Inertial Hegselmann-Krause Systems

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***Abstract*—We derive an energy bound for *inertial Hegselmann-Krause (HK) systems*, which are a variant of the classic *HK* model that allows agents to change weights as they please at each step. We use the bound to prove the convergence of *HK* systems with static agents, which settles a widely believed conjecture. This paper also introduces *anchored HK systems* and show their equivalence to the symmetric heterogeneous model.**

***Index Terms*—Hegselmann-Krause (HK), model organism.**

I. Introduction

The Hegselmann-Krause model of multiagent consensus has emerged as a "model organism" for opinion dynamics [1]. In an *HK* system, a collection of n agents, each one represented by a point in $\mathbb{R}^d$, evolves by applying the following rule at discrete times: move each agent to the mass center of all the agents within unit distance. It has been shown that the system always freezes eventually [2]–[6]. While the model has been the subject of numerous studies [7]–[12] and much is known about its convergence rate, its *heterogeneous* variant remains a mystery [13]–[17]. In that model, each agent can choose its own radius of confidence. In the *HK* model with static (sometimes called *closed-minded*) agents, all of the agents have radius either 1 or 0. While extensive simulations pointed to the convergence of that system [12], [14], [18], [17], a proof remained elusive. This shortcoming was widely held in the opinion-dynamics community as a glaring demonstration of our weak grasp of these deceptively simple distributed systems. (The open status of the closed-minded case was described by a leading researcher in the field as one of the outstanding gaps in our understanding of opinion dynamics [19].) This paper resolves this issue by settling the conjecture in the affirmative: *HK* systems with closed-minded agents always converge. Our proof entails making the problem a special case of a much broader class of dynamical systems, the *inertial HK systems* (more on which below).

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The relaxation time of the original *HK* model has been shown to be $n^{O(n)}$ in any fixed dimension [2], a bound later improved to a polynomial bound in both n and d [20]. For the particular case $d = 1$, a bound of $O(n^5)$ was established in [21], which was lowered to $O(n^4)$ in [22] and then to $O(n^3)$ in [20]. The model can be generalized in various ways, its ultimate expression being the grand unified model of *influence systems* [23], in which each agent gets to pick its neighbors by following its own distinct, arbitrary criteria. Oddly, even the most seemingly innocuous modifications of the original *HK* model have stumped researchers in the field. This is the case of *HK* systems with closed-minded agents, where any agent's radius of confidence is either 0 or 1. To prove that these systems always converge, we introduce the more general *inertial HK systems* and establish a bound on their kinetic 2-energy. We also introduce the *anchored* variant of *HK* systems and prove that it is equivalent to the symmetric heterogeneous model. This fairly surprising result sheds new light on the convergence properties of these systems.

A. Inertial HK Systems

Instead of being required to move to the mass center of its neighbors at each step, each agent of an *inertial HK* system may move toward it by any fraction of length; setting this fraction to zero makes the agent closed-minded, which means that it remains frozen in place. Formally, the system consists of n agents represented by points $x_1(t), \ldots, x_n(t)$ in $\mathbb{R}^d$ at time $t = 0, 1, 2,$ etc. Two agents i and j are said to be *neighbors* if they are within unit distance: $\|x_i(t) - x_j(t)\|_2 \le 1$. The neighbors of i form a set $N_i(t)$; these sets form an undirected communication network G_t with a self-loop at each of the n nodes. The dynamics of the system is specified by

$$x_i(t+1) = (1-\lambda)x_i(t) + \frac{\lambda}{|N_i|}\sum_{j\in N_i} x_j(t), \qquad (1)$$

where $\lambda \in [0, 1]$ is called the *inertia*; whenever the time is clearly understood from the context, we omit the argument t from $N_i(t)$ to alleviate the notation. Likewise, for convenience, we write λ instead of the more accurate $\lambda_i(t)$: indeed, not only the inertias λ need not have the same value for all the agents, but they can be reset to a different value with each application of (1). In this way, we can select any agent to be closed-minded by setting their inertia to 0. We can also retrieve the original *HK* model by turning all the inertias to 1. In its full generality, an inertial *HK* system is not necessarily deterministic: indeed, the numbers λ can be set ahead of time or they can be assigned probabilistically or adversarially at each time step. We tackle the issue of convergence by turning our attention to their *kinetic* *s-energy*. The concept was introduced in [2] as a generating function for studying averaging processes in dynamic networks. It is defined as follows:

$$K(s) = \sum_{t \geq 0} \sum_{i=1}^{n} \|x_i(t+1) - x_i(t)\|_2^s.$$

We provide an upper bound for the case $s = 2$.

Theorem 1.1: The kinetic s-energy of an n-agent inertial HK-system whose inertias are uniformly bounded from above by λ_0 satisfies $K(2) \leq \lambda_0 n^2/4$. The upper bound λ_0 can be any real number in $[0, 1]$.

We use this result to establish the convergence of *HK* systems with closed-minded agents. Note that the convergence is asymptotic: the system may never freeze into a complete stop. This is even true for $n = 2$, where a single closed-minded agent can pull the other one toward itself forever. Indeed, if the mobile agent is initialized close enough to the closed-minded one, it will eventually converge to it by halving its distance at each step. The network G_t becomes fixed in this case. In general, it changes with time, however. Interestingly, fixed-point attraction does not automatically imply the convergence of the communication network, so we address this issue separately.

Theorem 1.2: An HK system with any number of closed-minded agents converges asymptotically to a fixed-point attractor. The communication network converges for all initial conditions if $d = 1$ and for all initial conditions outside a set of measure zero if $d > 1$.

The specific meaning of this last clause is that, in dimension two and higher, as long as we perturb the closed-minded agents by an arbitrarily small amount at the beginning, the communication network G_t will settle to a fixed graph in finite time almost surely. The perturbation is likely an unnecessary artifact of the proof and it would be nice to settle this point. The main open problem, however, is to derive an effective upper bound on the relaxation time.

B. Anchored HK Systems

The original *HK* system fixes the same radius for each pair of agents. By contrast, in a *symmetric heterogeneous HK system*, each pair (i, j) is given its own threshold radius r_{ij}, so that agents i and j are neighbors at time t whenever $\|x_i(t) - x_j(t)\|_2 \leq r_{ij}$. We require $r_{ij} = r_{\mathrm{ji}}$ and $r_{\mathrm{ii}} \geq 0$ (the latter to create self-loops). Note that $r_{ij} = 0$ means that i and j are neighbors only when their positions coincide, while $r_{ij} < 0$ implies that i and j are never joined together.

As one might expect, heterogeneity adds considerable difficulty to the analysis. We show that, by lifting the system into higher dimension, we can go back to the original assumption that all radii are the same. To formalize this somewhat unexpected result, we define an *anchored HK system* as consisting of n agents, each one represented by a vector $z_k = (x_k(t), y_k)$. The vector is a combination of a mobile part $x_k(t) \in \mathbb{R}^d$ and a static part $y_k \in \mathbb{R}^{d'}$; the dimensions d and d' are the same for all the agents. Two agents i and j are neighbors if and only if $\|z_i(t) - z_j(t)\|_2 \leq r$, where r is a fixed positive constant. At each step, the mobile part of an agent moves to the mass center of all its neighbors while its anchored part remains fixed. (Note that the averaging is done one coordinate at a time, so the static coordinates affect only the neighborhood relationships and do not participate in the averaging itself.)

Anchored *HK* systems capture a notion of partial closed-mindedness: agents are closed-minded in some coordinates but open-minded in others. Both mobile and anchored parts, on the other hand, affect the communication network. Surprisingly, anchored and symmetric heterogeneous systems are conjugate: in other words, there exists a bijection between them that respects their dynamics and establishes the equivalence of the two systems. Specifically, we prove the following:

Theorem 1.3: Given any anchored HK system $z_k(t) = (x_k(t), y_k)$ in $\mathbb{R}^d \times \mathbb{R}^{d'}$, there exists a conjugate symmetric heterogeneous HK system $x'_k(t)$ in $\mathbb{R}^d$. Conversely, a symmetric heterogeneous HK system of n agents in $\mathbb{R}^d$ is conjugate to an anchored HK system $z_k(t) = (x_k(t), y_k)$ with agents in $\mathbb{R}^d \times \mathbb{R}^{n-1}$. In both cases, the conjugacy is formed by the trivial correspondence: $x_k(t) = x'_k(t)$ for any k and t. Both anchored and symmetric heterogeneous HK systems converge asymptotically to a fixed-point attractor. If there is no pair of agents (i, j) such that $\|y_i - y_j\|_2 = r$ in an anchored HK system or such that $r_{ij} = 0$ in a symmetric heterogeneous HK system, then the communication network converges to a fixed graph.

While the convergence of symmetric heterogeneous *HK* systems can be inferred directly from known results, the convergence of the communication networks requires special treatment, however. An interesting corollary of these results is the convergence of *HK* systems embedded within a social network [24]–[26]. Imagine that the existence of an edge between two agents i, j is a function not only of their relative distance but also of a predetermined, fixed relationship. By setting $r_{ij} < 0$, we can enforce the absence of an edge. In this way we can restrict the *HK* action to the edges of a fixed, arbitrary social network, and still assert convergence.

II. The Convergence of Inertial HK Systems

The purpose of this section is to prove Theorem 1.1. The proof is algorithmic: it is a message-passing protocol that simulates the update of a distributed Lyapunov function. It follows a line of reasoning borrowed from the field of *amortized analysis*, a subject of theoretical computer science. Algorithmic proofs for dynamical systems have been used before [2], but our approach is quite different from previous incarnations. The basic idea is to assign a certain quantity with each agent and update them at each step according to fixed rules. Because the quantities in question are nonnegative and subject to conservation constraints, it is natural to think of them as amounts of *money* and the rules governing their updates as a *trading mechanism*. Of course, the metaphor is used only for explanatory purposes. The mathematical reality is that each agent holds a certain positive number that can go up and down with time but whose sum can never increase. To maintain this last property, agents modify their associated number via exchanges: if one increases it by a certain amount, some other agent must decrease it by the same amount so as to keep the sum constant. The point of all this is that agents can also decrease their number by a certain quantity of interest: for example, their displacement. In this way, the total displacement of all the agents can never exceed the initial sum of the agents' numbers. This informal explanation highlights the benefits of using the language of "money," "trade," and "spending" while keeping the discussion mathematically rigorous.

Here are the details. We assign each agent i a certain amount of money, $C_i(0)$, at the beginning $(t=0)$ and specify a protocol for spending and exchanging it with other agents as time progresses. If we knew ahead of time the total contribution of agent i to the kinetic 2-energy, we could simply set $C_i(0)$ to that amount and let the agent "pay" for its contribution from its own pocket. This information is not available, however, so we take an initial guess and set up an exchange protocol so that no agent runs out of money. By giving money to their neighbors in a judicious manner, we show how each agent remains in a position to pay for its share of the 2-energy at each step. Our initial guess is

$$C_i(0) = \sum_{j=1}^{n} \min\Big\{ \|x_i(0) - x_j(0)\|_2^2,\, 1 \Big\}.$$

To specify the exchange protocol, we first simplify the notation as follows:

$$\begin{cases} \Delta_i = x_i(t+1) - x_i(t) \\ d_{ij} = x_i(t) - x_j(t) \\ d'_{ij} = x_i(t+1) - x_j(t+1). \end{cases}$$

The two rules below are applied to every agent i at any time step $t \geq 0$:

1) For every neighbor of j at time t (which includes i itself), agent i spends $\|\Delta_i + \Delta_j\|_2^2$ units of money and gives to agent j an amount equal to $2(d_{ij} - \Delta_j)^T \Delta_j$.
2) If agent j becomes a new neighbor of i at time $t+1$ or, conversely, ceases to be one, then agent i spends an amount of money equal to $|\|d'_{ij}\|_2^2 - 1|$.

Let $C_i(t)$ be the amount of money held by agent i at time t, and let N_i^{in} (resp. N_i^{out}) denote the set of agents that are neighbors of i at time $t+1$ (resp. t) but not at time t (resp. $t+1$). Using the symmetry of the neighbor relation, we express the cash flow at time t by

$$\begin{aligned} &C_i(t+1) - C_i(t) \\ &\quad = 2\sum_{j\in N_i} (d_{\text{ji}} - \Delta_i)^T \Delta_i - 2\sum_{j\in N_i} (d_{ij} - \Delta_j)^T \Delta_j \\ &\quad - \sum_{j\in N_i} \|\Delta_i + \Delta_j\|_2^2 - \sum_{j\in N_i^{\text{in}}\cup N_i^{\text{out}}} |\|d'_{ij}\|_2^2 - 1|. \end{aligned}$$

Since $(d_{\text{ji}} - \Delta_i)^T \Delta_i - (d_{ij} - \Delta_j)^T \Delta_j = d_{ij}^T(\Delta_i - \Delta_j) - 2d_{ij}^T \Delta_i + \|\Delta_j\|_2^2 - \|\Delta_i\|_2^2$ and, by (1), $\lambda \sum_{j\in N_i} d_{ij} = -|N_i|\Delta_i$, we have

$$\begin{aligned} &C_i(t+1) - C_i(t) \\ &\quad = \sum_{j\in N_i} \Big\{ 2d_{ij}^T(\Delta_i - \Delta_j) + \|\Delta_i - \Delta_j\|_2^2 - 4d_{ij}^T\Delta_i \Big\} \\ &\quad - 4|N_i|\|\Delta_i\|_2^2 - \sum_{j\in N_i^{\text{in}}\cup N_i^{\text{out}}} |\|d'_{ij}\|_2^2 - 1| \\ &\quad = \sum_{j\in N_i} \Big\{ 2d_{ij}^T(\Delta_i - \Delta_j) + \|\Delta_i - \Delta_j\|_2^2 \Big\} \\ &\quad + 4\big(\lambda^{-1} - 1\big)|N_i|\|\Delta_i\|_2^2 - \sum_{j\in N_i^{\text{in}}\cup N_i^{\text{out}}} |\|d'_{ij}\|_2^2 - 1|. \end{aligned}$$

Note that $\lambda = 0$ implies that $\Delta_i = 0$, so in that case it is understood that $(\lambda^{-1} - 1)|N_i|\|\Delta_i\|_2^2 = 0$ in the identity above. Since $d'_{ij} = d_{ij} + \Delta_i - \Delta_j$, the first summand in the last equality above is equal to $\|d'_{ij}\|_2^2 - \|d_{ij}\|_2^2$; therefore

$$\begin{aligned} &C_i(t+1) - C_i(t) \\ &\quad = \sum_{j\in N_i} \Big\{ \|d'_{ij}\|_2^2 - \|d_{ij}\|_2^2 \Big\} - \sum_{j\in N_i^{\text{in}}\cup N_i^{\text{out}}} |\|d'_{ij}\|_2^2 - 1| \\ &\quad + 4\big(\lambda^{-1} - 1\big)|N_i|\|\Delta_i\|_2^2. \end{aligned}$$

Fix any j $(1 \leq j \leq n)$ and consider the difference $D_j := \min\big\{ \|d'_{ij}\|_2^2, 1 \big\} - \min\big\{ \|d_{ij}\|_2^2, 1 \big\}$. If i and j are not neighbors at time t or $t+1$, then $D_j = 0$. If they are neighbors at times t and $t+1$, then $D_j = \|d'_{ij}\|_2^2 - \|d_{ij}\|_2^2$. If they are neighbors at time t but not $t+1$ (ie, $j \in N_i^{out}$), then $D_j = 1 - \|d_{ij}\|_2^2$, which can be rewritten as $\|d'_{ij}\|_2^2 - \|d_{ij}\|_2^2 - |\|d'_{ij}\|_2^2 - 1|$. Finally, if they are neighbors at time $t+1$ but not t (ie, $j \in N_i^{\text{in}}$), then $D_j = \|d'_{ij}\|_2^2 - 1$, which can be expressed as $-|\|d'_{ij}\|_2^2 - 1|$. Since j cannot be both in N_i and N_i^{in}, this implies trivially that

$$\begin{aligned} C_i(t+1) - C_i(t) &= \sum_{j=1}^{n} D_j + 4\left(\lambda^{-1} - 1\right)|N_i|\|\Delta_i\|_2^2 \\ &= \sum_{j=1}^{n} \min\big\{ \|d'_{ij}\|_2^2, 1 \big\} - \sum_{j=1}^{n} \min\big\{ \|d_{ij}\|_2^2, 1 \big\} \\ &+ 4\big(\lambda^{-1} - 1\big)|N_i|\|\Delta_i\|_2^2. \end{aligned}$$

Since $|N_i| > 0$ and $\lambda \leq \lambda_0$, it follows that

$$\begin{aligned} C_i(t) \geq &\sum_{j=1}^{n} \min\big\{ \|d_{ij}\|_2^2, 1 \big\} \\ &+ 4\big(\lambda_0^{-1} - 1\big) \sum_{k=0}^{t-1} \|x_i(k+1) - x_i(k)\|_2^2. \end{aligned}$$

Being its own neighbor, agent i spends at least $4\|\Delta_i\|_2^2$ money at each step. Summing up over all the agents, this amounts to $4K(2)$. This shows that the initial injection of money allows the system to spend $4K(2)$ and still be left with as much as $4\big(\lambda_0^{-1} - 1\big)K(2)$. Theorem 1.1 follows from the fact that the initial injection of money is at most n per agent, which is n^2 in total. ■

III. HK Systems With Closed-Minded Agents

This section proves Theorem 1.2. The bound on the kinetic 2-energy shows that the system eventually slows down to a crawl but it falls short of proving convergence. Indeed, an agent moving along a circle by $1/t$ at time t contributes finitely to the kinetic 2-energy yet travels an infinite distance. We prove that *HK* systems with closed-minded agents always converge asymptotically. We treat the one-dimensional separately for two reasons: the proof is entirely self-contained and the convergence of the communication network does not require perturbation. In dimension two and higher, we prove that the agents always converge to a fixed position: the system has a fixed-point attractor. We show how a tiny random perturbation ensures that the network eventually settles on a fixed graph.

A. The One-Dimensional Case

We begin with the one-dimensional case, which is particularly simple. By Theorem 1.1, we can choose a small enough $\varepsilon > 0$ and an integer t_ε large enough so that no agent moves by a distance of more than ε at any time $t \geq t_\varepsilon$. Fix $t > t_\varepsilon$ and let x_i (resp. N_i) denote the position (resp. neighbors) of agent i at time t; we use primes and double primes to indicate the equivalent quantities for time $t+1$ and $t+2$. The symmetric difference between N_i and N_i', if nonempty, is the disjoint union of a set L_i of agents located at $x_i - 1 \pm O(\varepsilon)$ at times t and $t+1$ and a set R_i at locations $x_i + 1 \pm O(\varepsilon)$. For each subset, we distinguish between the agents of N_i not in N_i' and vice-versa, which gives the disjoint partitions $L_i = L_i^{\text{in}} \cup L_i^{\text{out}}$ and $R_i = R_i^{\text{in}} \cup R_i^{\text{out}}$. The locations x_i' and x_i'' of agent i at times $t+1$ and $t+2$ are given by

$$\begin{cases} |N_i| x_i' = \left(\sum_{j \in N_i \cap N_i'} x_j \right) + \left(\sum_{j \in L_i^{\text{out}} \cup R_i^{\text{out}}} x_j \right) \\ |N_i'| x_i'' = \left(\sum_{j \in N_i \cap N_i'} x_j' \right) + \left(\sum_{j \in L_i^{\text{in}} \cup R_i^{\text{in}}} x_j' \right). \end{cases}$$

All x_k' and x_k'' are of the form $x_k \pm O(\varepsilon)$, so subtracting the two identities shows that

$$\begin{aligned} (|N_i'| - |N_i|) x_i = \; & (|L_i^{\text{in}}| - |L_i^{\text{out}}|)(x_i - 1) \\ & + (|R_i^{\text{in}}| - |R_i^{\text{out}}|)(x_i + 1) \pm O(\varepsilon n). \end{aligned}$$

Since the dynamics is translation-invariant, we can assume that $x_i = 0$. Setting ε small enough, the integrality of the set cardinalities implies that the net flow of neighbors on the left of agent i is the same as it is on the right:

$$|L_i^{\text{out}}| - |L_i^{\text{in}}| = |R_i^{\text{out}}| - |R_i^{\text{in}}|. \tag{2}$$

Among all the agents undergoing a change of neighbors between times t and $t+1$, pick the one that ends up the furthest to the right at time $t+1$, choosing the one of largest index i to break ties. We distinguish between two cases:

1) $x_i' \geq x_i$: No agent of R_i^{out} can be closed-minded; nor can it be mobile since, ranks being preserved, it would provide an agent undergoing a change of neighbors and landing to the right of i at time $t+1$, in contradiction with the definition of i. It follows that R_i^{out} is empty, which in turn implies that L_i^{in} is not, since by our choice of i not all four terms in (2) can be zero. Since agent i is not moving left, neither is any agent j of L_i^{in}. Its set N_j of neighbors changes between time t and $t+1$ and R_j^{out} is empty. To see why the latter is true, we first note that N_j cannot lose any closed-minded agent to the right. Also, since any mobile agent in R_j^{out} is to the left of i at time t, it stays to the left of it by conservation of ranks; hence the agent remains a neighbor of j, a contradiction. The argument so far uses the rightmost status of agent i only to assert that R_i^{out} is empty. This means we are back to square one and we can proceed inductively, eventually reaching a contradiction.
2) $x_i' < x_i$: The key observation is that our previous argument never uses time directionality, so we can exchange the role of t and $t+1$, which implies that now $x_i' > x_i$. Note that the superscripts *in* and *out* must be swapped. While we chose i as the mobile agent landing furthest to the right, by symmetry we must now choose the one starting the furthest to the right: of course, since mobile agents can never cross this make no difference.

We conclude that each agent is now endowed with a fixed set of neighbors, so the dynamics is specified by the powers of a fixed stochastic matrix with positive diagonal, which are well known to converge. The system is attracted to a fixed point at an exponential rate, but of course we have no a priori bound on the time it takes to fall into that basin of attraction. The communication network converges.

B. The Higher-Dimensional Case

Generalizing the previous argument to higher dimension fails on several counts, the most serious one being the loss of any left-right ordering. We follow a different tack, which begins with a distinction between two types of agents. An agent is *trapped* at time t if there exists a path in the current communication graph leading to a closed-minded agent; it is said to be *free* otherwise. There exists a time t_o after which the agents fall into two categories: some of them are never trapped past t_0 and are called *eternally free*; the others are *chronically trapped* (ie, trapped an infinite number of times). As we did before, we pick a parameter $\varepsilon > 0$ (to be specified below) and $t_\varepsilon > t_o$ large enough so that no agent moves by a distance of more than ε at any time $t \geq t_\varepsilon$. If two agents ever get to share the same position, their fates become completely tangled since they can never again get separated. Since such merges occur fewer than n times, we can make t_ε big enough, if necessary, so that all merges are in the past. To summarize, past t_ε, the mobile agents move by increments less than ε, no merging occurs, and the system consists only of eternally free and chronically trapped agents.

At any time, the state system is represented by a n-by-d matrix whose i-th row encode the position of agent i in $\mathbb{R}^d$. The matrix consists of two parts: x for the mobile agents and y for the closed-minded ones. A transition of the system is a linear map of the form $x \leftarrow Ax + By$, where each row of the nonnegative matrix $(A \,|\, B)$ sums up to 1.

Lemma 3.1: Past t_ε, no agent can move while free.

Proof: Fix $t \geq t_\varepsilon$ and consider a connected component $\mathcal{C}$ of the graph induced by the free agents. If z denotes its position matrix at time t and k its number of rows, then $z' = Cz$, where primes refer to time $t+1$ and C is a k-by-k stochastic matrix for a random walk in the undirected graph $\mathcal{C}$. Because the graph is connected, the eigenvalue 1 of C is simple, so the null space of $I - C$, and hence of $(I-C)^T(I-C)$, is spanned by $\mathbf{1}$. By Courant-Fischer, therefore, any vector u normal to $\mathbf{1}$ satisfies $\|(I-C)u\|_2 \geq \sigma\|u\|_2$, where σ is the smallest positive singular value of $I - C$. If we define $\bar{z} = z - \frac{1}{k}\mathbf{1}\mathbf{1}^T z$, it immediately follows that

$$\sigma\|\bar{z}\|_2 \leq \|(I-C)\bar{z}\|_2 = \|(I-C)z\|_2 = \|z - z'\|_2 \leq \varepsilon\sqrt{n}.$$

Setting $\varepsilon < \frac{1}{2}\sigma/\sqrt{n}$ ensures that any two of the k agents are within unit distance. It follows that $\mathcal{C}$ is the complete graph and $C = \frac{1}{k}\mathbf{1}\mathbf{1}^T$. Since the agents can no longer merge, the only option left is for all k of them to be already merged at time t, hence unable to move. ■

The lemma implies that eternally free agents can never move again past t_ε. Indeed, it shows that an eternally free agent can only move if it is joined to a trapped one, which, by definition, it cannot be. Since eternal freedom keeps the agents from playing any role after time t_ε, we might as well assume that all the mobile agents in the system are chronically trapped. This means that, at all instants, either an agent is trapped (ie, joined to a closed-minded agent via a path) or it is *isolated*, meaning that the other agents are either merged with it or at distance greater than one. An agent cannot move while isolated.

The position matrix z of the k trapped agents at time $t \geq t_\varepsilon$ satisfies the relation $z' = Tz + Uy$, where primes denote time $t+1$ and the k-by-n matrix $(T \,|\, U)$ has each row summing up to 1. Being trapped implies that U is not the null matrix. In fact, viewed as a Markov chain, the trapped agents correspond to transient states, which means that T^k tends to the null matrix as k goes to infinity. This shows that T cannot have 1 as an eigenvalue; therefore $I - T$ is nonsingular. Let μ be a uniform upper bound on the singular values of all the (so-called fundamental) matrices $(I-T)^{-1}$; since their number is finite, so is μ. Since $z' = Tz + Uy$ and $\|z' - z\|_2 \leq \varepsilon\sqrt{n}$, the matrix z is very close to $(I-T)^{-1}Uy$; specifically,

$$\begin{aligned} \|z - (I-T)^{-1}Uy\|_F &= \|(I-T)^{-1}(z - z')\|_F \\ &\leq \mu\|z - z'\|_F \leq \mu\varepsilon\sqrt{n}. \end{aligned} \tag{3}$$

A matrix of the form $(I-T)^{-1}Uy$ is called an *anchor*. Since the set of all possible anchors (for given y) is finite, the minimum (Frobenius-norm) distance r between any two distinct anchors is strictly positive. The value of r does not depend on ε, so we can always lower the value of the latter, if necessary, to ensure that $r > (1+2\mu)\varepsilon\sqrt{n}$.

By (3) and Lemma 3.1 we know that, at any time t past t_ε, any mobile agent is either stuck in place (if free) or at distance at most $\mu\varepsilon\sqrt{n}$ away from an anchor. As a result, no agent can ever change anchors since this would necessitate a one-step leap of at least $r - 2\mu\varepsilon\sqrt{n} > \varepsilon\sqrt{n}$ for the positional matrix, hence the displacement of an agent by a distance of at least ε, which has been ruled out. Since the argument holds for any ε small enough, each mobile agent is thus constrained to converge toward its chosen anchor. This concludes the proof that all agents converge to a fixed point in $\mathbb{R}^d$. The convergence is asymptotic and no bound can be inferred directly from our analysis.

The result does not imply that the communication network should also converge to a fixed graph. The lack of convergence points to a situation where the agents are still moving in increasingly small increments, yet edges of the network keep switching forever. This can only occur if at least one pair of anchor points are at distance 1: by anchor point, we mean the points formed by any row of an anchor matrix or of y. The key observation is that all the anchor points are convex combinations of the rows of y, so an interdistance of 1 is expressed by an equality of the form $\|v^T y\|_2 = 1$. There are only a finite set of such equalities to consider and each one denotes an algebraic surface of codimension 1. Any random perturbation of the closed-minded agents will result in the convergence of the communication network almost surely. This completes the proof of Theorem 1.2. ■

IV. Anchored and Symmetric Heterogeneous *HK* Systems

This section proves Theorem 1.3. We begin with a proof of the conjugacy between the two types of *HK* systems.

A. The Bijection Relation

To express an anchored *HK* system $z(t) = (x_k(t), y_k)$ as a symmetric heterogeneous one is straightforward. We have the equivalence

$$\begin{aligned} &\|z_i(t) - z_j(t)\|_2^2 \leq r^2 \\ \Leftrightarrow\ &\|x_i(t) - x_j(t)\|_2^2 \leq r^2 - \|y_i - y_j\|_2^2. \end{aligned} \tag{4}$$

We define $r_{ij} = \sqrt{r^2 - \|y_i - y_j\|_2^2}$ if the right hand side of (4) is non-negative, and $r_{ij} = -1$ otherwise. Then the system $x_k(t)$ together with thresholds r_{ij} forms a symmetric heterogeneous *HK* system. Notice that the equivalence (4) ensures that the communication graphs of the given anchored *HK* system and its corresponding symmetric heterogeneous *HK* counterpart are identical.

For the other direction, we need to lift the given symmetric heterogeneous *HK* system to an anchored *HK* version. We need the following lemma, whose proof can be found in the Appendix.

Lemma 4.1: For any n-by-n symmetric matrix $R = (r_{ij})$ with no negative terms in the diagonal, there exist $r > 0$ and vectors $y_k \in \mathbb{R}^{n-1} (1 \leq k \leq n)$, such that

$$\|y_i - y_j\|_2 = \sqrt{r^2 - r_{ij}^2 \operatorname{sign}(r_{ij})}, \tag{5}$$

for any $i \neq j$; here $\operatorname{sign}(x) = 1$ if $x \geq 0$ and -1 otherwise.

Given a symmetric heterogeneous *HK* system $x_k(t)$, we choose the anchors y_k by appealing to Lemma 4.1. For any $r_{ij} \geq 0$, it then follows that

$$\begin{aligned} &\|x_i(t) - x_j(t)\|_2^2 \leq r_{ij}^2 \\ \Leftrightarrow\ &\|x_i(t) - x_j(t)\|_2^2 + \|y_i - y_j\|_2^2 \leq r^2, \end{aligned} \tag{6}$$

and for any $r_{ij} < 0$, and we always have

$$\|x_i(t) - x_j(t)\|_2^2 + \|y_i - y_j\|_2^2 > r^2, \tag{7}$$

for any $i \neq j$, which prevents any edge between i and j. This means that the dynamics of the symmetric heterogenous *HK* system coincides precisely with that of the mobile part of the lifted anchored system.

Remark: Lemma 4.1 asserts that, given $(n-1)n/2$ lengths d_{ij} $(i \neq j)$ of the form $(r^2 - r_{ij}^2 \operatorname{sign}(r_{ij}))^{1/2}$, we can find n points $y_k \in \mathbb{R}^{n-1}$ such that the pairwise distance $\|y_i - y_j\|_2 = d_{ij}$. Notice that, if d_{ij} itself is arbitrary, this is not always possible. For example, in the case $n = 3$, the problem is equivalent to finding a triangle in $\mathbb{R}^2$ with each side length given. The problem is solvable if and only if the three lengths satisfy the triangle inequality. In our case, however, there is an extra parameter r that we can use. Intuitively, if we choose a large r such that all the $|r_{ij}|$ are relatively small, then the problem of finding y_k is equivalent to finding an almost regular polytope, each edge of which is roughly of the same length r.

B. Proof of Convergence

The fixed-point attraction of symmetric heterogeneous *HK* systems can be inferred directly from known results about infinite products of type-symmetric stochastic matrices [2]–[5]. The same holds of anchored systems. In both cases, given any $\varepsilon > 0$ and any initial condition, the n agents will eventually reach a ball of radius ε that they will never leave; we call this ε-*convergence*. We study the conditions for this to imply that the corresponding communication networks themselves converge to a fixed graph. It suffices to consider the case of a symmetric heterogeneous *HK* system. Consider a connected component $\mathcal{C}$ of the graph and let z and $z' = Cz$ denote the corresponding position matrices at time t and $t+1$, where C is the corresponding k-by-k stochastic matrix associated with $\mathcal{C}$. As we did in the proof of Lemma 3.1 we define σ to be a uniform lower bound on any positive singular value of $I - C$ for any such matrix C. Setting

$$\varepsilon = \frac{\sigma}{2\sqrt{n}} \min_{r_{ij}>0} r_{ij}$$

implies that

$$\begin{aligned} \|\bar{z}\|_2 \leq\ & \frac{1}{\sigma}\|(I-C)\bar{z}\|_2 = \frac{1}{\sigma}\|z - z'\|_2 \\ \leq\ & \frac{\sqrt{n}\epsilon}{\sigma} \leq \frac{1}{2}\min_{r_{ij}>0} r_{ij}, \end{aligned}$$

where $\bar{z} = z - \frac{1}{k}\mathbf{1}\mathbf{1}^T z$ is the projection of z onto the orthogonal space of $\mathbf{1}$. It follows that, for any pair (i,j) in $\mathcal{C}$ such that $r_{ij} > 0$, there will be an edge between i and j. With the assumption $r_{ij} \neq 0$, the communication graph is now fixed and convergence proceeds at an exponential rate from that point on. The bijection result of the previous section shows that the condition $r_{ij} = 0$ corresponds to $\|y_i - y_j\|_2 = r$ in the case of anchored systems. This concludes the proof of Theorem 1.3. ■

Appendix

Our proof of Lemma 4.1 relies on two technical facts. For convenience, we use bold letters to denote vectors; for example, u_k denotes the k-th coordinate of vector $\boldsymbol{u}$.

Fact A: There exist $n+1$ vectors $\boldsymbol{u}^{(k)} \in \mathbb{R}^n$ $(0 \leq k \leq n)$ such that $\|\boldsymbol{u}^{(i)} - \boldsymbol{u}^{(j)}\|_2 = 1 (0 \leq i < j \leq n)$, $u_i^{(k)} = 0$ for $i > k \geq 0$ and all $u_k^{(k)}$ exceed $1/\sqrt{2}$ and decrease as k grows.

Proof: Proceeding by induction, we write $\boldsymbol{u}^{(0)} = \mathbf{0}$, $\boldsymbol{u}^{(1)} = \boldsymbol{e}_1$ and $\boldsymbol{u}^{(2)} = \frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2$, where $\boldsymbol{e}_i$ is the unit vector in the i-th dimension. Assume we already constructed $\boldsymbol{u}^{(k)}$ $(0 \leq k \leq m < n)$ such that $u_i^{(k)} = 0$ for $i > k$ and $u_k^{(k)} > 1/\sqrt{2}$. Then we can write $\boldsymbol{u}^{(k)}$ as

$$\boldsymbol{u}^{(k)} = \sum_{i=1}^{k} u_i^{(k)} \boldsymbol{e}_i, \quad k = 1, 2, \ldots, m.$$

We define

$$\begin{aligned} \boldsymbol{u}^{(m+1)} =\ & \sum_{i=1}^{m-1} u_i^{(m)} \boldsymbol{e}_i + \Big(u_m^{(m)} - \frac{1}{2u_m^{(m)}}\Big)\boldsymbol{e}_m \\ & + \sqrt{1 - \frac{1}{4\big(u_m^{(m)}\big)^2}}\, \boldsymbol{e}_{m+1}. \end{aligned}$$

Since $u_m^{(m)} > 1/\sqrt{2}$, we have

$$u_{m+1}^{(m+1)} > \sqrt{1 - \frac{1}{4(1/\sqrt{2})^2}} = \frac{1}{\sqrt{2}}.$$

For $k = 0, 1, \ldots, m$,

$$\begin{aligned} \|\boldsymbol{u}^{(m+1)} - \boldsymbol{u}^{(k)}\|_2^2 =\ & \|\boldsymbol{u}^{(m)} - \boldsymbol{u}^{(k)}\|_2^2 + u_m^{(k)}/u_m^{(m)} \\ =\ & (1 - \delta_{\mathrm{km}}) + u_m^{(k)}/u_m^{(m)} = 1. \end{aligned}$$

Notice that, for $0 \leq k < n$,

$$\begin{aligned} \big(u_{k+1}^{(k+1)}\big)^2 - \big(u_k^{(k)}\big)^2 =\ & \left(1 - \frac{1}{4\big(u_k^{(k)}\big)^2}\right) - \big(u_k^{(k)}\big)^2 \\ =\ & -\left(u_k^{(k)} - \frac{1}{2u_k^{(k)}}\right)^2 \leq 0, \end{aligned}$$

which proves the monotonicity claim.

Fact B: For any integer $n > 0$, there is a positive number γ depending on n such that, for any t_{ij} satisfying $|1 - t_{ij}| \leq \gamma$ and $t_{ij} = t_{\mathrm{ji}}$ $(0 \leq i < j \leq n)$, there exist vectors $\boldsymbol{y}^{(k)} \in \mathbb{R}^n (0 \leq k \leq n)$ such that $\|\boldsymbol{y}^{(i)} - \boldsymbol{y}^{(j)}\|_2 = t_{ij}$, for $0 \leq i < j \leq n$.

Proof: We make repeated use of the matrix infinity norm. Recall that if M is a p-by-q matrix, its infinity norm is defined as the maximum absolute row sum of M:

$$\|M\|_\infty := \max_{1 \leq i \leq p} \sum_{j=1}^{q} |m_{ij}|.$$

As one would expect of a matrix norm, the infinity norm is submultiplicative:

$$\|MN\|_\infty \leq \|M\|_\infty \|N\|_\infty,$$

for any p-by-q matrix M and q-by-r matrix N. We define a constant

$$\alpha = 5n + \max_{1 \leq k \leq n} \|C_k^{-1}\|_\infty,$$

where C_k is the k-by-k matrix whose i-th row consists of the first k elements of the vector $\boldsymbol{u}^{(i)}$ in Fact A. Note that C_k is lower-triangular and invertible. Let $\gamma = \alpha^{-4n}$. The intuition of the proof is that the vectors $\boldsymbol{y}^{(k)}$ we are seeking should be close to the vectors $\boldsymbol{u}^{(k)}$. We build the desired vectors by induction. Let $\boldsymbol{y}^{(0)} = \boldsymbol{0}$ and $\boldsymbol{y}^{(1)} = t_{01}\boldsymbol{e}_1$. Then it is obvious that $\|\boldsymbol{y}^{(0)} - \boldsymbol{y}^{(1)}\|_2 = t_{01}$ and $\boldsymbol{y}^{(0)}$ and $\boldsymbol{y}^{(1)}$ are close to the vectors from Fact A:

$$\|\boldsymbol{y}^{(0)} - \boldsymbol{u}^{(0)}\|_\infty = 0 < \gamma,$$
$$\|\boldsymbol{y}^{(1)} - \boldsymbol{u}^{(1)}\|_\infty = |t_{01} - 1| \le \gamma \le \alpha^4\gamma.$$

Suppose $\boldsymbol{y}^{(0)}, \boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(k-1)}$ have been specified such that $y_i^{(j)} = 0$ for $i > j$,

$$\|\boldsymbol{y}^{(i)} - \boldsymbol{u}^{(i)}\|_\infty \le \alpha^{4i}\gamma \quad (0 \le i \le k-1), \tag{8}$$

and

$$\|\boldsymbol{y}^{(i)} - \boldsymbol{y}^{(j)}\|_2 = t_{ij} \quad (0 \le i < j \le k-1).$$

We need to show is that there exists a vector $\boldsymbol{y}^{(k)}$ such that $y_i^{(k)} = 0$ for $i > k$,

$$\|\boldsymbol{y}^{(k)} - \boldsymbol{u}^{(k)}\|_\infty \le \alpha^{4k}\gamma \tag{9}$$

and

$$\|\boldsymbol{y}^{(k)} - \boldsymbol{y}^{(i)}\|_2 = t_{\rm ik} \quad (0 \le i \le k-1). \tag{10}$$

This last relation is equivalent to

$$\sum_{j=1}^{i} \left(y_j^{(k)} - y_j^{(i)}\right)^2 + \sum_{j=i+1}^{k} \left(y_j^{(k)}\right)^2 = t_{\rm ik}^2 \tag{11}$$

for $0 \le i \le k-1$. By subtracting the equations for $1 \le i \le k-1$ from the one for $i = 0$, we get a linear system for $\hat{\boldsymbol{y}} := (y_1^{(k)}, y_2^{(k)}, \dots, y_{k-1}^{(k)})^T$:

$$A\hat{\boldsymbol{y}} = \boldsymbol{b}.$$

Here the $(k-1) \times (k-1)$ matrix A is a lower triangular matrix where $A_{ij} = y_j^{(i)}$ $(i \ge j)$ and $\boldsymbol{b}$ is a $(k-1)$ dimensional column vector where

$$b_i = \frac{1}{2}\Big(t_{0k}^2 - t_{\rm ik}^2 + \sum_{j=1}^{i} \left(y_j^{(i)}\right)^2\Big).$$

We derive similar relations from Fact A:

$$\sum_{j=1}^{i} \left(u_j^{(k)} - u_j^{(i)}\right)^2 + \sum_{j=i+1}^{k} \left(u_j^{(k)}\right)^2 = 1 \quad (0 \le i \le k-1), \tag{12}$$

which implies a linear system $C\hat{\boldsymbol{u}} = \boldsymbol{d}$ for

$$\hat{\boldsymbol{u}} := (u_1^{(k)}, u_2^{(k)}, \dots, u_{k-1}^{(k)})^T,$$

where C is shorthand for C_{k-1} and $d_i = \frac{1}{2}\sum_{j=1}^{i}\left(u_j^{(i)}\right)^2$. We already observed that C is nonsingular; we note that, by (8) and $u_i^{(i)} > 1/\sqrt{2}$, the same is true of A. Next, we derive upper bounds on the length of the vector $\boldsymbol{b}$ and its distance from $\boldsymbol{d}$. By $|1 - t_{ij}| \le \gamma$ and $\gamma < 1/2$,

$$|t_{0k}^2 - t_{\rm ik}^2| = |t_{0k} + t_{\rm ik}||t_{0k} - t_{\rm ik}| \le (2 + 2\gamma) \cdot 2\gamma < 6\gamma. \tag{13}$$

By our induction hypothesis (8), the fact that $|y_j^{(i)}| \le 1 + \gamma$, and the definition of γ, we have

$$\begin{aligned} \left|\left(y_j^{(i)}\right)^2 - \left(u_j^{(i)}\right)^2\right| &= |y_j^{(i)} + u_j^{(i)}||y_j^{(i)} - u_j^{(i)}| \\ &\le (2 + \alpha^{4i}\gamma) \cdot \alpha^{4i}\gamma \\ &< 3\alpha^{4(k-1)}\gamma. \end{aligned} \tag{14}$$

Thus, by (13), (14),

$$\|\boldsymbol{b} - \boldsymbol{d}\|_\infty \le 3(1 + n\alpha^{4(k-1)}/2)\gamma. \tag{15}$$

By inequality (13) and the fact that γ is small enough, we have

$$\begin{aligned} \|\boldsymbol{b}\|_\infty &\le \frac{1}{2}\Big(\max_{1\le i\le k} |t_{0k}^2 - t_{\rm ik}^2| + \max_{1\le i\le k} \|\boldsymbol{y}^{(i)}\|_2^2\Big) \\ &< \frac{1}{2}(6\gamma + (1+\gamma)^2) < 1. \end{aligned} \tag{16}$$

We also claim that

$$\|A^{-1} - C^{-1}\|_\infty \le 2n\alpha^{4k-2}\gamma. \tag{17}$$

Here is why. First, notice that (8) implies $\|A - C\|_\infty \le n\alpha^{4(k-1)}\gamma$. Then based on the definition of α, we have $\|C^{-1}\|_\infty < \alpha$, and hence

$$\|C^{-1}(A - C)\|_\infty \le \|C^{-1}\|_\infty \|A - C\|_\infty < n\alpha^{4k-3}\gamma. \tag{18}$$

The right hand side of the above inequality is smaller than $1/2$ based on the definition of γ, which allows us to expand the matrix inverse $[I + C^{-1}(A - C)]^{-1}$ as

$$[I + C^{-1}(A - C)]^{-1} = I + \sum_{i=0}^{\infty} (-1)^i [C^{-1}(A - C)]^i,$$

from which it follows that

$$\|[I + C^{-1}(A - C)]^{-1}\|_\infty \le 2. \tag{19}$$

Notice that

$$A^{-1} - C^{-1} = [I + C^{-1}(A - C)]^{-1} C^{-1}(C - A)C^{-1},$$

then inequality (17) directly follows from inequalities (18) and (19). By (15), (16), (17) and the fact that $\|C^{-1}\|_\infty < \alpha$, finally we have

$$\begin{aligned} &\|\hat{\boldsymbol{y}} - \hat{\boldsymbol{u}}\|_\infty \\ &= \|A^{-1}\boldsymbol{b} - C^{-1}\boldsymbol{d}\|_\infty \\ &= \|(A^{-1} - C^{-1})\boldsymbol{b} + C^{-1}(\boldsymbol{b} - \boldsymbol{d})\|_\infty \\ &\le \|(A^{-1} - C^{-1})\|_\infty \|\boldsymbol{b}\|_\infty + \|C^{-1}\|_\infty \|(\boldsymbol{b} - \boldsymbol{d})\|_\infty \\ &\le 2n\alpha^{4k-2}\gamma + 3(1 + n\alpha^{4(k-1)}/2)\alpha\gamma < \alpha^{4k-1}\gamma. \end{aligned}$$

This shows that

$$|y_j^{(k)} - u_j^{(k)}| \le \alpha^{4k-1}\gamma \quad (1 \le j \le k-1). \tag{20}$$

In turn, this implies that

$$\left|\left(y_j^{(k)}\right)^2 - \left(u_j^{(k)}\right)^2\right| = |y_j^{(k)} + u_j^{(k)}||y_j^{(k)} - u_j^{(k)}| < (2+\alpha^{4k-1}\gamma)\alpha^{4k-1}\gamma < 3\alpha^{4k-1}\gamma. \quad (21)$$

It suffices now to set the remaining (nonzero) coordinate of $\boldsymbol{y}^{(k)}$ yet to be specified, which is $y_k^{(k)}$. Recall that it must satisfy

$$\sum_{j=1}^{k}\left(y_j^{(k)}\right)^2 = t_{0,k}^2$$

and, by our construction, this single equality suffices to imply all of (10). This implies a unique setting of (positive) $y_k^{(k)}$, so we need only be concerned with (9) and the positivity of $\left(y_k^{(k)}\right)^2$. Since $|1-t_{0k}^2| = |1-t_{0k}||1+t_{0k}| \le \gamma(2+\gamma) < 3\gamma$, inequality (12) for $i=0$, combined with (14), establishes that

$$\left|\left(y_k^{(k)}\right)^2 - \left(u_k^{(k)}\right)^2\right| \le \sum_{i=1}^{k-1}\left|\left(y_i^{(k)}\right)^2 - \left(u_i^{(k)}\right)^2\right| + |1-t_{0k}^2| \le 3(1+n\alpha^{4k-1})\gamma.$$

Since $u_k^{(k)} > 1/\sqrt{2}$, it follows that

$$\left(y_k^{(k)}\right)^2 > \frac{1}{2} - 3(1+n\alpha^{4k-1})\gamma > 0.$$

Furthermore,

$$|y_k^{(k)} - u_k^{(k)}| = \frac{\left|\left(y_k^{(k)}\right)^2 - \left(u_k^{(k)}\right)^2\right|}{y_k^{(k)} + u_k^{(k)}} \le 3\sqrt{2}(1+n\alpha^{4k-1})\gamma < \alpha^{4k}\gamma. \quad (22)$$

In conjunction with (20), this establishes (9), and completes the inductive construction. ■

It should be noted that Fact B can also be proven via the implicit function theorem and a perturbation argument based on Fact A. The benefit of the proof given above is to provide an explicit construction.

Lemma 4.1: For any n-by-n symmetric matrix $R = (r_{ij})$ with no negative terms in the diagonal, there exist $r > 0$ and vectors $y_k \in \mathbb{R}^{n-1} (1 \le k \le n)$, such that

$$\|y_i - y_j\|_2 = \sqrt{r^2 - r_{ij}^2 \,\text{sign}\,(r_{ij})}, \quad (23)$$

for any $i \ne j$; here sign $(x) = 1$ if $x \ge 0$ and -1 otherwise.

Proof: Choose a sufficiently large r such that

$$\max_{i,j}|r_{ij}| < \gamma r,$$

where γ is the small positive constant from Fact B. We set t_{ij} to $\sqrt{1 - r_{ij}^2\,\text{sign}(r_{ij})/r^2}$ and easily verify that $|1 - t_{ij}| \le \gamma$. Fact B guarantees the existence of vectors $\boldsymbol{z}_k \in \mathbb{R}^{n-1}$ $(1 \le k \le n)$ such that $\|\boldsymbol{z}_i - \boldsymbol{z}_j\|_2 = t_{ij}$. Setting $y_k = r\boldsymbol{z}_k$ satisfies the requirements. ■

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