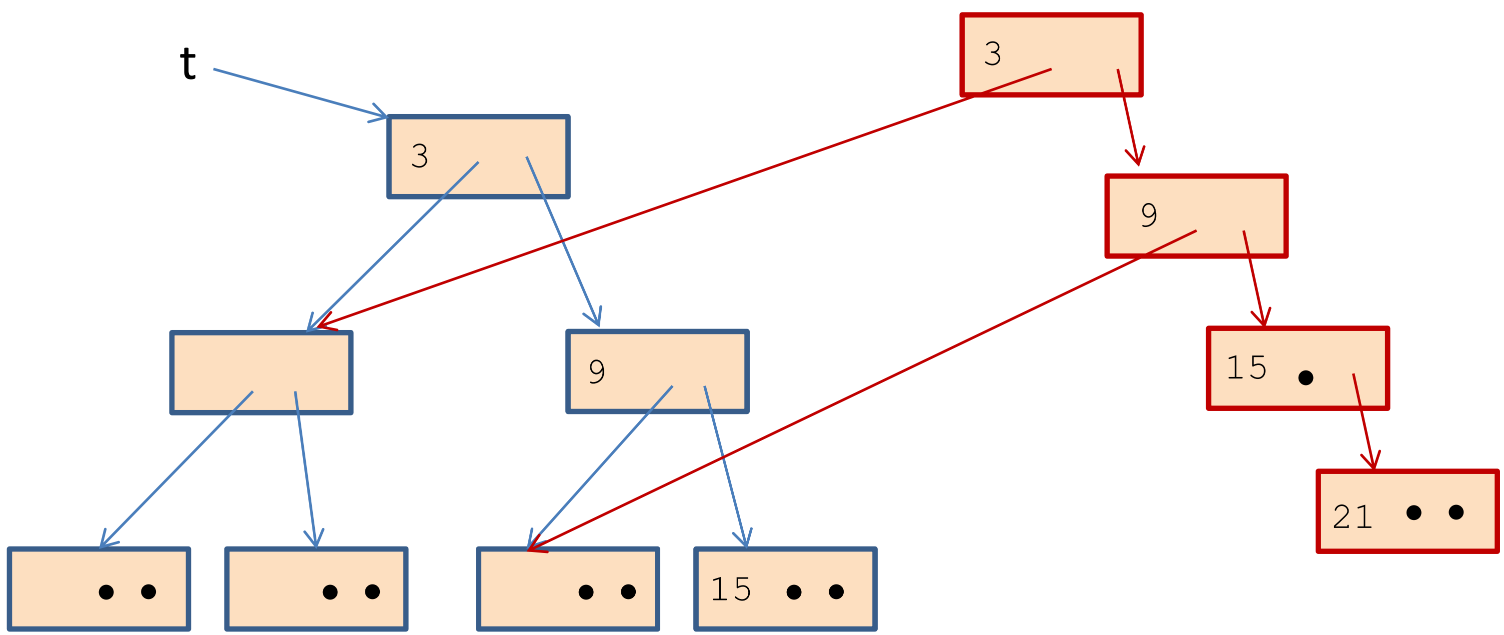
Total space allocated is proportional to the height of the tree.

~ log n, if tree with n nodes is balanced



Compare

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let check_option (o:int option) : int option =
  match o with
    Some _ -> o
  | None -> failwith "found none"
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let cadd (c1:int*int) (c2:int*int) : int*int =
  let (x1,y1) = c1 in
  let (x2,y2) = c2 in
  (x1+x2, y1+y2)
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let double (c1:int*int) : int*int =
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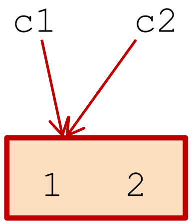


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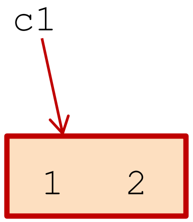


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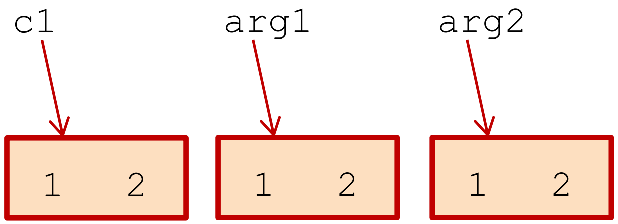

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cadd allocates
double allocates 2 pairs

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cadd allocates
double does not

extracts components; does not allocate

FUNCTION CLOSURES

Closures

Consider the following program:

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let choose (arg:bool * int * int) : int -> int =
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  if b then
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choose (true, 1, 2);;
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Substitution and Compiled Code

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compile →

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choose:
  mov rb r_arg[0]
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  compare rb 0
  ...
  jmp ret

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  ...
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choose:
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```
choose_subst:
  mov rb 0xF8[0]
  mov rx 0xF8[4]
  mov ry 0xF8[8]
  compare rb 0
  ...
  jmp ret
```

```
0xF8: 0
      1
      2
```

Substitution and Compiled Code

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main:
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  jmp choose
```

```
choose_subst:
  mov rb 0xF8[0]
  mo
  mo
  co
  ..
  jm
```

```
0xF8: 0
      1
```

```
choose_subst2:
  compare 1 0
  ...
  jmp ret
```

What we aren't going to do

The substitution model of evaluation is *just a model*. It says that we generate new code at each step of a computation. We don't do that in reality. Too expensive!

The substitution model is a faithful model for reasoning about program correctness but it doesn't help us understand what is going on at the machine-code level

- that's a good thing! *abstraction!!*
- *you should almost never think about machine code when writing a program. We invented high-level programming languages so you don't have to.*

Still, we need to have a more faithful space model in order to understand how to write efficient algorithms.

Some functions are easy to implement

```
let add (x:int*int) : int =
  let (y,z) = x in
  y + z
;;
```

```
# argument in r1
# return address in r0

add:
  ld r2, r1[0]      # y in r2
  ld r3, r1[4]      # z in r3
  add r4, r2, r3    # sum in r4
  jmp r0
```

If no functions in ML were nested then compiling ML would be just like compiling C. (Take COS 320 to find out how to do that...)

How do we implement functions?

Let's remove the nesting and compile them like we compile C.

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x)
  else
    (fun n -> n + y)
;;
```

?

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    f1
  else
    f2
;;
```

?

```
let f1 n = n + x;;
```

?

```
let f2 n = n + y;;
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let f2 n = n + y;;
```

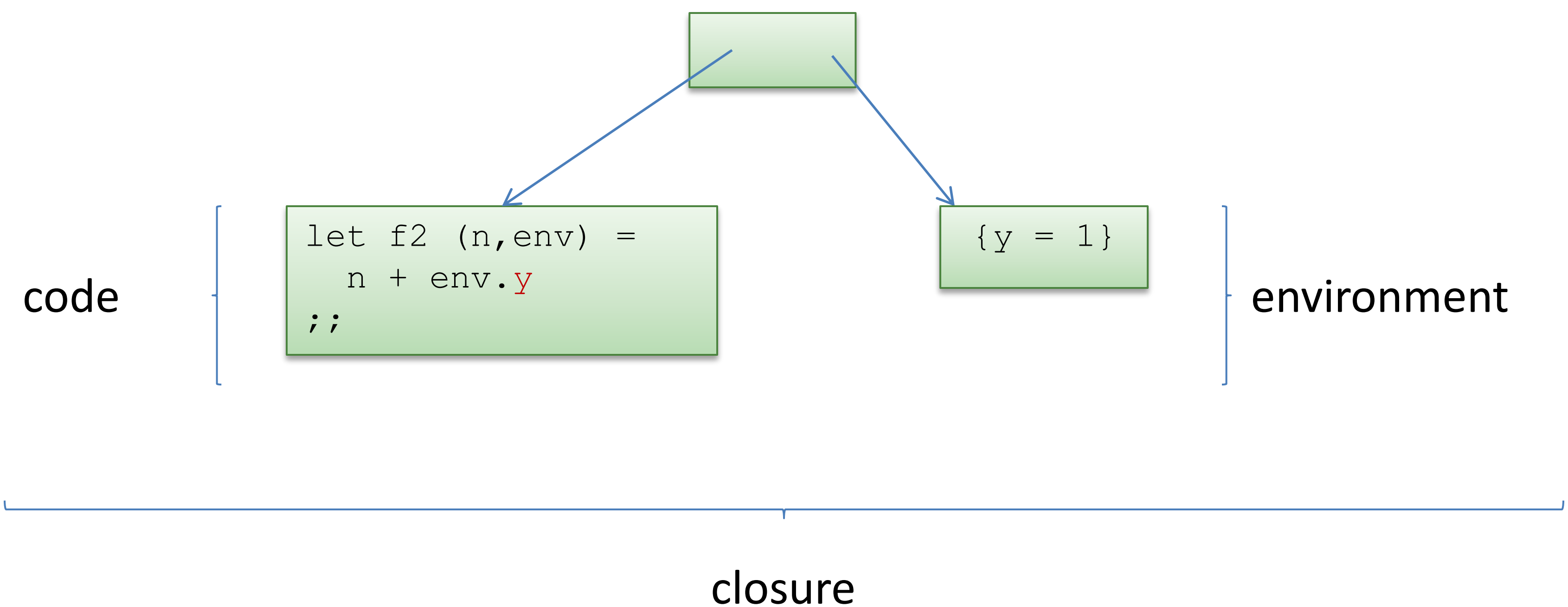
Darn! *Doesn’t work naively*. Nested functions contain *free variables*.
Simple unnesting leaves them undefined.

How do we implement functions?

We can’t define a function like the following using code alone:

```
let f2 n = n + y;;
```

A *closure* is a pair of some code and an environment:



Closure Conversion

Closure conversion (also called lambda lifting) converts open, nested functions in to closed, top-level functions.

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let choose arg =
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;;
```

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

add environment parameter

create closures

use environment variables instead of free variables

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;;
```

add environment parameter

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

create closures

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

use environment variables instead of free variables

```
(choose (true,1,2)) 3
```

```
let c_closure        = (choose, ())                in (* create closure *)
let (c_code, c_cenv) = c_closure                   in (* extract code, env *)
let (f_code, f_env)  = c_code ((true,1,2), c_env) in (* call choose code, extract f code, env *)
  f_code (3, f_env)                                   (* call f code *)
;;
```

Closure Conversion

Closure conversion (also called lambda lifting) converts open, nested functions in to closed, top-level functions.

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x + y)
  else
    (fun n -> n + y)
;;
```

```
(choose (true,1,2)) 3
```

add environment parameter

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

create closures

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

use environment variables instead of free variables

```
let c_closure          = (choose, ())                in (* create closure *)
let (c_code, c_cenv) = c_closure                     in (* extract code, env *)
let (f_code, f_env)  = c_code ((true,1,2), c_env) in (* call choose code, extract f code, env *)
  f_code (3, f_env)                                     (* call f code *)
;;
```

Closure Conversion

Closure conversion (also called lambda lifting) converts open, nested functions in to closed, top-level functions.

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x + y)
  else
    (fun n -> n + y)
;;
```

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

add environment parameter

create closures

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

use environment variables instead of free variables

```
(choose (true,1,2)) 3
```

```
let c_closure        = (choose, ())                in (* create closure *)
let (c_code, c_cenv) = c_closure                   in (* extract code, env *)
let (f_code, f_env)  = c_code ((true,1,2), c_env) in (* call choose code, extract f code, env *)
  f_code (3, f_env)                                   (* call f code *)
;;
```

Closure Conversion

Closure conversion (also called lambda lifting) converts open, nested functions in to closed, top-level functions.

```
let choose arg =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x + y)
  else
    (fun n -> n + y)
;;
```

add environment parameter

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

create closures

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

use environment variables instead of free variables

```
(choose (true,1,2)) 3
```

```
let c_closure        = (choose, ())                in (* create closure *)
let (c_code, c_cenv) = c_closure                   in (* extract code, env *)
let (f_code, f_env)  = c_code ((true,1,2), c_env)  in (* call choose code, extract f code, env *)
  f_code (3, f_env)                                   (* call f code *)
;;
```

One Extra Note: Typing

Even though the original, non-closure-converted code was well-typed, the closure-converted code isn't because the environments are different

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, F1 {xe=x; ye=y})
  else
    (f2, F2 {ye=y})
;;
```

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

```
type f1_env = {x1:int; y1:int}      type f1_clos = (int * f1_env -> int) * f1_env

type f2_env = {y2:int}              type f2_clos = (int * f2_env -> int) * f2_env
```

One Extra Note: Typing

Even though the original, non-closure-converted code was well-typed, the closure-converted code isn't because the environments are different

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, F1 {x1=x; y2=y})
  else
    (f2, F2 {y2=y})
;;
```

```
let f1 (n,env) =
  match env with
    F1 e -> n + e.x1 + e.y2
  | F2 _ -> failwith "bad env!"
;;
```

```
let f2 (n,env) =
  match env with
    F1 _ -> failwith "bad env!"
  | F2 e -> n + e.y2
;;
```

```
type f1_env = {x1:int; y1:int}      type f1_clos = (int * f1_env -> int) * f1_env

type f2_env = {y2:int}              type f2_clos = (int * f2_env -> int) * f2_env
```

fix I:

```
type env = F1 of f1_env | F2 of f2_env
type f1_clos = (int * env -> int) * env
type f2_clos = (int * env -> int) * env
```

One Extra Note: Typing

Even though the original, non-closure-converted code was well-typed, the closure-converted code isn't because the environments are different

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

```
type f1_env = {xe:int; ye:int}     type f1_clos = (int * f1_env -> int) * f1_env

type f2_env = {xe:int}             type f2_clos = (int * f2_env -> int) * f2_env
```

fix II:

```
type f1_env = {xe:int; ye:int}
type f2_env = {xe:int}
type f1_clos = exists env.(int * env -> int) * env
type f2_clos = exists env.(int * env -> int) * env
```

One Extra Note: Typing

Even though the original, non-closure-converted code was well-typed, the closure-converted code isn't because the environments are different

```
let choose (arg,env) =
  let (b, x, y) = arg in
  if b then
    (f1, {xe=x; ye=y})
  else
    (f2, {ye=y})
;;
```

```
let f1 (n,env) =
  n + env.xe + env.ye
;;
```

```
let f2 (n,env) =
  n + env.ye
;;
```

"*From System F to Typed Assembly Language*,"
-- Morrisett, Walker et al.

```
type f1_env = {xe:int; ye:int}      type f1_clos = (int * f1_env -> int) * f1_env

type f2_env = {xe:int}              type f2_clos = (int * f2_env -> int) * f2_env
```

fix II:

```
type f1_env = {xe:int; ye:int}
type f2_env = {xe:int}
type f1_clos = exists env.(int * env -> int) * env
type f2_clos = exists env.(int * env -> int) * env
```

Aside: Existential Types

map has a *universal* polymorphic type:

map : ('a -> 'b) -> 'a list -> 'b list "for *all* types 'a and for *all* types 'b, …"

when we closure-convert a function that has type int -> int, we get a function with *existential* polymorphic type:

exists 'a. ((int * 'a) -> int) * 'a "there *exists some* type 'a such that, …"

In OCaml, we can approximate existential types using datatypes (a data type allows you to say "there exists a type 'a drawn from one of the following finite number of options." In Haskell, you've got the real thing.

Closure Conversion: Summary

(before)	(after)

All function definitions equipped with extra env parameter:

`let f arg = ...`	`let f_code (arg, env) = ...`

All free variables obtained from environment:

`x`	`env.cx`

All functions values paired with environment:

`f`	`(f_code, {ve1=v1; ...; ven=vn})`

All function calls extract code and environment and call code:

`f e`	`let (f_code, f_env) = f in` `f_code (e, f_env)`

The Space Cost of Closures

The space cost of a closure

= the cost of the pair of code and environment pointers

+ the cost of the data referred to by function free variables

TAIL CALLS AND CONTINUATIONS

Some Innocuous Code

```
(* sum of 0..n *)

let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;

let big_int = 1000000;;

sum big_int;;
```

Let’s try it.

(Go to tail.ml)

Some Other Code

Four functions: Green works on big inputs; Red doesn't.

```
let sum_to2 (n: int) : int =
  let rec aux (n:int) (a:int) : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

```
let rec sum2 (l:int list) : int =
  match l with
      [] -> 0
    | hd::tail -> hd + sum2 tail
;;
```

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;
```

```
let sum (l:int list) : int =
  let rec aux (l:int list) (a:int) : int =
    match l with
        [] -> a
      | hd::tail -> aux tail (a+hd)
  in
  aux l 0
;;
```

Some Other Code

Four functions: Green works on big inputs; Red doesn't.

```
let sum_to2 (n: int) : int =
  let rec aux (n:int) (a:int) : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

```
let rec sum2 (l:int list) : int =
  match l with
      [] -> 0
    | hd::tail -> hd + sum2 tail
;;
```

```
let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;
```

```
let sum (l:int list) : int =
  let rec aux (l:int list) (a:int) : int =
    match l with
        [] -> a
      | hd::tail -> aux tail (a+hd)
  in
  aux l 0
;;
```

code that works:
no computation after recursive function call

Tail Recursion

A *tail-recursive function* is a function that does no work after it calls itself recursively.

```
(* sum of 0..n *)

let sum_to2 (n: int) : int =
  let rec aux (n:int)(a:int)
                : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

Tail Recursion

A *tail-recursive function* is a function that does no work after it calls itself recursively.

Tail-recursive:

```
    sum_to2 1000000
-->
    aux 1000000 0
-->
    aux 99999 1000000
-->
    aux 99998 1999999
-->
    ...
-->
    aux 0 (-363189984)
-->
    -363189984
```

```
(* sum of 0..n *)

let sum_to2 (n: int) : int =
  let rec aux (n:int)(a:int)
                  : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

constant size expression

(addition overflow occurred at some point)

Tail Recursion

A *tail-recursive function* is a function that does no work after it calls itself recursively.

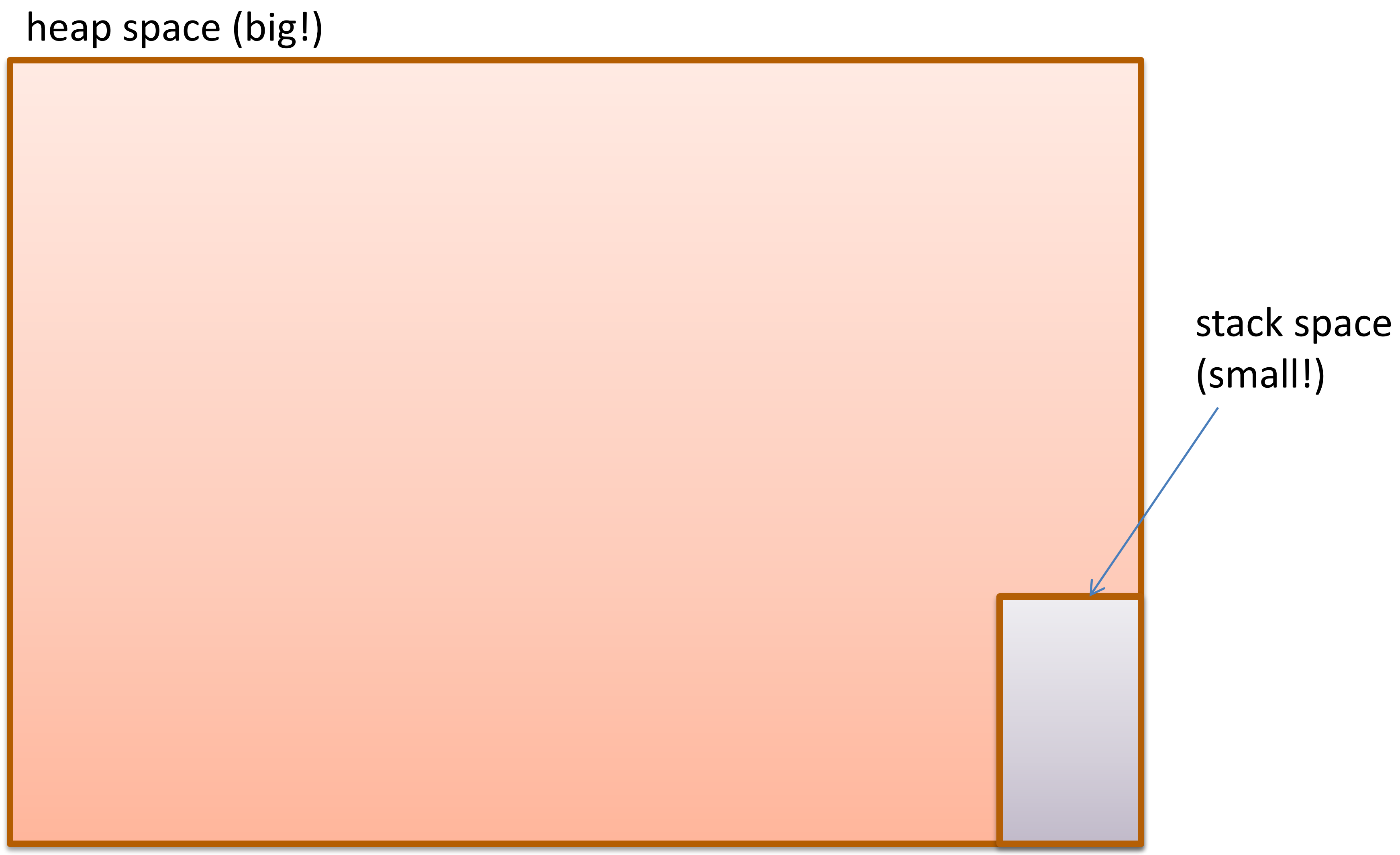
Not tail-recursive:

```
    sum_to 1000000
-->
    1000000 + sum_to 99999
-->
    1000000 + 99999 + sum_to 99998
-->
    ...
-->
    1000000 + 99999 + 99998 + ... + sum_to 0
-->
    1000000 + 99999 + 99998 + ... + 0
-->
    ... add it all back up ...
```

```
(* sum of 0..n *)

let rec sum_to (n:int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else 0
;;

let big_int = 1000000;;

sum big_int;;
```

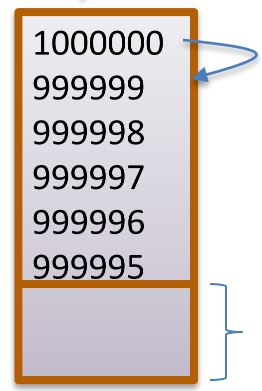
expression grows
at every recursive call

Memory is partitioned: Stack and Heap



Data Needed on Return Saved on Stack

```
    sum_to 1000000
-->
    ...
-->
    1000000 + 99999 + 99998 + 99997 + ... +
-->
    ...
-->
    ...
```



the stack

every non-tail call puts the data from the calling context on the stack

not much space left! will run out soon!

Question

Can any non-tail-recursive function be transformed in to a tail-recursive one?

```
let rec sum_to (n: int) : int =
  if n > 0 then
    n + sum_to (n-1)
  else
    0
;;
```

human ingenuity

```
let sum_to2 (n: int) : int =
  let rec aux (n:int)(a:int) : int =
    if n > 0 then
      aux (n-1) (a+n)
    else a
  in
  aux n 0
;;
```

not only is sum2 tail-recursive but it reimplements an algorithm that took *linear space* (on the stack) using an algorithm that executes in *constant space*!

Question

Can any non-tail-recursive function be transformed in to a tail-recursive one? Yes, if we can capture the *differential* between a tail-recursive function and a non-tail-recursive one.

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

Idea: Focus on what happens after the recursive call.

Question

Can any non-tail-recursive function be transformed in to a tail-recursive one? Yes, if we can capture the *differential* between a tail-recursive function and a non-tail-recursive one.

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

what happens next

Idea: Focus on what happens after the recursive call.

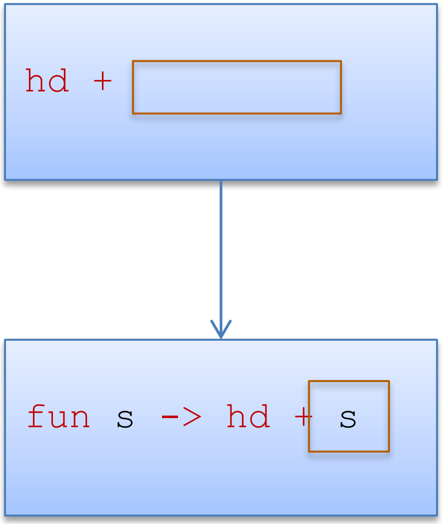
Extracting that piece:

```
hd + [      ]
```

How do we capture it?

Question

How do we capture that computation?



Question

How do we capture that computation?

```
hd +
```

```
fun s -> hd + s
```

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> ???) ;;
```

Question

How do we capture that computation?

```
hd + [          ]
```

```
fun s -> hd + s
```

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;
```

Question

How do we capture that computation?

```
hd + 
```

```
fun s -> hd + s
```

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = ??
```

Question

How do we capture that computation?

```
hd +
```

```
fun s -> hd + s
```

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
```

Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
```

Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
```

Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
-->
    sum_cont [] (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s))
```

Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
-->
    sum_cont [] (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s))
-->
    (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s)) 0
```

Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
-->
    sum_cont [] (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s))
-->
    (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s)) 0
-->
    (fun s -> (fun s -> s) (1 + s)) (2 + 0))
```

Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
-->
    sum_cont [] (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s))
-->
    (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s)) 0
-->
    (fun s -> (fun s -> s) (1 + s)) (2 + 0))
-->
    (fun s -> s) (1 + (2 + 0))
```

Execution

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
-->
    sum_cont [] (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s))
-->
    (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s)) 0
-->
    (fun s -> (fun s -> s) (1 + s)) (2 + 0))
-->
    (fun s -> s) (1 + (2 + 0))
-->
    1 + (2 + 0)
-->
    3
```

Question

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum (l:int list) : int = sum_cont l (fun s -> s)
```

```
    sum [1;2]
-->
    sum_cont [1;2] (fun s -> s)
-->
    sum_cont [2] (fun s -> (fun s -> s) (1 + s));;
-->
    sum_cont [] (fun s -> (fun s -> (fun s -> s) (1 + s)) (2 + s))
-->
    ...
-->
    3
```

Where did the stack space go?

CPS

CPS:

- short for *Continuation-Passing Style*
- Every function takes a continuation as an argument that expresses "what to do next"
- CPS functions only call other functions as the last thing they do

CORRECTNESS OF A CPS TRANSFORM

Are the two functions the same?

```
type cont = int -> int;;

let rec sum_cont (l:int list) (k:cont): int =
  match l with
    [] -> k 0
  | hd::tail -> sum_cont tail (fun s -> k (hd + s)) ;;

let sum2 (l:int list) : int = sum_cont l (fun s -> s)
```

```
let rec sum (l:int list) : int =
  match l with
    [] -> 0
  | hd::tail -> hd + sum tail
;;
```

Here, it is really pretty tricky to be sure you've done it right if you don't prove it. Let's try to prove this theorem and see what happens:

```
for all l:int list,
  sum_cont l (fun x => x) == sum l
```

Attempting a Proof

```
for all l:int list, sum_cont l (fun s -> s) == sum l

Proof: By induction on the structure of the list l.

case l = []
  ...

case: hd::tail
  IH: sum_cont tail (fun s -> s) == sum tail
```

Attempting a Proof

```
for all l:int list, sum_cont l (fun s -> s) == sum l

Proof: By induction on the structure of the list l.

case l = []
  ...

case: hd::tail
  IH: sum_cont tail (fun s -> s) == sum tail

   sum_cont (hd::tail) (fun s -> s)
==
```

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== sum_cont tail (fn s' -> (fn s -> s) (hd + s'))   (eval)
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== sum_cont tail (fn s' -> hd + s')                 (eval -- hd + s' valuable)
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Need to Generalize the Theorem and IH

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== sum_cont tail (fn s' -> (fn s -> s) (hd + s'))   (eval)
== sum_cont tail (fn s' -> hd + s')                 (eval -- hd + s' valuable)

== darn!
```

we'd like to use the IH, but we can't!
we might like:

sum_cont tail (fn s' -> hd + s') == sum tail

... but that's not even true

not the identity continuation (fun s -> s) like the IH requires

Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)
```

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  must prove:  for all k:int->int, sum_cont [] k == k (sum [])
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  must prove:  for all k:int->int, sum_cont [] k == k (sum [])

  pick an arbitrary k:
```

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Proof: By induction on the structure of the list l.

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  must prove:  for all k:int->int, sum_cont [] k == k (sum [])

  pick an arbitrary k:

     sum_cont [] k
```

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```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = []

  must prove:  for all k:int->int, sum_cont [] k == k (sum [])

  pick an arbitrary k:

     sum_cont [] k
  == match [] with [] -> k 0 | hd::tail -> ...      (eval)
  == k 0                                            (eval)
```

Need to Generalize the Theorem and IH

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  for all k:int->int, sum_cont l k == k (sum l)

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  pick an arbitrary k:

       sum_cont [] k
  == match [] with [] -> k 0 | hd::tail -> ...      (eval)
  == k 0                                            (eval)



  == k (sum [])
```

Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = []

  must prove:  for all k:int->int, sum_cont [] k == k (sum [])

  pick an arbitrary k:

       sum_cont [] k
  == match [] with [] -> k 0 | hd::tail -> ...       (eval)
  == k 0                                             (eval)

  == k (0)                                           (eval, reverse)
  == k (match [] with [] -> 0 | hd::tail -> ...)     (eval, reverse)
  == k (sum [])

case done!
```

Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = [] ===> done!

case l = hd::tail

  IH:   for all k':int->int, sum_cont tail k' == k' (sum tail)

  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))
```

Need to Generalize the Theorem and IH

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  Pick an arbitrary k,

     sum_cont (hd::tail) k
```

Need to Generalize the Theorem and IH

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  Pick an arbitrary k,

     sum_cont (hd::tail) k
  == sum_cont tail (fun s -> k (hd + x))      (eval)
```

Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

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  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))

  Pick an arbitrary k,

     sum_cont (hd::tail) k
  == sum_cont tail (fun s -> k (hd + x))      (eval)

  == (fun s -> k (hd + s)) (sum tail)         (IH with IH quantifier k'
                                               replaced with (fun x -> k (hd+x))
```

Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = [] ===> done!

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  IH:   for all k':int->int, sum_cont tail k' == k' (sum tail)

  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))

  Pick an arbitrary k,

     sum_cont (hd::tail) k
  == sum_cont tail (fun s -> k (hd + x))      (eval)

  == (fun s -> k (hd + s)) (sum tail)         (IH with IH quantifier k'
                                               replaced with (fun x -> k (hd+x))
  == k (hd + (sum tail))                      (eval, since sum total and
                                                     and sum tail valuable)
```

Need to Generalize the Theorem and IH

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)

Proof: By induction on the structure of the list l.

case l = [] ===> done!

case l = hd::tail

  IH:   for all k':int->int, sum_cont tail k' == k' (sum tail)

  Must prove:  for all k:int->int, sum_cont (hd::tail) k == k (sum (hd::tail))

  Pick an arbitrary k,

     sum_cont (hd::tail) k
  == sum_cont tail (fun s -> k (hd + x))      (eval)

  == (fun s -> k (hd + s)) (sum tail)         (IH with IH quantifier k'
                                               replaced with (fun x -> k (hd+x))
  == k (hd + (sum tail))                      (eval, since sum total and
                                                     and sum tail valuable)
  == k (sum (hd:tail))                        (eval sum, reverse)

case done!
QED!
```

Finishing Up

Ok, now what we have is a proof of this theorem:

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)
```

We can use that general theorem to get what we really want:

```
for all l:int list,
   sum2 l
== sum_cont l (fun s -> s)       (by eval sum2)
== (fun s -> s) (sum l)          (by theorem, instantiating k with (fun s -> s)
== sum l
```

So, we've show that the function sum2, which is tail-recursive, is functionally equivalent to the non-tail-recursive function sum.

SUMMARY

Summary of the CPS Proof

We tried to prove the *specific* theorem we wanted:

```
for all l:int list, sum_cont l (fun s -> s) == sum l
```

But it didn't work because in the middle of the proof, *the IH didn't apply* -- inside our function we had the wrong kind of continuation -- not (fun s -> s) like our IH required. So we had to *prove a more general theorem* about *all* continuations.

```
for all l:int list,
  for all k:int->int, sum_cont l k == k (sum l)
```

This is a common occurrence -- *generalizing the induction hypothesis* -- and it requires human ingenuity. It's why proving theorems is hard. It's also why writing programs is hard -- you have to make the proofs and programs work more generally, around every iteration of a loop.

Overall Summary

We developed techniques for reasoning about the space costs of functional programs

- the cost of *manipulating data types* like tuples and trees
- the cost of allocating and *using function closures*
- the cost of *tail-recursive* and non-tail-recursive *functions*

We also talked about some important program transformations:

- *closure conversion* makes nested functions with free variables in to pairs of closed code and environment
- the *continuation-passing style* (CPS) transformation turns non-tail-recursive functions in to tail-recursive ones that use no stack space
  - the stack gets moved in to the function closure
- since stack space is often small compared with heap space, it is often necessary to use *continuations and tail recursion*
  - but full CPS-converted programs are unreadable: use judgement

Challenge: CPS Convert the incr function

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

(see solution after the next slide)

END

CPS Convert the incr function

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

```
type cont = tree -> tree ;;

let rec incr_cps (t:tree) (i:int) (k:cont) : tree =
  match t with
    Leaf -> k Leaf
  | Node (j,left,right) -> ...
;;
```

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

first continuation:

```
Node (i+j, ___________ , incr right i)
```

second continuation:

```
Node (i+j, left_done, ______________ )
```

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr i left, incr i right)
;;
```

first continuation:

```
fun left_done -> Node (i+j, left_done , incr right i)
```

second continuation:

```
fun right_done -> k (Node (i+j, left_done, right_done))
```

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

second continuation
inside
first continuation:

```
fun left_done ->
  let k2 =
    (fun right_done ->
       k (Node (i+j, left_done, right_done))
     )
  in
  incr right i k2
```

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
    Leaf -> Leaf
  | Node (j,left,right) -> Node (i+j, incr left i, incr right i)
;;
```

```
type cont = tree -> tree ;;

let rec incr_cps (t:tree) (i:int) (k:cont) : tree =
  match t with
    Leaf -> k Leaf
  | Node (j,left,right) ->
      let k1 = (fun left_done ->
                  let k2 = (fun right_done ->
                              k (Node (i+j, left_done, right_done)))
                  in
                  incr_cps right i k2
                )
      in
      incr_cps left i k1
;;


let incr_tail (t:tree) (i:int) : tree = incr_cps t i (fun t -> t);;
```