

Hoare Logic: Loops & Framing

COS 441 Slides 12

Agenda

- Last few lectures
 - Hoare Logic:
 - $\{P\}C\{Q\}$
 - If P is true in the initial state s. And C in state s evaluates to s'. Then Q must be true in s'.
 - Rules of Hoare logic:
 - rule of consequence
 - assignment rule, skip rule, sequence rule, if rule
- This time:
 - While Loops

HOARE LOGIC: WHILE LOOPS

While Statements

- Rule for while statements

If ???

then { P } while (e > 0) do C { Q }

While Statements

- **Bogus** rule for while statements

If $\{ P \ \& \ e > 0 \}$ C $\{ Q \}$

then $\{ P \}$ while $(e > 0)$ do C $\{ Q \}$

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then $\{ P \}$ while $(e > 0)$ do $C \{ Q \}$



basic problem:
this rule only
captures 1 iteration
of the loop,
not all of them

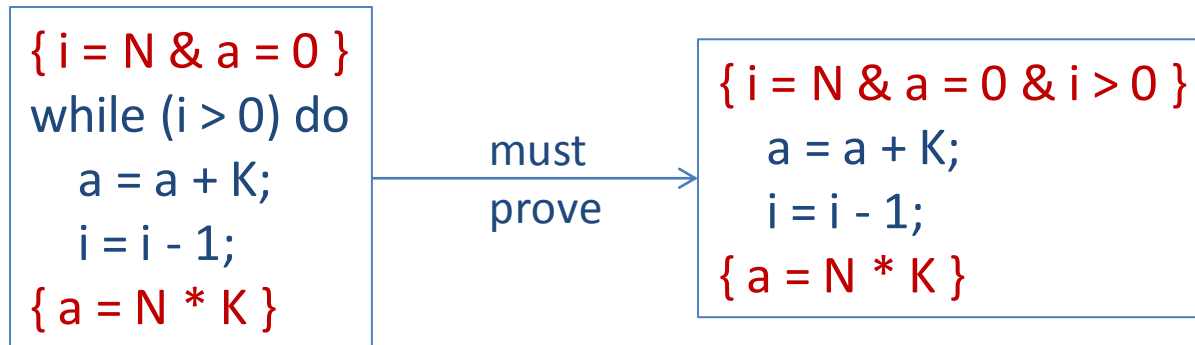
```
{ i = N & a = 0 }  
while (i > 0) do  
  a = a + K;  
  i = i - 1;  
{ a = N * K }
```

While Statements

- **Bogus** rule for while statements

If $\{ P \ \& \ e > 0 \}$ C $\{ Q \}$

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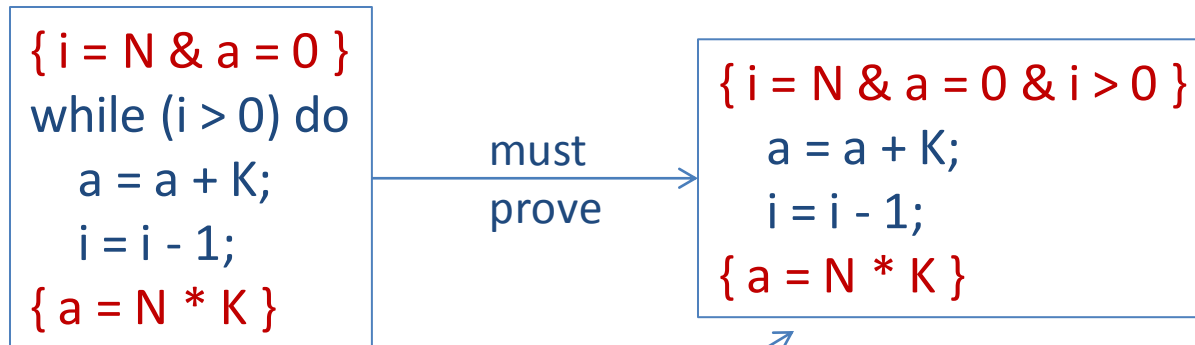


While Statements

- **Bogus** rule for while statements

If $\{P \ \& \ e > 0\} \ C \ \{Q\}$

then $\{P\} \ \text{while} \ (e > 0) \ \text{do} \ C \ \{Q\}$



this isn't even
close to a valid triple!
With that precondition,
 $a = K$ at the end!

While Statements

- **Problem:** We need to verify **all** iterations of a loop and we need to do it with a finite amount of work
- **Solution:** We will come up with an **invariant** that holds at the beginning and end of all iterations.
 - We prove that the loop body preserves the invariant **every** time around
- Unfortunate reality: Inferring invariants automatically is undecidable.
 - This puts significant limits on the degree to which we can automate verification.

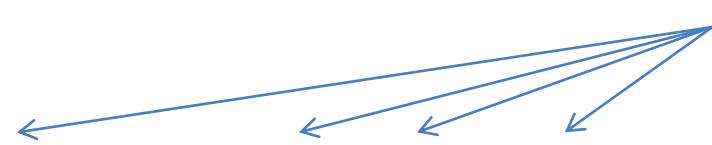
While Statements

- While rule:

loop invariant I

If $P \Rightarrow I$ and $\{e > 0 \ \& \ I\} C \{I\}$ and $I \ \& \ \sim(e > 0) \Rightarrow Q$

then $\{P\}$ while $(e > 0)$ do $C \{Q\}$



While Statements

- While rule:

loop invariant I

If $P \Rightarrow I$ and $\{e > 0 \ \& \ I\} C \{I\}$ and $I \ \& \ \sim(e > 0) \Rightarrow Q$

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- Inference rule notation:

$$\frac{P \Rightarrow I \quad \{e > 0 \ \& \ I\} C \{I\} \quad I \ \& \ \sim(e > 0) \Rightarrow Q}{\{P\} \text{ while } (e > 0) \text{ do } C \{Q\}}$$

While Statements

$$\frac{P \Rightarrow I \quad \{e > 0 \ \& \ I\} C \{I\} \quad I \ \& \ \sim(e > 0) \Rightarrow Q}{\{P\} \text{ while } (e > 0) \text{ do } C \{Q\}}$$

$\{i = 7 \ \& \ a = 0\} \text{ while } (i > 0) \text{ do } a = a+K; i = i-1; \{a = 7 \cdot K\}$

While Statements

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$$\frac{i = 7 \ \& \ a = 0 \Rightarrow I \quad \{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\} \quad I \ \& \ \sim(i > 0) \Rightarrow a = 7*K}{\{i = 7 \ \& \ a = 0\} \text{ while } (i > 0) \text{ do } a = a+K; i = i-1; \{a = 7*K\}}$$

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What works as I?

- true initially
- true before/after each iteration
- must imply Q when loop terminates

$$i = 7 \ \& \ a = 0 \Rightarrow I \quad \{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\} \quad I \ \& \ \sim(i > 0) \Rightarrow a = 7*K$$

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Invariant I is $(a = (7-i)*K) \ \& \ i \geq 0$

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- true initially
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- must imply Q when loop terminates

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$\{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\}$

$I \ \& \ \sim(i > 0) \Rightarrow a = 7*K$

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Invariant I is $(a = (7-i)*K) \ \& \ i \geq 0$

Checking I:

- $i = 7 \ \& \ a = 0 \Rightarrow (a = (7-i)*K) \ \& \ i \geq 0$

substitute:

7 for i

0 for a

$$7 \geq 0$$

$$0 = (7 - 7)*K$$

$$i = 7 \ \& \ a = 0 \Rightarrow I \quad \{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\} \quad I \ \& \ \sim(i > 0) \Rightarrow a = 7*K$$

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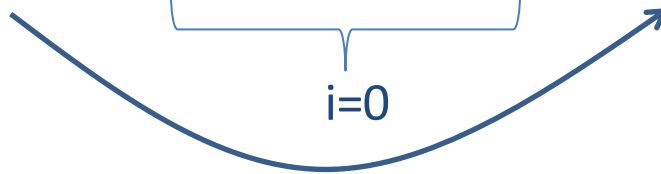
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Checking I :

- $i = 7 \ \& \ a = 0 \Rightarrow (a = (7-i)*K) \ \& \ i \geq 0$
- $(a = (7-i)*K) \ \& \ i \geq 0 \ \& \ \sim(i > 0) \Rightarrow a = 7*K$
- validate the triple: $\{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\}$

$$\frac{i = 7 \ \& \ a = 0 \Rightarrow I \quad \{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\} \quad I \ \& \ \sim(i > 0) \Rightarrow a = 7*K}{\{i = 7 \ \& \ a = 0\} \text{ while } (i > 0) \text{ do } a = a+K; i = i-1; \{a = 7*K\}}$$

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$a = (7-(i-1))*K \ \& \ i-1 \geq 0$

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Invariant I is $(a = (7-i)*K) \ \& \ i \geq 0$

$$\begin{array}{c}
 a + K = (7-(i-1))*K \ \& \ i-1 \geq 0 \\
 \swarrow \quad \searrow \\
 \frac{i > 0 \ \& \ I \Rightarrow Q}{\{i > 0 \ \& \ I\} a = a+K \{P\}} \quad \frac{\{Q\} a = a+K \{P\}}{\{P\} i = i-1; \{I\}} \\
 \swarrow \quad \searrow \\
 \frac{i = 7 \ \& \ a = 0 \Rightarrow I \quad \{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\}}{\{i = 7 \ \& \ a = 0\} \text{ while } (i > 0) \text{ do } a = a+K; i = i-1; \{a = 7*K\}} \quad I \ \& \ \sim(i > 0) \Rightarrow a = 7*K
 \end{array}$$

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$$i > 0 \ \& \ a = (7-i)*K \ \& \ i \geq 0 \Rightarrow a + K = (7-(i-1))*K \ \& \ i-1 \geq 0$$

$$i > 0 \ \& \ I \Rightarrow Q$$

$$\frac{}{\{Q\} a = a+K \{P\}}$$

$$\frac{\{i > 0 \ \& \ I\} a = a+K \{P\} \quad \{P\} i = i-1; \{I\}}{\{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\}}$$

$$i = 7 \ \& \ a = 0 \Rightarrow I$$

$$\{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\}$$

$$I \ \& \ \sim(i > 0) \Rightarrow a = 7*K$$

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$$\begin{aligned} i > 0 \ \& \ a = (7-i)*K \ \& \ i \geq 0 &\Rightarrow a + K = (7-(i-1))*K \ \& \ i-1 \geq 0 \\ &== a + K = (7-i)*K + K \ \& \ i-1 \geq 0 \\ &== a = (7-i)*K \ \& \ i-1 \geq 0 \end{aligned}$$

$$i > 0 \ \& \ I \Rightarrow Q$$

$$\frac{}{\{Q\} a = a+K \{P\}}$$

$$\frac{\{i > 0 \ \& \ I\} a = a+K \{P\} \quad \{P\} i = i-1; \{I\}}{\{i > 0 \ \& \ I\} a = a+K; i = i-1; \{I\}}$$

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Another Example

- What is the loop invariant?

$\{x = 0\}$ while $(x < 15)$ $\{ x = x + 2 \}$ $\{ \text{even}(x) \}$

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$\{x = 0\}$ while $(x < 15)$ $\{ x = x + 2 \}$ $\{ \text{even}(x) \}$

Invariant: $\text{even}(x)$

- $x = 0 \Rightarrow \text{even}(x)$
- $\text{even}(x) \ \& \ x \geq 15 \Rightarrow \text{even}(x)$
- $\{ \text{even}(x) \ \& \ x < 15 \} \ x = x + 2 \ \{ \text{even}(x) \}$

Another Example

- What is the loop invariant?

$\{x = 0 \ \& \ y = N \ \& \ N > 0\}$ while $(y > 0)$ $\{y = y - 1; x = x + 1\}$ $\{x = N\}$

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Invariant: $x + y = N$

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Invariant: $x + y = N$

- $x = 0 \ \& \ y = N \ \& \ N > 0 \Rightarrow x + y = N$
- $y \leq 0 \ \& \ x + y = N \Rightarrow x = N$

nope!



Another Example

- What is the loop invariant?

$\{x = 0 \ \& \ y = N \ \& \ N > 0\}$ while $(y > 0)$ $\{y = y - 1; x = x + 1\}$ $\{x = N\}$

Invariant: $x + y = N \ \& \ y \geq 0$

• $x = 0 \ \& \ y = N \ \& \ N > 0$

$\Rightarrow x + y = N \ \& \ y \geq 0$

• $y \leq 0 \ \& \ x + y = N \ \& \ y \geq 0$

$\Rightarrow x = N$

add constraint to
invariant

easier to establish
postcondition

more difficult
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- What is the loop invariant?

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Invariant: $x + y = N \ \& \ y \geq 0$

- $x = 0 \ \& \ y = N \ \& \ N > 0 \quad \Rightarrow \quad x + y = N \ \& \ y \geq 0$
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Invariant: $x + y = N \ \& \ y \geq 0$

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- $y \leq 0 \ \& \ x + y = N \ \& \ y \geq 0 \quad \Rightarrow \quad x = N$
- $\{x + y = N \ \& \ y \geq 0 \ \& \ y > 0\} \ y = y - 1; x = x + 1 \ \{x + y = N \ \& \ y \geq 0\}$

While Statements: Summary

- Given a Hoare triple for a while loop:
 - $\{ P \} \text{ while } (e > 0) \text{ do } C \{ Q \}$
- We prove it correct by:
 - guessing an invariant I (this is the hard part)
 - proving I holds initially: $P \Rightarrow I$
 - showing the loop body preserves I :
 - $\{ e > 0 \ \& \ I \} C \{ I \}$
 - showing the postcondition holds on loop termination:
 - $I \ \& \ \sim(e > 0) \Rightarrow Q$
- As a rule:
$$\frac{P \Rightarrow I \quad \{ e > 0 \ \& \ I \} C \{ I \} \quad I \ \& \ \sim(e > 0) \Rightarrow Q}{\{ P \} \text{ while } (e > 0) \text{ do } C \{ Q \}}$$
- Note: one often adds I as an annotation on the loop:
 - $\text{ while } [I] (e > 0) \text{ do } C$

FRAMING & MODULARITY

Another Issue: Framing

- Another valid triple:

$$\{x = 9 \ \& \ y = 7 \ \& \ z = 23\} \ x = x + 1 \ \{x = 10 \ \& \ y = 7 \ \& \ z = 23\}$$

- Proving it using the rules:

Another Issue: Framing

- Another valid triple:

$$\{x = 9 \ \& \ y = 7 \ \& \ z = 23\} x = x + 1 \{x = 10 \ \& \ y = 7 \ \& \ z = 23\}$$

- Proving it using the rules:

(1) $\{x + 1 = 10 \ \& \ y = 7 \ \& \ z = 23\} x = x + 1 \{x = 10 \ \& \ y = 7 \ \& \ z = 23\}$ (assignment rule)

(2) $x = 9 \ \& \ y = 7 \ \& \ z = 23 \Rightarrow x + 1 = 10 \ \& \ y = 7 \ \& \ z = 23$ (implication)

(3) $\{x = 9 \ \& \ y = 7 \ \& \ z = 23\} x = x + 1 \{x = 10 \ \& \ y = 7 \ \& \ z = 23\}$ (by (1), (2), consequence)

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(3) $\{x = 9 \ \& \ y = 7 \ \& \ z = 23\} x = x + 1 \{x = 10 \ \& \ y = 7 \ \& \ z = 23\}$ (by (1), (2), consequence)

- Note: Formulae not involving x are just propagated
- More generally, conjuncts not involving variables that are not *modified* are just propagated
- Can we factor those expressions out of most of the proof?

The Simple Frame Rule

- The Simple Frame Rule (also called the rule of constancy)

$$\frac{\{P\} C \{Q\} \quad C \text{ does not modify the (free) variables of } R}{\{P \& R\} C \{Q \& R\}}$$

- What counts as “modifying”?
 - In our simple language, the only way a variable may be modified is if it appears on the left in an assignment statement
 - In languages with functions or methods, calling one of them may have a modification effect
 - In C, you might be able to intentionally modify variables on the stack
 - In C, you might also have a buffer overflow ... yikes!
- The frame rule is a way of *simplifying* proofs
- Why are Haskell proofs so easy? Nothing is modified!

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- The Simple Frame Rule (also called the rule of constancy)

$$\frac{\{P\} C \{Q\}}{\{P \& R\} C \{Q \& R\}} \quad \text{C does not modify the (free) variables of R}$$

- Example:

$$\{x = 6 \& y = 7 \& z = 23\} x = x + 1; x = x * 2; x = x - 4; \{x = 10 \& y = 7 \& z = 23\}$$

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- Example:

$$\frac{\{x = 6\} x = x + 1; x = x * 2; x = x - 4; \{x = 10\} \quad x = x + 1; x = x * 2; x = x - 4; \text{ does not modify } y \text{ or } z}{\{x = 6 \& y = 7 \& z = 23\} x = x + 1; x = x * 2; x = x - 4; \{x = 10 \& y = 7 \& z = 23\}}$$

SUMMARY!

Summary

- **States** map variables to values
- **Formulae** describe states:
 - semantics in Haskell: `fsem :: State -> Form -> Maybe Bool`
 - semantics in Math: $[[f]]s$
 - formulae and states we deal with are well-formed
 - well-formedness is a very simple syntactic analysis
 - $P \Rightarrow Q$ means P describes a subset of the states that Q does
- **Hoare Triples** characterize program properties
 - $\{ P \} C \{ Q \}$ – know when it is **valid**
 - know the statement rules you can use to conclude $\{ P \} C \{ Q \}$
 - understand the structural rules:
 - rule of consequence
 - frame rule
 - know how to build formal proofs using inference rules