On edge colouring, fractionally colouring and partitioning graphs

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Abstract

We present an assortment of results in graph theory. First, Tutte conjectured that every two-edge-connected cubic graph with no Petersen graph minor is three-edge-colourable. This generalizes the four-colour theorem. Robertson et al. had previously shown that to prove Tutte's conjecture, it was enough to prove it for doublecross graphs. We provide a proof of the doublecross case.

Seymour conjectured the following generalization of the four-colour theorem. Every d-regular planar graph can be d-edge-coloured, provided that for every odd-cardinality set X of vertices, there are at least d edges with exactly one end in X. Seymour's conjecture was previously known to be true for values of $d \leq 7$. We provide a proof for the case $d = 8$.

We then discuss upper bounds for the fractional chromatic number of graphs not containing large cliques. It has been conjectured that each graph with maximum degree at most Δ and no complete subgraph of size Δ has fractional chromatic number bounded below Δ by at least a constant $f(\Delta)$. We provide the currently best known bounds for $f(\Delta)$, for $4 \leq \Delta \leq 10^3$. We also give a general upper bound for the fractional chromatic number in terms of the sizes of cliques and maximum degrees in local areas of a graph.

Next, we give a result that says, roughly, that a graph with sufficiently large treewidth contains many disjoint subgraphs with 'good' linkedness properties. A similar result was a key tool in a recent proof of a polynomial bound in the excluded grid theorem. Our theorem is a quantitative improvement with a new proof.

Finally, we discuss the p-centre problem, a central NP-hard problem in graph clustering. Here we are given a graph and an integer p, and need to identify a set of p vertices, called centres, so that the maximum distance from a vertex to its closest centre (the p-radius) is minimized. We give a quasilinear time approximation algorithm to solve p-centres when the hyperbolicity of the graph is fixed, with a small additive error on the p-radius.

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Chapter 1

Overview

This thesis is based on a collection of papers. The theme is broadly graph theory, but we explore different areas, namely edge colouring, fractional vertex colouring and graph structure. In this chapter, we give a brief overview of what is covered. The results we present are grouped into chapters according to topic, and each of Chapters 3-7 can be read independently of the others. In Chapter 2 we give a basic set of definitions that will be used throughout. (The reader can look ahead to that chapter for any definitions needed to follow this section.)

Chapters 3 and 4 are each about different generalizations of the four-colour theorem (4CT) which is usually stated as follows.

Theorem 1.0.1 (Four-colour theorem)**.** *Every planar graph is* 4*-colourable.*

Proposed by Guthrie in 1852, its statement is remarkably simple. And yet, a correct proof eluded mathematicians for over a century before it was proved in 1977 by Appel and Haken. The search for a solution over the years gave rise to many important and sophisticated tools in graph theory, including the method of reducible configurations and discharging. Building on these tools, my coauthors and I have worked on two generalizations of the four-colour theorem. To describe them, it is convenient to work with an equivalent edge-colouring formulation of the four-colour theorem, given by Tait [81] in 1880: Every two-edge-connected, three-regular planar graph has chromatic index 3.

1.0.1 Tutte's three-edge-colouring conjecture

In 1966, Tutte conjectured that every two-edge-connected cubic graph with no Petersen graph minor is three-edge-colourable [82]. This class of graphs properly contains the class of planar graphs, so Tutte was proposing a generalization of the four-colour theorem. Robertson et al. [74] had previously shown that to prove the conjecture, it was enough to prove it for doublecross graphs. [1] In joint work with Sanders, Seymour and Thomas [37], we solved the doublecross case, thereby placing the final piece in the puzzle of Tutte's conjecture. We give this proof in Chapter 3. The proof method is based on the proof of the four-colour theorem; it is computer-assisted and relies on the method of reducible configurations and discharging.

1.0.2 Seymour's planar d-edge-colouring conjecture

Here is another strengthening of the four-colour theorem. The four-colour theorem says that the edges of any 3-regular planar graph can be partitioned into three perfect matchings, provided the necessary condition that there is no set X of vertices with just one edge between X and its complement. Seymour conjectured in 1979 that the natural generalization of this statement to d-regular graphs holds [79]. That is, every d-regular planar graph can be d-edge-coloured, provided that for every odd-cardinality set X of vertices, there are at least d edges with exactly one end in X. Seymour's conjecture was previously known to be true for values of $d \leq 6$. As a masters student, the author worked with Kawarabayashi to prove the case $d = 7$. Later, in joint work with Chudnovsky and Seymour, we found a simpler proof which we were able to extend to prove the case $d = 8$[19], [20]. This proof also relies on reducibility and discharging arguments, but is short enough to describe without the assistance of computers. The conjecture remains open for larger values of d. We give the $d = 8$ proof in Chapter 4.

1.0.3 Fractional extensions of Brooks' theorem

We then shift our focus to vertex colourings, in particular fractional colourings. A classical theorem of Brooks says that every graph with maximum degree Δ has chromatic number at most $\Delta + 1$. Furthermore, the only graphs that achieve this upper bound are the complete graphs and odd

[1] Doublecross graphs are those graphs which can be drawn in the plane with two crossings that both lie in the same region.

cycles [9]. That is, by excluding a complete subgraph of size $\Delta + 1$ (also known as $K_{\Delta+1}$) one reduces the upper bound in Brooks' theorem by 1 (when $\Delta \geq 3$). If a graph has a K_Δ subgraph, then its chromatic number is at least Δ. What about if it doesn't? A lovely conjecture of Borodin and Kostochka says that if one also excludes K_Δ, then the graph has chromatic number at most $\Delta - 1$ (for sufficiently large Δ) [7]. This question is open and notoriously difficult, but in joint work with King, we proved versions of Borodin and Kostochka's question for the fractional chromatic number [35, 34]. [2]

It has been conjectured that with the exception of two graphs, every K_Δ-free graph has fractional chromatic number strictly less than Δ and tight upper bounds have been conjectured. When $\Delta = 3$, the question has been resolved; Dvořák et al. showed that every subcubic triangle-free graph has chromatic number at most $\frac{14}{5}$ [29]. It has also been solved when Δ is very large, as Reed showed that the Borodin-Kostochka conjecture holds for values of $\Delta \geq 10^3$ [68]. It remains open to find tight upper bounds when Δ takes values in between. In Chapter 5 we prove the currently best known upper bounds for each $4 \leq \Delta \leq 10^3$. The proof relies on structural reductions and the analysis of a randomized fractional colouring scheme.

In the last part of Chapter 5, we present a different upper bound on the fractional chromatic number. Reed's ω, Δ, χ conjecture proposes that every graph satisfies $\chi \leq \lceil \frac{1}{2}(\omega(G)+\Delta(G)+1) \rceil$[69]. This conjecture is known for some classes of graphs, but is open in general. However, the upper bound is known to hold (without the roundup) for the fractional chromatic number χ_f. Thus, if a graph has a big χ_f, then the average of its clique number and its maximum degree is big. McDiarmid proved a 'local strengthening' of Reed's bound, namely $\chi_f \leq \max_{v \in V(G)} \frac{1}{2}(\omega(v) + d(v) + 1)$, where $\omega(v)$ denotes the size of the largest clique containing v and $d(v)$ its degree [65]. This says that if a graph has a big χ_f, then it has a witness inside the closed neighbourhood of a vertex, i.e. a vertex whose degree averaged with the size of the largest clique it belongs to is big. The fractional chromatic number can be arbitrarily far from the bound supplied by either of these theorems (consider the star $K_{1,r}$). A natural extension of the local strengthening is to ask for two adjacent vertices to witness a big fractional chromatic number. Our theorem is that indeed, one can improve

[2] A fractional colouring is a relaxation of traditional colouring where we assign rational weights to the stable sets of a graph (instead of integer weights) so that for any given vertex the total weight on stable sets it is at least one. The fractional chromatic number is the least total weight needed so that this is possible, and it is at most the chromatic number.

the local bound as follows. Every graph satisfies $\chi_f \leq \max_{uv \in E(G)} \frac{1}{4}(\omega(u)+d(u)+1+\omega(v)+d(v)+1)$.

1.0.4 On the excluded grid theorem

In the Graph Minors papers, Robertson and Seymour introduced the notions of treewidth and tree decomposition of graphs which have become important tools in structural graph theory as well as in algorithmic applications. One of the many celebrated results from this series is the so-called Excluded Grid Theorem [75]. This says that for every k, there exists $f(k)$ such that every graph not containing a $k \times k$-grid as a minor has treewidth at most $f(k)$. In Robertson and Seymour's original proof, they showed that one can take $f(k)$ to be a certain extremely large function of k containing iterated exponential towers and later, with Thomas, improved this to about 20^{2k^5} [73]. At the same time, they suggested that this relationship might be tightened to take $f(k) = O(k^2 \log k)$. Finding a polynomial upper bound for $f(k)$ was an open problem for many years, though it did receive some attention. A series of improvements were proved by various researchers before Chekuri and Chuzhoy gave the first polynomial upper bound for $f(k)$ in 2013. They showed that one can take $f(k) = O(k^{99})$ in [13],[12], and later improved this to $f(k) = O(k^{20})$ in [21] [22].

A key tool in Chekuri and Chuzhoy's proof is the existence, in graphs with large treewidth, of many disjoint subgraphs with 'good' linkedness properties. This involves a rather technical definition which we make precise in Chapter 6. Roughly speaking though, for parameters r, h and α, where r is the desired number of such subgraphs, h measures the number of disjoint paths between any two of them and α measures the well-linkedness, inside the subgraph, of its vertices with neighbours outside, we give sufficient conditions on r, h, α for existence of the subgraphs. Our contribution in Chapter 6 is a quantitative improvement of Chekuri and Chuzhoy's result. Our proof also has the advantage of being self-contained, while the result in [12] is implicitly shown inside the proof of a complex algorithm. Our result can be applied to achieve a modest improvement on the first bound given by Chekuri and Chuzhoy. We use some of the same ideas as they do in [12] to obtain our main theorem, but in several places we use different techniques. In particular, in Section 6.6.2 we prove the following result about partitioning a graph into parts with relatively more edges inside than leaving each part, which may be of independent interest. For each r, every graph G with maximum degree Δ and at least $225r^2\Delta$ edges has a partition of its vertices into r

parts $X_1, \ldots, X_r$ so that for each i, $|E(G[X_i])| \geq \frac{1}{4(r-1)}|\delta(X_i)|$.

1.0.5 Clustering in δ-hyperbolic graphs

Finally, in Chapter 7 we show how to exploit the hyperbolicity of a graph to cluster its vertices efficiently. The hyperbolicity of a graph is an invariant which roughly measures the hyperbolicity of the metric space induced by its associated distance metric. More concretely, the hyperbolicity constant of a graph can be expressed as a 'four-point condition' as follows. A graph is δ-hyperbolic if for every choice of four vertices $u, v, x, y \in V(G)$ with $d(u,v) + d(x,y) \geq d(u,x) + d(v,y) \geq d(u,y) + d(v,x)$ we have $d(u,v) + d(x,y) - d(u,x) - d(v,y) \leq 2\delta$. In particular, every tree is 0-hyperbolic. It has been shown experimentally that real-world networks (social networks, for example) tend to have small constant hyperbolicity; this is in contrast with random graphs which tend to have logarithmic hyperbolicity. Such graphs also tend to be so large that quadratic time algorithms are highly impractical, so it is interesting to design fast approximation algorithms for optimization problems on graphs with a fixed hyperbolicity. One line of attack is to exploit the 'treelikeness' of such graphs, since many NP-complete problems are solvable in polynomial time on trees. Taking this approach, with Kennedy and Saniee, we considered the p-centre problem, a popular approach to graph clustering [33]. In this problem, we are given a graph G and integer p, and want to identify a set of p vertices (called centres) such that the maximum distance from a vertex to its closest centre is minimized. This optimal distance is called the p-radius (denoted r_p), and in general it is hard to compute.

We give a quasilinear time approximation algorithm for with an additive error at most 3δ. Specifically, for a graph G with n vertices, m edges and hyperbolic constant δ, our algorithm constructs p-centers in time $O(p(\delta+1)(n+m)\log(n))$ with radius not exceeding $r_p+\delta$ when $p \leq 2$ and $r_p+3\delta$ when $p \geq 3$, where r_p are the optimal radii. This improves on a previously known algorithm for p-centers with accuracy $r_p+\delta$ but with time complexity $O((n^3 \log n + n^2 m)\log(\mathrm{diam}(G)))$ which is impractical for large graphs.

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The results in this thesis have been obtained in collaboration with various coauthors, to whom the author is extremely grateful. Most of them appear in articles that have been published or submitted for publication. In many places there is an overlap between the text of this thesis and the papers mentioned below.

The results in Chapter 3 is joint with Daniel Sanders, Robin Thomas, and Paul Seymour and and has been published in [37]. The associated computer programs can be found online [36].

The results in Chapter 4 are joint with Maria Chudnovsky and Paul Seymour. That paper has also been published in [20].

The results presented in Chapter 5 are joint with Andrew King. These have been published in [35], [34] and [17].

The results in Chapter 6 is joint with Paul Seymour. These results have not appeared elsewhere, nor have they been submitted for publication.

Finally, the results in Chapter 7 are the result of joint work with Sean Kennedy and Iraj Saniee. These have not appeared elsewhere yet, but have been submitted for publication. A manuscript is available at [33].

Chapter 2

Definitions

In this thesis we use standard graph-theoretic terminology, but we take the opportunity in this chapter to remind the reader of a few definitions and concepts we will be using often. A thorough introduction to graph theory can be found in [84] or [26]. We will give additional definitions as the need arises and repeat information from below as we see fit.

2.1 Basics

A graph $G = (V, E)$ consists of a set V of *vertices* and E of vertex pairs called *edges*. For a given graph G we write $V(G)$ and $E(G)$ to denote its sets of vertices and edges, respectively. When the reader encounters quantities n or m, she may safely assume they refer to the number of vertices or edges of the graph being discussed. Graphs in this thesis are always finite and undirected. A *simple graph* is a graph without parallel edges or loops. In most cases, we will deal with simple graphs, though sometimes we will explicitly allow parallel edges and, rarely, loops.

To refer to edges in $E(G)$ we write $e = uv$. In this case, the edge e has *ends* u and v. The vertices u and v are said to be *adjacent*, or *neighbours*, and the edge e is *incident* with both u and v. Two edges are incident if they share a common end. The *degree* of a vertex in G is its number of incident edges, with loops counted twice, and is denoted $d_G(v)$ or simply $d(v)$. We use $\Delta(G)$ to denote the maximum degree. A graph is *d-regular* if every vertex has degree d. A *cubic graph* is simply a 3-regular graph, and a *subcubic* graph is one with maximum degree at most 3. The *neighbourhood* of a vertex v is the set of its neighbours, and the *closed neighbourhood* is the neighbourhood, along with v. These are denoted $N(v)$ and $\tilde{N}(v)$, respectively. For $S \subseteq V(G)$, the vertex v is *complete* to S if $S \subseteq N(v)$. It is *anticomplete* to S if $S \cap N(v) = \emptyset$.

2.2 Colouring

A *proper k-colouring* is a map $\phi : V(G) \to \{1, \ldots, k\}$ with $\phi(u) \neq \phi(v)$ for each edge $uv \in E(G)$. In other words, ϕ is a partition of the vertices into stable sets. If such a colouring exists, we say that G is *k-colourable*. We generally drop the 'proper' designation as these are the only colourings in consideration. The *chromatic number* of G is the least integer k for which a proper k-colouring of G exists.

A set $M \subseteq E(G)$ of edges is called a *matching* if no two edges in M are incident. The matching M is *perfect* if every vertex is incident with some edge in M. A *proper k-edge-colouring* is a map $\phi : E(G) \to \{1, \ldots, k\}$ so that $\phi(e) \neq \phi(f)$ for each pair of incident edges. If G has such a colouring, we say it is *k-edge-colourable*. The *chromatic index* of G is the least integer k for which a proper k-colouring exists.

2.3 Subgraphs and minors

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The graph H is an *induced subgraph* if for each edge $uv \in E(G)$ we have $uv \in E(H)$ if and only if $u, v \in V(H)$. We write $G[S]$ to denote the induced subgraph of G with vertex set S. We say that G is *H-free* if it has no induced subgraph isomorphic to H.

To *delete* a vertex v is to remove v from $V(G)$ and all edges incident to v from $E(G)$. To delete an edge e is to remove it from $E(G)$. The resulting subgraphs are respectively denoted $G \setminus v$ and $G \setminus e$. To *contract* an edge e is to identify its ends; the resulting graph is denoted G/e. A graph H is a *minor* of G if H can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. We say that G is *H-minor-free* if it has no minor isomorphic to H.

If $S \subseteq V(G)$, we say that S is a *stable set* if no two vertices in S are adjacent. If every pair of vertices in S is adjacent, then $G[S]$ is called a *clique*, or a complete subgraph. We denote the complete graph on n vertices by K_n. We write $\alpha(G)$ to denote the maximum size of a stable set in

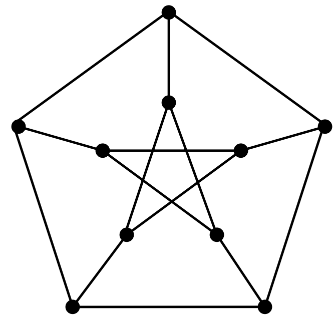

Figure 2.1: The Petersen graph.

G and $\omega(G)$ to denote the size of a maximum clique. These invariants are respectively called the *stability number* and the *clique number*.

A graph P is a *path* if it is isomorphic to the graph with vertices $V = \{v_1, \ldots, v_k\}$ and edges $E = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$. Then P is a $v_1 - v_k$-path and the vertices v_1 and v_k are its *ends*. Two paths are *vertex-disjoint* if they have no vertices in common; they are *edge-disjoint* if they have no edges in common. A graph is a *cycle* if it is isomorphic to the graph with $V = \{v_1, \ldots, v_k\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\}$.

The Petersen graph is the cubic graph depicted in Figure 2.1. It has 10 vertices, 15 edges and chromatic index 4.

2.4 Connectivity

A graph G is *connected* if for every pair of vertices $u, v \in V$, there exists a path in G with ends u, v. Otherwise, G is *disconnected*. The *connected components* of G are its maximal connected subgraphs. A set $S \subseteq V$ is a vertex cutset if $G \setminus S$ is disconnected. The graph is *k-connected* if it has at least $k+1$ vertices, and there is no set of $k-1$ vertices whose removal leaves a disconnected graph. A *separation* is a pair (S, T) of subsets of vertices with $S \cup T = V(G)$, such that there are no edges between $S \setminus T$ and $T \setminus S$. The *order* of (S, T) is $|S \cap T|$.

A *cut* is a partition $(S, V(G) \setminus S)$ of $V(G)$. The set of edges that have exactly one end in S is denoted $\delta_G(S)$ (equivalently, $\delta_G(V(G) \setminus S)$). We drop the subscript G if it is clear which graph we are referring to. A set of edges $F \subseteq E(G)$ and called an *edge cut* if $F = \delta(S)$ for some $S \subseteq V(G)$. A graph is *k-edge-connected* if the size of a minimum edge cut is at least k. An edge cut of size one is called a *cut-edge* or a *bridge*. A 2-edge-connected graph is also called *bridgeless*.

For distinct nonadjacent vertices s and t, an $s-t$*-cut* is, depending on the context, either a vertex cut or an edge cut S so that s and t belong to different components of $G \setminus S$. It is a classical theorem of Menger that the maximum number of vertex-disjoint $s-t$-paths is equal to the size of the smallest $s-t$ vertex cut [64]. Similarly, the maximum number of edge-disjoint $s-t$-paths is equal to the size of a smallest $s-t$-edge cut.

2.5 Planarity

A *planar graph* is a graph that can be drawn in the plane without edge crossings. Formally, we define a drawing of a graph as they do in [71]: Let Σ be a fixed 2-sphere; a *line* is a subset of Σ homeomorphic to the unit interval, an *open disc* a subset homeomorphic to the real plane $\mathbb{R}^2$ and a *closed disc* a subset homeomorphic to a unit circle in $\mathbb{R}^2$. A *drawing* D of $G = (V, E)$ consists of a closed set $U \subseteq \Sigma$ and $V \subseteq U$ of vertices. (We use V to denote the vertices in both the graph and the drawing via the natural bijection between the two.) For each edge $e = uv \in E$, there is a corresponding line, also called an edge, in $U \setminus V$ with ends u, v. A pair of edges in D may intersect only in their common ends.

Let D be a drawing of a 2-edge-connected planar graph. A *region* is a connected component of $\Sigma \setminus U$. A region is *finite* if it is a closed disc, and *infinite* otherwise. Any drawing has exactly one infinite region. Each region is *bounded* by a cycle of G in the natural sense. The *length* of a region is the length of its boundary cycle. A vertex or edge is *incident* with a region if it belongs to the region's boundary cycle. A region with length 3 is a *triangle*. A *triangulation* is a non-null drawing in which every region is a triangle and a *near-triangulation* is a non-null drawing in which every finite region is a triangle. Let $F(G)$ denote the set of regions in a drawing of a connected graph G. Euler's formula states that $|V(G)| + |F(G)| - |E(G)| = 2$.

Let G be a connected graph drawn in the plane. The *dual graph* of G is the graph with vertex set $F(G)$ and which has an edge ff' for each pair of regions f, f' and edge e such that f, f' are both incident with e. The dual is again a planar graph, and its dual is G.

Chapter 3

Three-edge-colouring cubic doublecross graphs

3.1 Introduction

Recall Tait's formulation of the four-colour theorem (4CT) [81], that every two-edge-connected cubic planar graph is three-edge-colourable. In 1966, Tutte [82] proposed that every two-edge-connected cubic graph with no Petersen graph minor is three-edge-colourable. Tutte's conjecture strengthens the 4CT since the Petersen graph, being non-planar, is not a minor of any planar graph. It is a proper strengthening since there do exist non-planar two-edge-connected cubic graphs without a Petersen minor; consider for example the complete bipartite graph $K_{3,3}$. The result presented in this chapter is what was the last remaining unproved step in the proof of Tutte's conjecture, which is now officially a theorem.

Theorem 3.1.1. *Every two-edge-connected cubic graph without the Petersen graph as a minor is three-edge-colourable.*

Tutte's conjecture precedes the proof of the 4CT, due to Appel and Haken [3, 4] in 1974, and its later simplification by Robertson, Seymour, Sanders and Thomas in 1997 [71]. Our methods depend heavily on those developed in the latter proof. The work in this chapter is joint with Paul Seymour, Daniel Sanders and Robin Thomas, and has been published in [37].

In this discussion, all graphs are finite and simple. A graph G is *apex* if $G \setminus v$ is planar for some vertex v ; and a graph G is *doublecross* if it can be drawn in the plane with only two crossings, both on the infinite region in the natural sense.

It is easy to check that apex and doublecross graphs do not contain the Petersen graph as a minor; but there is also a converse. Let us say a graph G is *theta-connected* if G is cubic and has girth at least five, and $|\delta_G(X)| \geq 6$ for all $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 6$. ($\delta_G(X)$ denotes the set of edges of G with one end in X and one end in $V(G) \setminus X$.) Robertson, Seymour and Thomas proved in [72] that every theta-connected graph with no Petersen graph minor is either apex or doublecross (with one exception, that is three-edge-colourable); and in [74] that every minimal counterexample to Tutte's conjecture was either apex or theta-connected. It follows that every minimal counterexample to Tutte's conjecture is either apex or doublecross, and so to prove the conjecture in general, it suffices to prove it for apex graphs and for doublecross graphs. Sanders and Thomas proved in [77] that every two-edge-connected apex cubic graph is three-edge-colourable, so all that remains is the doublecross case, which is the objective of this chapter. Our main theorem is:

Theorem 3.1.2. *Every two-edge-connected doublecross cubic graph is three-edge-colourable.*

The proof method is by modifying the proof of the 4CT given in [71]. Again we give a list of reducible configurations (the definition of "reducible" has to be modified to accommodate the two pairs of crossing edges), and a discharging procedure to prove that one of these configurations must appear in every minimal counterexample (and indeed in every non-apex theta-connected doublecross graph). This will prove that there is no minimal counterexample, and so the theorem holds. Happily, the discharging rules given in [71] still work without any modification.

3.2 Crossings

We are only concerned with graphs that can be drawn in the plane with two crossings, and one might think that these are not much different from planar graphs, and perhaps one could just *use* the 4CT rather than going to all the trouble of repeating and modifying its proof. For graphs with one crossing this is true: here is a pretty theorem of Jaeger [51]:

Theorem 3.2.1. *Let G be a two-edge-connected cubic graph, drawn in the plane with one crossing. Then G is three-edge-colourable.*

Proof. Let e, f be the two edges that cross one another, and let $e = z_1z_3$ and $f = z_2z_4$ say. Let H be obtained from G by deleting e, f, adding four new vertices $y_1, \ldots, y_4$, and edges $g_i = y_iz_i$ for $1 \leq i \leq 4$. Thus every vertex of H has degree three, except for $y_1, \ldots, y_4$ which have degree one; and H can be drawn in a closed disc with $y_1, \ldots, y_4$ drawn in the boundary of the disc in order. By adding four edges y_1y_2, y_2y_3, y_3y_4 and y_4y_1, we obtain a two-edge-connected cubic planar graph, which therefore is three-edge-colourable, by the 4CT. Consequently H is also three-edge-colourable; let $\phi : E(H) \to \{1, 2, 3\}$ be a three-edge-colouring of H. Since each colour appears at every vertex of H different from $y_1, \ldots, y_4$, and there are an even number of such vertices, it follows that each of the three colours appears on an even number of $g_1, \ldots, g_4$. In particular, if $\phi(g_1) = \phi(g_3)$ then $\phi(g_2) = \phi(g_4)$, giving a three-edge-colouring of G as required. We may assume then that $\phi(g_1) \neq \phi(g_3)$, and similarly $\phi(g_2) \neq \phi(g_4)$. From the symmetry we may therefore assume that $\phi(g_i) = 1$ for $i = 1, 2$, and $\phi(g_i) = 2$ for $i = 3, 4$. Let J be the subgraph of H with vertex set $V(H)$ and edge set all edges e of H with $\phi(e) \in \{1, 2\}$. Every vertex of H different from $y_1, \ldots, y_4$ therefore has degree two in J, and $y_1, \ldots, y_4$ have degree one; and so two components of J are paths with ends in $\{y_1, \ldots, y_4\}$, and all other components are cycles. Let the two components which are paths be P_1, P_2; and we may assume that y_1 is an end of P_1. The second end of P_1 cannot be y_3, since then P_2 would have ends y_2, y_4, which is impossible by planarity. So the second end of P_1 is one of y_2, y_4; and in either case, if we exchange colours 1 and 2 on the edges of P_1, and otherwise leave ϕ unchanged, we obtain a new three-edge-colouring ϕ' of H, in which $\phi'(g_1) = \phi'(g_3)$ and $\phi'(g_2) = \phi'(g_4)$, which therefore gives a three-edge-colouring of G. This proves Theorem 3.2.1. $\square$

We have tried (hard!) to do something similar to handle the doublecross case, but failed; it seems necessary to do it the long way, modifying the proof of the 4CT. Fortunately that is not as difficult as it was for the apex case in [77].

3.3 Discharging and a sketch of the proof

Our proof closely follows Robertson et al.'s proof of the four-colour theorem. For the uninitiated reader, in this section we give a quick overview of the ideas in that proof. It makes use of the *discharging method*, a technique which we will also need in Chapter 4. The discharging method is a powerful technique that has been around for over a century. Its invention has been credited to Wernicke who used it to prove that every planar graph with minimum degree 5 has two adjacent vertices, at most one of which has degree 6 [83]. Its application in early attacks on the 4CT goes back to the work of Heesch.

Details vary depending on the application, but the essential formula is the following. Let's say we want to prove that all planar graphs belonging to a subclass $\mathcal{X}$ satisfy some property. [1] There are two phases. In the first, called *unavoidability*, we show that every graph in $\mathcal{X}$ must contain at least one of a list of forbidden subgraphs. To do this, we initially assign numbers, called *charges*, to vertices and/or regions in such a way that the total sum of charges distributed is positive. Then we move charges around between vertices and regions, in a way that conserves the total amount of charge in the graph, according to a list of rules called the *discharging rules.* We then prove that any vertex or region that has positive charge after the rules have been applied must contain one of the forbidden subgraphs. The particulars of the charge assignment and the rules vary depending on the problem at hand.

The second phase is called *reducibility.* Here, we need to show that no minimum counterexample to our purported theorem can contain any of the forbidded subgraphs in the list.

Let us see how this is applied in the 4CT [71]. Their proof works with planar triangulations, rather than their cubic duals. As Tait showed, it is sufficient to find a colouring of the edges with three colours so that the three edges incident with any face are all different. It was known that every minimum counterexample to the 4CT must be an internally 6-connected triangulation [6], meaning it has minimum degree 5, and any vertex-cutset of order 5 is the neighbourhood of a vertex. The unavoidability phase, while very intricate, is standard. The reducibility phase, however, is less straightforward.

The forbidden subgraphs in this case are a list of 633 *reducible configurations.* Each reducible

[1] Actually, we work with drawings of graphs but we will make that precise later.

configuration consists of a near-triangulation C bounded by a cycle R, called its *ring*. Let T be a triangulation drawn in the plane and assume it's a counterexample to the 4CT. Suppose C appears in T, then we can think of the drawing as having two parts, inside and outside the ring. If we replace the piece of T inside the ring (i.e. C) with a smaller triangulation, then the resulting triangulation T' is colourable by minimality. This means there is a nonempty set $\mathcal{C}(R)$ of colourings of R which can be extended to proper colourings of the part of T on the outside. Moreover if $\mathcal{C}(R)$ contains some colouring of R, then it must also contain any colouring that could be obtained from it by switching on 2-coloured paths in the dual with ends in R and interior outside R. More precisely, fix two colours, a and b say, and a colouring of T'. Starting from an edge in R coloured a and following a sequence of edges of colours alternating between a and b, so that any two in succession belong to the same region, we inevitably end up at an edge of R. Such an alternating path is called a *Kempe chain*. Moreover by planarity, no two such paths can cross. Thus, while we do not know what T' looks like, we do know that for any colouring $\phi \in \mathcal{C}(R)$ and pair of colours, there is some partition of $E(R)$ into pairs that are 'planar', such that $\mathcal{C}(R)$ contains any colouring obtained from ϕ by switching colours on any subset of these pairs. This is called being *consistent*. We also know that $\mathcal{C}(R)$ doesn't contain any colouring of R that extends to a colouring of C, since T is a counterexample. Using a computer (or by hand, if time is not a valuable resource) one can compute all consistent subsets of colourings of R. If each of these contains a colouring which extends to C, then we have found that C is a reducible configuration. (The set of colourings extendible to C is easy to compute, since C is small.) We say then that C is *D-reducible* or *C-reducible*, depending on the subgraph used to replace C. These concepts of reducibility go back to the work of Kempe, Birkhoff, Heesch and others, and are used in both established proofs of the 4CT [6], [55].

To adapt the proof of the 4CT to handle doublecross graphs, we need a little bit of extra work. For one, doublecross graphs are not planar, and so it doesn't make sense to take the dual triangulation. To circumvent this problem, we replace our doublecross graph with the planar graph obtained by subdividing each of the four crossing edges and identifying their midpoints. The dual of this graph is a near-triangulation, with one region of length eight. We will fill this region with a suitable graph to obtain a triangulation to work with. We'll also want to make sure that we can find a reducible configuration that is far away from this length-8 cycle and the triangulation we filled it with, in order to be sure the configuration actually belongs to the original graph.

We will also need a new, modified definition of consistency for our reductions. This is because Kempe chains can cross in a doublecross graph, so we need a definition of consistency that allows for that. For a fixed colouring in $\mathcal{C}(R)$ and pair of colours, there are now more possibilities for the pairing of ring edges, and so it is easier for $\mathcal{C}(R)$ to be consistent. Not all of the 633 configurations from the 4CT are reducible according to our new definition, but we found a list of 756 that are and for which we did not need to modify the discharging rules from the 4CT at all. In Sections 3.4 to 3.6 we describe the modified triangulation and the discharging argument. In Section 3.7 we describe the reducibility argument. We complete the proof in Section 3.8.

3.4 XX-good configurations and a discharging argument

Let us start being more precise. To prove the four-colour theorem, one can express the problem in terms of planar triangulations, or in dual form, in terms of planar cubic graphs. But in practise it is easier to work with triangulations; for instance, it is much easier to present long lists of reducible configurations if we present them as subgraphs of triangulations than as subgraphs of cubic graphs. For the present doublecross problem, it is still most convenient to present the list of reducible configurations as subgraphs of triangulations, even though most of the argument is done in terms of cubic graphs.

A drawing is defined as in Chapter 2 and [71], and therefore has no "crossings". (Sometimes we speak of a graph as being "drawn with crossings", but we omit the formal definition of this.) Recall, a triangulation T means a non-null drawing in a 2-sphere such that every region is bounded by a cycle of length three, and a near-triangulation is a non-null connected drawing in the plane such that every finite region is a triangle. If T is a near-triangulation, its infinite region is bounded by a cycle if and only if T is two-connected; and if so, we denote this cycle by T_∞. A *configuration* K consists of a near-triangulation G_K together with a map $\gamma_K : V(G_K) \to \mathbb{Z}$ ($\mathbb{Z}$ denotes the set of all integers) with the following properties:

- $|V(G_K)| \geq 2$;
- for every vertex v, $G_K \setminus v$ has at most two components, and if there are two, then $\gamma_K(v) = d(v) + 2$ (where $d(v)$ is the degree of v in G_K);

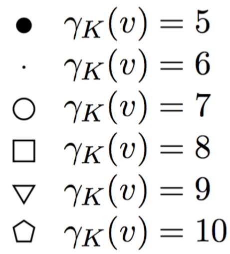

Figure 3.1: The shapes of vertices.

- for every vertex v, if v is not incident with the infinite region then $\gamma_K(v) = d(v)$; and otherwise $\gamma_K(v) > d(v)$, and in either case $\gamma_K(v) \geq 5$.

- K has ring-size at least two, where the *ring-size* of K is defined to be $\sum_v(\gamma_K(v) - d(v) - 1)$, summed over all vertices v incident with the infinite region such that $G_K \setminus v$ is connected. (In fact, all configurations used in this proof have ring-size at least six.)

We use the same conventions as in [71] to describe configurations, and in particular, we use the same vertex shapes in drawings to represent the numbers $\gamma_K(v)$.

Two configurations K, L are *isomorphic* if there is a homeomorphism of the plane mapping G_K to G_L and mapping γ_K to γ_L. In the Appendix to this thesis there are 756 configurations. Any configuration isomorphic to one of these is called an *XX-good configuration*. Note that every XX-good configuration K has the property that $\gamma_K(v) \leq 11$ for every $v \in V(G_K)$.

A triangulation T (in a 2-sphere Σ) or a near-triangulation (in a plane Σ) is *internally six-connected* if for every cycle C of T with length at most five, either some open disc in Σ, bounded by C, contains no vertex of T, or C has length five and some such open disc contains a unique vertex of T.

We say a configuration K *appears* in a triangulation T if

- G_K is an induced subgraph of T; and

- for each $v \in V(G_K)$, $\gamma_K(v)$ equals the degree of v in T.

Let T be a two-connected near-triangulation. We say a configuration K *appears internally* in T if

- G_K is an induced subgraph of $T \setminus V(T_\infty)$;
- for each $v \in V(G_K)$, $\gamma_K(v)$ equals the degree of v in T; and
- every vertex or edge of T that does not belong to G_K is drawn within the infinite region of G_K.

The main result of this section is the following.

Theorem 3.4.1. *Let J be a two-connected, internally six-connected near-triangulation. Suppose that J_∞ is an induced subgraph of J, and there are at least $4|V(J_\infty)|-11$ edges in J between $V(J_\infty)$ and $V(J) \setminus V(J_\infty)$. Then some XX-good configuration appears internally in J.*

To prove this, we use the discharging procedure from [71], so now we turn to that. A *discharging function* in a triangulation T means a map ϕ from the set of all ordered pairs of adjacent vertices of T into $\mathbb{Z}$, such that $\phi(u,v) + \phi(v,u) = 0$ for all adjacent u, v. In [71], we defined an explicit discharging function, for every internally six-connected triangulation. Since it is rather complicated, we refer the reader to [71] for the details of its definition. We need the following two properties; the first can be verified by hand, but the second needs a computer.

Theorem 3.4.2. *Let T be an internally six-connected triangulation, and let ϕ be the discharging function defined in* [71]*. Then*

- *for every edge uv, if $\phi(u,v) > 5$ then some XX-good configuration appears in T and contains u*
- *for every vertex u of G, if $10(6 - d_T(u)) > \sum_v \phi(u,v)$ (where the sum is over all vertices v adjacent to u) then some XX-good configuration K appears in T, and moreover either u or some neighbour of u is a vertex of G_K.*

Both of these statements are minor variants of theorems proved in [71] (Theorems 4.7 and 4.4 of that paper, respectively) and the methods of proof are unchanged. The proof of the first statement is virtually identical with the proof of Theorem 4.7 of [71], because all the “good configurations” used in that proof are also XX-good, except for one, the configuration called conf(2,10,6) in that paper, which is not XX-good. This is only needed at one step of the proof, and at that step we can use a different configuration instead, which is XX-good, the second on line 5 of page 3 of the Appendix. In the proof in [71] of Theorem 4.7 of that paper, it is not included in the statement of the theorem that the good configuration we find contains u, as we are claiming now; but in fact the proof shows that.

The proof of the second needs analogues of Theorems 4.5, 4.6, 4.8, 4.9 of the same paper. Again, in [71] it is not included in the statement of the theorem that the good configuration we find contains u or one of its neighbours; but this is implied by the fact that the configuration is always found within a "cartwheel" centred on u. The proofs of the analogues of 4.5, 4.6 and 4.8 are unchanged, because all the good configurations used for those proofs in [71] are also XX-good. The analogue of 4.9 is proved by computer. The computer program just checks a machine-readable proof of unavoidability, and is the same as was used in [71]; we just changed its two inputs, the list of configurations we want to prove unavoidable, and the files containing the machine-readable proofs. We are making the program and the computer-readable proofs available on the arXiv [36].

Proof of Theorem 3.4.1. Let $C = J_\infty$, and take a drawing of J in a 2-sphere such that C bounds some region r_0. Thus one region of J has a boundary of length $|V(C)|$, and all others have length three. Let J have r regions; then by Euler's formula, $|V(J)| - |E(J)| + r = 2$, and so

$$\sum_{v \in V(J)} 6 - d_J(v) = 6|V(J)| - 2|E(J)| = 6(|E(J)| + 2 - r) - 2|E(J)| = 4|E(J)| - 6r + 12.$$

The sum of the lengths of the regions of J is $|V(C)| + 3(r-1)$, and so $2|E(J)| = |V(C)| + 3(r-1)$. We deduce that

$$\sum_{v \in V(J)} 6 - d_J(v) = 4|E(J)| - 6r + 12 = 2(|V(C)| + 3(r-1)) - 6r + 12 = 2|V(C)| + 6.$$

Let there be k edges between $V(J) \setminus V(C)$ and $V(C)$. It follows that $\sum_{v \in V(C)} d_J(v) = k +$

$2|V(C)|$. Consequently

$$\sum_{v\in V(C)} 6-d_J(v) = 6|V(C)| - (k+2|V(C)|) = 4|V(C)| - k$$

and so

$$\sum_{u\in V(J)\setminus V(C)} 6-d_J(u) = (2|V(C)|+6) - (4|V(C)|-k) = k+6-2|V(C)|.$$

Now extend the drawing of J to a drawing of an internally six-connected triangulation T, by adding more vertices and edges drawn within r_0, in such a way that every vertex in $V(C)$ has degree in T at least 12. (It is easy to see that this is possible, since J is internally six-connected.)

(1) *If there is an XX-good configuration K that appears in T such that some vertex in $V(J)\setminus V(C)$ either belongs to G_K or has a neighbour in G_K, then K appears internally in J.*

No vertex in $V(C)$ belongs to $V(G_K)$, since $\gamma_K(v)\leq 11$ for every vertex v of K, and $d_T(v)\geq 12$ for every $v\in V(C)$. From the hypothesis, it follows that some vertex of G_K belongs to $V(J)\setminus V(C)$, and hence $V(G_K)\subseteq V(J)\setminus V(C)$ since G_K is connected. Moreover, every finite region of G_K is a finite region of J, since J is internally six-connected. But every vertex in $V(J)\setminus V(C)$ has the same degrees in T and in J, and so K appears internally in J. This proves (1).

Let ϕ be the discharging function on T defined in [71]. Suppose first that $\phi(u,v)>5$ for some edge uv of T with $u\in V(J)\setminus V(C)$ and $v\in V(C)$. By the first statement of Theorem 3.4.2, some XX-good configuration K appears in T and contains u, and the result follows from (1).

Thus we may assume that there is no such edge uv. Let there be k edges uv in J with $u\in V(J)\setminus V(C)$ and $v\in V(C)$. Consequently the sum of $\phi(u,v)$, over all edges uv of T with $u\in V(J)\setminus V(C)$ and $v\in V(C)$, is at most $5k$. But this equals the sum over all $u\in V(J)\setminus V(C)$, of the sum of $\phi(u,v)$ over all neighbours v of u, since $\phi(u,v)=-\phi(v,u)$ for all u,v. Therefore the sum over all $u\in V(J)\setminus V(C)$ of

$$10(6-d_J(u)) - \sum_{uv\in E(J)} \phi(u,v)$$

is at least $10(k+6-2|V(C)|)-5k$. Since $k \geq 4|V(C)|-11$ by hypothesis, the last is positive, and so there exists $u \in V(J) \setminus V(C)$ such that

$$10(6-d_J(u)) - \sum_{uv \in E(J)} \phi(u,v) > 0.$$

For such a vertex u, its degrees in J and T are the same; and so by the second assertion of Theorem 3.4.2, there is an XX-good configuration K that appears in T, such that either u or some neighbour of u is a vertex of G_K. But then again, the result follows from (1). This proves Theorem 3.4.1. □

3.5 The doublecross edges

By a *minimal counterexample*, we mean a cubic two-edge-connected doublecross graph G which is not three-edge-colourable, and such that every cubic two-edge-connected minor of G is three-edge-colourable except G itself.

Proposition 3.5.1. *Let G be a minimal counterexample. Then*

- *G is theta-connected;*
- *there are four edges $g_1, \ldots, g_4$ of G, and the graph $G \setminus \{g_1, g_2, g_3, g_4\}$ can be drawn in the plane such that its infinite region is bounded by a cycle Z;*
- *there are eight vertices $z_1, \ldots, z_8$ of Z, distinct and in order, such that $g_1 = z_1z_3$, $g_2 = z_2z_4$, $g_3 = z_5z_7$ and $g_4 = z_6z_8$.*

Proof. By the result of [77], G is not apex since it is not three-edge-colourable. Since G is also a minimal counterexample to Tutte's conjecture, the result of [74] implies that G is theta-connected. Now G can be drawn in the plane with only two crossings both on the infinite region. Let the crossing pairs of edges be (g_1, g_2) and (g_3, g_4). Since G is not apex, it follows that $\{g_1, g_2\} \neq \{g_3, g_4\}$, and indeed g_1, g_2 are disjoint from g_3, g_4. If g_1 shares an end with g_2, then the drawings of g_1, g_2 can be rearranged to eliminate their crossing, and again G is apex, which is again impossible (indeed, in this case G has crossing number at most one, and so we could use Theorem 3.2.1 instead of the result of [77]). So g_1, g_2, g_3, g_4 are disjoint. The graph obtained by deleting the edges $g_1, \ldots, g_4$ is two-edge-connected since G is theta-connected, and it is drawn in the plane such that g_1, g_2, g_3, g_4 are all drawn within its infinite region. Since it is two-edge-connected, its infinite region is bounded by a cycle. This proves Proposition 3.5.1. $\square$

To prove Theorem 3.1.2, we use the same approach as the proof of the four-colour theorem, proving the existence of an unavoidable set of reducible subgraphs. Some of these reducible subgraphs use all four of the edges $g_1, \ldots, g_4$, and the others use none of them. The length of the cycle Z of Proposition 3.5.1 is the deciding factor here; if $|E(Z)| \leq 20$ we will show the existence of a reducible subgraph using $g_1, \ldots, g_4$, while if $|E(Z)| \geq 21$ we will show the presence of one of the other kind. In this section we handle the case when $|E(Z)| \leq 20$.

Lemma 3.5.2. *If G is a minimal counterexample and Z is the cycle as in Proposition 3.5.1, then $|E(Z)| \geq 21$.*

Proof. Let $z_1, \ldots, z_8$ be as in Proposition 3.5.1, and for $1 \leq i \leq 8$ let Z_i be the path of Z with ends z_i, z_{i+1} that contains no other member of $\{z_1, \ldots, z_8\}$ (where z_9 means z_1). For $1 \leq i \leq 8$, let L_i denote $|E(Z_i)|$. We observe:

- $L_1, \ldots, L_8 \geq 1$, because $z_1, \ldots, z_8$ are all distinct;
- $L_1 + L_2 \geq 4$ since every cycle of G has length at least five; and for the same reason $L_2 + L_3, L_5 + L_6, L_6 + L_7 \geq 4$ and $L_1 + L_3, L_5 + L_7 \geq 3$;
- $L_1 + L_2 + L_3 \geq 7$ since there are at least six edges with exactly one end in $V(Z_1 \cup Z_2 \cup Z_3)$ (because G is theta-connected); and similarly $L_5 + L_6 + L_7 \geq 7$;
- $L_1 + L_8 \geq 3$, because if Z_1, Z_8 both have length one then G is apex (deleting the end of Z_8 not in Z_1 makes the graph planar); and similarly $L_3 + L_4, L_4 + L_5, L_7 + L_8 \geq 3$.

A choice of the 8-tuple $(L_1, \ldots, L_8)$ is called *plausible* if it satisfies the conditions just listed. Suppose that $|E(Z)| \leq 20$; then there are only finitely many plausible choices for $(L_1, \ldots, L_8)$, and we handle them one at a time. Now, therefore, we assume that we are dealing with some such plausible choice, and so we know the lengths $L_1, \ldots, L_8$. Let G^- be the graph obtained from G by deleting the four crossing edges $g_1 = z_1z_3$, $g_2 = z_2z_4$, $g_3 = z_5z_7$ and $g_4 = z_6z_8$. Then Z is a cycle of G^-, bounding a region in a planar drawing of G^-. Every vertex of Z different from $z_1, \ldots, z_8$ is incident with an edge of G^- that does not belong to $E(Z)$. Let the vertices of Z different from $z_1, \ldots, z_8$ be $v_1, \ldots, v_k$ say, numbered in circular order (starting from some arbitrary first vertex), and for $1 \le i \le k$ let f_i be the edge of G incident with v_i and not in $E(Z)$. Note that $f_1, \ldots, f_k$ might not all be distinct (because for instance some f_i might be incident with a vertex of the interior of Z_2 and incident with a vertex of the interior of Z_6). Let $F = \{f_1, \ldots, f_k\}$, and let $\mathcal{D}$ be the set of all maps from F to $\{1, 2, 3\}$.

A subset $\mathcal{C} \subseteq \mathcal{D}$ is said to be *consistent* if it has the following property. For all distinct $x, y \in \{1, 2, 3\}$, and each $\phi \in \mathcal{C}$, let $F_{x,y}$ be the set of all $f \in F$ with $\phi(f) \in \{x, y\}$; then there is a partition Π of $F_{x,y}$ into sets of size one and two, with the following properties:

- for $f \in F_{x,y}$, the member of Π containing f has size one if and only if both ends of f belong to $V(Z)$;
- for $1 \le a < b < c < d \le k$, not both $\{f_a, f_c\}, \{f_b, f_d\} \in \Pi$; and
- $\phi' \in \mathcal{C}$ for every subset $F' \subseteq F_{x,y}$ which is expressible as a union of members of Π, where ϕ' is defined by

$$\phi'(f) = \begin{cases} \phi(f) & \text{if } f \in F \setminus F' \\ y & \text{if } f \in F' \text{ and } \phi(f) = x \\ x & \text{if } f \in F' \text{ and } \phi(f) = y. \end{cases}$$

For any graph H with $F \subseteq E(H)$, we denote by $\mathcal{C}_H$ the set of all members of $\mathcal{D}$ that can be extended to a three-edge-colouring of H. Let $J = G^- \setminus E(Z)$, and let K be the subgraph of G formed by the edges in $F \cup E(Z) \cup \{g_1, g_2, g_3, g_4\}$ and their incident vertices. Since $|E(Z)| \le 20$, it follows that $|F| \le 12$.

For each plausible choice of $(L_1, \ldots, L_8)$ (except one, that we handle separately), there are three steps to be carried out on a computer, which we explain now. All three involve computation with subsets of $\mathcal{D}$, but since $|F| \le 12$ all three steps are easily implemented.

Step 1: Compute $\mathcal{C}_K$.

Step 2: Compute the maximal consistent subset $\mathcal{C}$ of $\mathcal{D} \setminus \mathcal{C}_K$. (The union of any two consistent sets is consistent, and so there is a unique maximal consistent subset of any set.)

Step 3: Verify that there is a graph K', obtained from K by deleting one or two of the edges $g_1, \ldots, g_4$, and suppressing the vertices of degree two that arise, such that $\mathcal{C} \cap \mathcal{C}_{K'} = \emptyset$. (This involves looking at the handful of possibilities for K' and computing $\mathcal{C}_{K'}$ for each of them.)

The exception is the 8-tuple $(4, 1, 4, 1, 4, 1, 4, 1)$, which is plausible but for which there is no K' as in step 3 above. For this 8-tuple let K' be obtained from K by deleting the unique edge in Z_2 and suppressing the vertices of degree two that arise. Again we check that $\mathcal{C} \cap \mathcal{C}_{K'} = \emptyset$.

In all cases, since G is not three-edge-colourable, and $J \cup K = G$, it follows that $\mathcal{C}_J$ and $\mathcal{C}_K$ are disjoint. The planarity of J implies that $\mathcal{C}_J$ is consistent (this is easy to see, and is a standard argument in proving the four-colour theorem – see [71]). Consequently $\mathcal{C}_J \subseteq \mathcal{C}$, and so $\mathcal{C}_J \cap \mathcal{C}_{K'} = \emptyset$. It follows that $J \cup K'$ is not three-edge-colourable. But $J \cup K'$ can be obtained from G by deleting either one or two disjoint edges, and suppressing the vertices of degree two that arise; and so $J \cup K'$ is two-edge-connected (since G is theta-connected), doublecross, and smaller than our supposedly minimal counterexample, which is impossible. This proves Lemma 3.5.2. □

We have omitted the details of the computer checking; this is all straightforward. (The program is available on the arXiv.) There are 2957 plausible 8-tuples to check, up to symmetry, but the program only takes about a minute to do them all, so we were content with that. If desired, running through all possible choices of $(L_1, \ldots, L_8)$ could be made more efficient at the cost of complicating the proof. For instance, we could quickly dispose of the case when $\min(L_1, L_2, L_3) = \min(L_5, L_6, L_7) = 1$, because in this case G contains a "C-reducible" subgraph (of a different kind), no matter what the other six lengths are. But we are aiming for simplicity rather than speed here.

3.6 Islands

An *island* means a graph I drawn in the plane, with the following properties:

- I is two-connected;
- every vertex has degree two or three; and
- every vertex of degree two is incident with the infinite region.

Let I be an island, and J be a geometric dual, where $j \in V(J)$ corresponds to the infinite region of I. For each $v \in V(J) \setminus \{j\}$, let $\gamma(v)$ be the length of the region of I that corresponds to v. Then the pair $(J \setminus \{j\}, \gamma)$ might or might not be a configuration; but more important, for every configuration K, there is an island that gives rise to it in this way, unique up to homeomorphism of the plane. (We leave checking this to the reader. One way is to go to the "free completion" of K defined in [71], take a dual, and delete the vertex corresponding to the infinite region.) We call this the *island of* K, and denote it by $I(K)$.

We need to work mostly with the islands of the XX-good configurations, but it is more compact to draw the configurations themselves. Sometimes we need to refer to an edge e of one of the islands, say of $I(K)$. Now e corresponds to some edge f of J under the duality (where J is as before), and if $f \in E(G_K)$ then we can refer to e by defining it as the edge dual to f. But sometimes the edge f is not an edge of G_K. For this reason, in the list of XX-good configurations, some vertices are drawn with extra "half-edges". These indicate some of the edges of J that are not edges of G_K, for convenience in referring to certain edges of $I(K)$.

Lemma 3.6.1. *Let G be a minimal counterexample, and let $Z, g_1, \ldots, g_4$ be as in Proposition 3.5.1. Then there is a cycle of $G \setminus \{g_1, \ldots, g_4\}$, bounding a closed disc Δ, such that the subgraph of G formed by the vertices and edges drawn in Δ is an island of some XX-good configuration.*

Proof. Let $G^- = G \setminus \{g_1, \ldots, g_4\}$; and let us extend the drawing of G^- by adding one new vertex z_∞ (drawn within the infinite region of G^-) and eight new edges, joining z_∞ to the eight ends of $g_1, \ldots, g_4$, forming G^+ say. Thus in G^+, every vertex has degree three except for z_∞, which has degree eight. (We can think of G^+ as obtained from G by subdividing the four edges $g_1, \ldots, g_4$ and identifying the four new vertices.)

Now take a geometric dual T of G^+, such that z_∞ belongs to the infinite region of T. It follows that:

- T is a near-triangulation;
- the cycle T_∞ is an induced subgraph of T;
- T is internally six-connected (since G is theta-connected); and

- the number of edges of T between $V(T_\infty)$ and $V(T) \setminus V(T_\infty)$ is at least 21, since $|E(Z)| \geq 21$ by Lemma 3.5.2, and for each $e \in E(Z)$, the corresponding edge of T has one end in $V(T_\infty)$ and the other in $V(T) \setminus V(T_\infty)$.

Since $|V(T_\infty)| = 8$ and so $4|V(T_\infty)| - 11 = 21$, Theorem 3.4.1 implies that some XX-good configuration K appears internally in T. Consequently the union of the closures of the regions of G^- that correspond to vertices in G_K is a closed disc that defines an island satisfying the theorem. This proves Lemma 3.6.1. $\square$

3.7 Reducibility

It remains to show that the outcome of Lemma 3.6.1 is impossible, but for that we need to discuss reducibility further.

If a, b, c, d are integers and $1 \leq a < b < c < d$, we call $\{\{a, c\}, \{b, d\}\}$ a *cross*. Let Π be a finite set of finite sets of positive integers, each of cardinality two and pairwise disjoint. We say that Π is *doublecross* if the following conditions hold:

- at most two subsets of Π are crosses
- if $A, B, C, D \in \Pi$ are distinct, and $\{A, B\}, \{C, D\}$ are crosses, let $X = A \cup B \cup C \cup D$ (so $|X| = 8$). Then for all $P \in \Pi$ with $P \cap X = \emptyset$ there do not exist $x_1, x_2 \in X$ such that $\{P, \{x_1, x_2\}\}$ is a cross.

This is equivalent to the following geometric condition, which may be easier to grasp: choose $k \geq 3$ such that $A \subseteq \{1, \ldots, k\}$ for each $A \in \Pi$. Take a regular k-vertex polygon in the plane, with vertices $v_1, \ldots, v_k$ in order. For each $A \in \Pi$ draw a line segment L_A between v_i, v_j, where $A = \{i, j\}$. Then we ask that

- no point of the plane belongs to more than two of L_A ($A \in \Pi$);
- at most two points of the plane belong to more than one of these lines;
- if there are two points x, y each belonging to two of the lines L_A ($A \in \Pi$) say, then either some L_A contains them both, or no L_A intersects the interior of the line segment between x, y.

We leave the equivalence to the reader.

Let $k \geq 1$, and let $\mathcal{D}$ be the set of all maps $\phi : \{1, \ldots, k\} \to \{1, 2, 3\}$. We say a subset $\mathcal{C}$ of $\mathcal{D}$ is *XX-consistent* if it has the following property. For all distinct $x, y \in \{1, 2, 3\}$, and each $\phi \in \mathcal{C}$, let $R_{x,y}$ be the set of all $i \in \{1, \ldots, k\}$ with $\phi(i) \in \{x, y\}$; then there is a doublecross partition Π of $R_{x,y}$, such that $\phi' \in \mathcal{C}$ for every subset $F' \subseteq R_{x,y}$ which is expressible as a union of members of Π, where ϕ' is defined by

$$\phi'(f) = \begin{cases} \phi(f) & \text{if } f \in \{1, \ldots, k\} \setminus F' \\ y & \text{if } f \in F' \text{ and } \phi(f) = x \\ x & \text{if } f \in F' \text{ and } \phi(f) = y. \end{cases}$$

Let I be the island of a configuration K. We say $F \subseteq E(I)$ is a *matching* if no two edges in F have a common end, and $V(F)$ denotes the set of vertices incident with an edge in F. A *three-edge-colouring modulo* F of I means a map $\phi :\ E(I) \setminus F \to \{1, 2, 3\}$, such that for all distinct edges $e, f \in E(I) \setminus F$ with a common end v say, $\phi(e) = \phi(f)$ if and only if $v \in V(F)$.

Let the vertices with degree two in I be $v_1, \ldots, v_k$ in order on the boundary of the infinite region. With $\mathcal{D}$ as before, let $\mathcal{C}_K$ be the set of all $\psi \in \mathcal{D}$ such that there is a three-edge-colouring ϕ of I with $\phi(e) \neq \psi(i)$ for $1 \leq i \leq k$ and for each edge e of I incident with v_i. We say that K is *XXD-reducible* if there is no non-null XX-consistent subset of $\mathcal{D} \setminus \mathcal{C}_K$. We say that K is *XXC-reducible* if there is a matching F of I with the following properties:

- $1 \leq |F| \leq 4$.
- If $|F| = 4$, then either some finite region of I is incident with at least three members of F, or there are two finite regions of I, say r, s, such that some edge of I is incident with both r, s, and every edge of F is incident with one of them.
- Let $\mathcal{C}_F$ be the set of all $\psi \in \mathcal{D}$ such that there is a three-edge-colouring modulo F of I, say ϕ, such that for $1 \leq i \leq k$ and every edge $e \in E(I) \setminus F$, $\phi(e) = \psi(i)$ if and only if $v_i \in V(F)$. Then every XX-consistent subset of $\mathcal{D} \setminus \mathcal{C}_K$ is disjoint from $\mathcal{C}_F$.

We call such a set F a *reducer* for K. (We will show that if K appears in a minimal counterexample G, then deleting from G the edges in F and suppressing the resultant vertices of degree two will make a smaller counterexample, which is impossible; and so K cannot appear.) We need:

Theorem 3.7.1. *For every XX-good configuration K, either K is XXD-reducible, or K is XXC-reducible.*

Proof. The proof uses a computer. Each XX-good configuration K is drawn in the Appendix to this thesis, and in that drawing sometimes some edges are drawn thickened. If no edges are thickened then we claim K is XXD-reducible, and otherwise we claim it is XXC-reducible, and the corresponding reducer F corresponds to the set of thickened edges and half-edges of G_K under planar duality. To show this, for each XX-good configuration K in turn, we carry out two steps:

Step 1: Compute $\mathcal{C}_K$.

Step 2: Compute the maximal XX-consistent subset $\mathcal{C}$ of $\mathcal{D} \setminus \mathcal{C}_K$. (The union of any two XX-consistent sets is XX-consistent, and so there is a unique maximal XX-consistent subset of any set.)

If $\mathcal{C}$ is empty we have verified that K is XXD-reducible and we stop here. Otherwise, we carry out:

Step 3: Let F be the set of edges of $I(K)$ that correspond under geometric duality to the thickened edges and half-edges of G_K given in the Appendix, and verify that F is a reducer for K.

This is just the same process as in the proof of the four-colour theorem [71], and is carried out on a computer the same way; we omit further details. (Again, we are making the program available on the arXiv [36].) □

3.8 Assembling the pieces

Now we combine these various lemmas to prove Theorem 3.1.2.

Proof of Theorem 3.1.2. Suppose the result is false; then there is a minimal counterexample G. Let $Z, g_1, \ldots, g_4$ be as in Proposition 3.5.1, and let $G^- = G \setminus \{g_1, g_2, g_3, g_4\}$. By Lemma 3.6.1, there is a cycle D of G^-, bounding a closed disc Δ, such that the subgraph of G formed by the vertices and edges drawn in Δ is an island I of some XX-good configuration K say. Let $v_1, \ldots, v_k$ be the vertices of D that have degree two in I, numbered in order on D, and for $1 \leq i \leq k$, let e_i be the edge of G incident with v_i and not in $E(I)$. Note that $e_1, \ldots, e_k$ need not all be distinct, because some e_i might have both ends in $V(D)$.

Now let $\mathcal{D}$ be the set of all maps from $\{1, \ldots, k\}$ to $\{1, 2, 3\}$. We say a map $\phi : E(G) \setminus E(I) \to \{1, 2, 3\}$ is a *three-edge-colouring of* $\overline{I}$ if $\phi(e) \neq \phi(f)$ for every two distinct edges $e, f \in E(G) \setminus E(I)$ with a common end in G. Let $\mathcal{C}$ be the set of $\psi \in \mathcal{D}$ such that there is a three-edge-colouring ϕ of $\overline{I}$ with $\psi(i) = \phi(e_i)$ for $1 \leq i \leq k$. We claim:

(1) *$\mathcal{C}$ is non-null and XX-consistent.*

Clearly it is non-null, from the minimality of G. Let $\psi \in \mathcal{C}$, and choose a three-edge-colouring ϕ of $\overline{I}$ with $\psi(i) = \phi(e_i)$ for $1 \leq i \leq k$. Let $x, y \in \{1, 2, 3\}$ be different, and let H be the subgraph of G formed by the edges $e \in E(G) \setminus E(I)$ with $\phi(e) \in \{x, y\}$ and their ends. It follows that every component of H is either a cycle, or a path with distinct ends both in $V(D)$. Let Π be the set of all $\{i, j\}$ such that $1 \leq i < j \leq k$ and some component of H is a path with end-edges e_i, e_j. (Possibly $e_i = e_j$, and this path has only one edge.) Then we can switch colours x, y on any subset of $\{1, \ldots, k\}$ that is expressible as a union of members of Π, by exchanging the colours x, y on the corresponding components of H. It remains to show that Π is doublecross. To see this, note first that if $\{\{a, b\}, \{c, d\}\} \subseteq \Pi$ is a cross, and P, Q are the components of H with end-edges e_a, e_b and e_c, e_d respectively, then either P contains one of g_1, g_2 and Q contains the other, or P contains one of g_3, g_4 and Q contains the other. Since G can be drawn with no crossing pairs of edges except g_1, g_2 and g_3, g_4, and the two crossings they form are on a common region, it follows that Π is doublecross. This proves (1).

Now let $\mathcal{C}_K$ be as in Theorem 3.7.1; that is, $\mathcal{C}_K$ is the set of all $\psi \in \mathcal{D}$ such that there is a three-edge-colouring ϕ of I with $\phi(e) \neq \psi(i)$ for $1 \leq i \leq k$ and for each edge e of I incident with v_i.

(2) $\mathcal{C}_K \cap \mathcal{C} = \emptyset$.

For suppose that $\psi \in \mathcal{C}_K \cap \mathcal{C}$. Choose a three-edge-colouring ϕ_1 of $\overline{I}$ such that $\psi(i) = \phi_1(e_i)$ for $1 \leq i \leq k$. Choose a three-edge-colouring ϕ_2 of I such that $\phi_2(e) \neq \psi(i)$ for $1 \leq i \leq k$ and each edge e of I incident with v_i. For each edge e of G, let

$$\phi(e) = \begin{cases} \phi_1(e) & \text{if } e \notin E(I) \\ \phi_2(e) & \text{if } e \in E(I). \end{cases}$$

We claim that ϕ is a three-edge-colouring of G. For let $e, f \in E(G)$ be distinct with a common end v say. If $e, f \in E(I)$ then

$$\phi(e) = \phi_2(e) \neq \phi_2(f) = \phi(f)$$

since ϕ_2 is a three-edge-colouring of I; and similarly $\phi(i) \neq \phi(j)$ if $i, j \notin E(I)$. We may therefore assume that $e \in E(I)$ and $f \notin E(I)$; and consequently v is one of $v_1, \ldots, v_k$, say v_i. From the choice of ϕ_1 it follows that $\psi(i) = \phi_1(e_i) = \phi(f)$; and from the choice of ϕ_2, $\phi(e) = \phi_2(e) \neq \psi(i)$. It follows that $\phi(e) \neq \phi(f)$. This proves (2).

By (1) and (2), it follows that K is not XXD-reducible. Since K is XX-good, it is therefore XXC-reducible by Theorem 3.7.1. Let $F \subseteq E(I)$ be a reducer. Let $\mathcal{C}_F$ be the set of all $\psi \in \mathcal{D}$ such that there is a three-edge-colouring modulo F of I, say ϕ, such that for $1 \leq i \leq k$ and every edge $e \in E(I) \setminus F$ incident with v_i, $\phi(e) = \psi(i)$ if and only if $v_i \in V(F)$. From the definition of a reducer, it follows that every XX-consistent subset of $\mathcal{D} \setminus \mathcal{C}_K$ is disjoint from $\mathcal{C}_F$, and so in particular, $\mathcal{C} \cap \mathcal{C}_F = \emptyset$, by (1) and (2).

(3) *The graph $G \setminus F$ has a cutedge.*

Suppose it does not. Then from the minimality of G, there is a map $\phi : E(G) \setminus F \to \{1, 2, 3\}$, such that for all distinct edges $e, f \in E(G) \setminus F$ with a common end v, $\phi(e) = \phi(f)$ if and only if $v \in V(F)$. (To see this, suppress the vertices of degree two.) For $1 \leq i \leq k$, let $\psi(i) = \phi(e_i)$. Then $\psi \in \mathcal{C} \cap \mathcal{C}_F$, which is impossible. This proves (3).

(4) $|F| = 4$, *and there is a cycle W of G of length five, such that $F \subseteq \delta_G(V(W))$.*

Let f_0 be a cutedge of $G \setminus F$. Consequently there exists $Y \subseteq V(G)$, such that $f_0 \in \delta_G(Y) \subseteq F \cup \{f_0\}$.

By replacing Y by its complement if necessary, we may assume that $|Y| \leq |V(G) \setminus Y|$. Suppose first that $|\delta_G(Y)| \leq 4$. Since G is theta-connected, it follows that $|Y| \leq 2$. Since $F \cap \delta_G(Y)$ is a matching, and G is three-connected, this is impossible. Thus $|\delta_G(Y)| \geq 5$, and so $|F| = 4$, and $\delta_G(Y) = F \cup \{f_0\}$.

Since G is theta-connected and $|Y| \leq |V(G) \setminus Y|$, it follows that $|Y| \leq 5$. Now $|Y| \geq 4$ since F is a matching; and $|Y|$ is odd since $|\delta_G(Y)|$ is odd; so $|Y| = 5$. Since $|\delta_G(Y)| = 5$ and so there are five edges of G with both ends in Y, it follows that there is a cycle W of G with $V(W) \subseteq Y$; and since G is theta-connected, W has length five. This proves (4).

We remark that the edges in F all belong to $E(I)$, but some of the other six edges of G with an end in the cycle W of (4) might not belong to $E(I)$, and indeed might not belong to $E(G^-)$. We recall that from the choice of F, we have:

(5) *Either there exists a finite region of I incident with three edges in F, or there are two finite regions r, r' of I, such that some edge of I is incident with both r, r', and every edge in F is incident with one of r, r'.*

Let W be as in (4), with vertices $w_1, \ldots, w_5$ in order, and for $1 \leq i \leq 5$ let h_i be the edge of G incident with w_i and not in $E(W)$, where $F = \{h_1, \ldots, h_4\}$.

(6) *W contains at least one of $g_1, \ldots, g_4$.*

For suppose not; then W is a cycle of G^-, and consequently bounds a finite region of G^- since G is theta-connected. For $1 \leq i \leq 5$, let r_i be the second region of G^- incident with the edge w_iw_{i+1}, where w_6 means w_1. Now $r_4 \neq r_5$, since h_5 is not a cut-edge of G^-, and so r_1, r_2, r_3 are the only regions of G^- incident with two of $h_1, \ldots, h_4$. If some finite region of G^- is incident with three of $h_1, \ldots, h_4$, then two of r_1, r_2, r_3 are equal and finite, contradicting the theta-connectivity of G. By (5), r_1, r_3 are finite regions of G^-, and there is an edge of G^- incident with r_1, r_3, again contrary to the theta-connectivity of G. This proves (6).

Let $z_1, \ldots, z_8$ be as in Proposition 3.5.1, and for $1 \leq i \leq 8$, let Z_i be the path of Z between

z_i, z_{i+1} containing no other vertex in $\{z_1, \ldots, z_8\}$ (where z_9 means z_1).

(7) W *contains two of* $g_1, \ldots, g_4$.

For suppose W only contains one. From the symmetry we may assume that $g_1 \in E(W)$. Consequently $W \setminus g_1$ is a four-edge path of G^- between z_1, z_3. Since z_1, z_3 have degree two in G^-, the first and last edges of this path belong to the cycle Z. If the edge of $W \setminus g_1$ incident with z_1 belongs to Z_8, then there is a three-edge path in G^- between $Z_8 \setminus z_1$ and z_3, contrary to the theta-connectivity of G; so the first edge of $W \setminus g_1$ belong to Z_1 and similarly the last edge belongs to Z_2, respectively. Since G is theta-connected, it follows that the middle vertex of the path $W \setminus g_1$ also belongs to $Z_1 \cup Z_2$; and so $W \setminus g_1 = Z_1 \cup Z_2$. In particular, $z_2 \in V(W)$, and so $g_2 \in \delta_G(V(W))$. Since every edge in F belongs to $E(G^-)$, it follows that $g_2 = h_5$, and so $z_2 = w_5$. Now since G is theta-connected, there are five edge-disjoint paths of G between $V(W)$ and $Z_5 \cup Z_6 \cup Z_7$. One of them uses g_2, but the other four are paths of G^-, and start at distinct vertices of W. Their first edges are the four edges in F. Let these paths be $P_1, \ldots, P_4$ say, numbered so P_i has first vertex w_i and first edge h_i. It follows that no region of G^- is incident with three of $h_1, \ldots, h_4$, because of the paths $P_1, \ldots, P_4$; and similarly no two regions with a common edge are together incident with all of $h_1, \ldots, h_4$, contrary to (5). This proves (7).

Since $g_1, \ldots, g_4$ are pairwise vertex-disjoint, and W has length five, it follows that W contains exactly two of $g_1, \ldots, g_4$. Let W contain g_1 and g_i say. The other three edges of W are edges of G^- incident with vertices in $\{z_1, \ldots, z_8\}$, and so all three belong to $E(Z)$. Consequently $i = 2$, and one of Z_1, Z_3 has length one, and the other has length two, and from the symmetry we may assume that Z_1 has length one and Z_3 length two. From the theta-connectivity of G, there are five edge-disjoint paths of G^- from $V(W)$ to $Z_5 \cup Z_6 \cup Z_7$, and their first edges are the five edges in $\delta_G(V(W))$. Again this contradicts (5), and completes the proof of Theorem 3.1.2. $\square$

Chapter 4

Edge-colouring d-regular planar graphs

4.1 Introduction

Recall once more the form of the four-colour theorem, due to Tait [81], which asserts that a 3-regular planar graph can be 3-edge-coloured if and only if it is 2-edge-connected. In Chapter 3 we considered Tutte's strengthening of the statement, with 'planar' replaced with 'Petersen minor-free'. In this chapter, we consider the question: when can d-regular planar graphs be d-edge-coloured?

Let G be a graph. (Graphs in this chapter may have loops or parallel edges.) Recall that if $X \subseteq V(G)$, $\delta_G(X) = \delta(X)$ denotes the set of all edges of G with an end in X and an end in $V(G) \setminus X$. We say that G is *oddly d-edge-connected* if $|\delta(X)| \geq d$ for all odd subsets X of $V(G)$. Since every perfect matching contains an edge of $\delta(X)$ for every odd set $X \subseteq V(G)$, it follows that every d-regular d-edge-colourable graph is oddly d-edge-connected. (Note that for a 3-regular graph, being oddly 3-edge-connected is the same as being 2-edge-connected, because if $X \subseteq V(G)$, then $|\delta(X)| = 1$ if and only if $|X|$ is odd and $|\delta(X)| < 3$.) The converse is false, even for $d = 3$ (the Petersen graph is a counterexample); but for planar graphs perhaps the converse is true. That is the content of the following conjecture [79], proposed by Seymour in about 1975.

Conjecture 4.1.1. *If G is a d-regular planar graph, then G is d-edge-colourable if and only if G is oddly d-edge-connected.*

Some special cases of this conjecture have been proved.

- For $d = 3$ it is the four-colour theorem, and was proved by Appel and Haken [3, 4, 71];
- for $d = 4, 5$ it was proved by Guenin [45];
- for $d = 6$ it was proved by Dvořák, Kawarabayashi and Král' [28];
- for $d = 7$ it was proved by Kawarabayashi and the present author, and appears in the Master's thesis [32] of the latter. The methods we use in this chapter can also be applied to the $d = 7$ case, resulting in a proof somewhat simpler than the original, and this simplified proof for the $d = 7$ case has been published in a four-author paper [19].

Here we prove the next case, namely:

Theorem 4.1.2. *Every* 8*-regular oddly* 8*-edge-connected planar graph is* 8*-edge-colourable.*

All these proofs (for $d > 3$), including ours, proceed by induction on d. Thus we need to assume the truth of the result for $d = 7$. The proof of Theorem 4.1.2 that follows is joint work with Maria Chudnovsky and Paul Seymour and has been published in [20].

4.2 T-joins

Before we get to the proof of Theorem 4.1.2, we discuss a connection between Conjecture 4.1.1 and packing T-joins. Let G be a graph, and $T \subseteq V(G)$ a set of vertices with even cardinality. We call the pair (G, T) a *graft*.

Definition 4.2.1. *A T-join is a minimal set of edges $J \subseteq E(G)$ such that for each vertex $v \in V(G)$ we have $|J \cap \delta(v)|$ odd if and only if $v \in T$.*

Denote by $\nu(G, T)$ the maximum number of edge-disjoint T-joins in G.

Definition 4.2.2. *A T-cut is an edge cut $\delta(S)$ with $|T \cap S|$ odd.*

Denote by $\tau(G, T)$ the number of edges in a smallest T-cut. The following easy observation appears in [23].

Lemma 4.2.3. *Let (G,T) be a graft. Let J be a T-join and $S \subseteq V(G)$. Then $|J \cap \delta(S)|$ is odd if and only if $\delta(S)$ is a T-cut.*

Lemma 4.2.3 implies that the size smallest T-cut is an upper bound for the size of any collection of edge-disjoint T-joins.

Corollary 4.2.4. *For any graft (G,T), $\nu(G,T) \leq \tau(G,T)$.*

When does Corollary 4.2.4 hold with equality? The following conjecture appears in [45]. By the *parity* of a cut $\delta(S)$, we mean the parity of $|\delta(S)|$.

Conjecture 4.2.5. *Let (G,T) be a graft with G planar such that all T-cuts have the same parity. Then $\nu(G,T) = \tau(G,T)$.*

It is not difficult to see that Conjecture 4.2.5 implies Conjecture 4.1.1:

Proof of Conjecture 4.1.1 assuming Conjecture 4.2.5. Let G be a d-regular planar graph. Assume that G is oddly d-edge-connected, we shall show it is d-edge-colourable. (The reverse implication is easy and we omit it.) Let $T = V(G)$ and consider the graft (G,T). We claim that all the T-cuts have odd parity. To see this, let $S \subseteq V(G)$ have odd cardinality. Denoting by $E(S)$ the set of edges with both ends in S, we have $d|S| = 2|E(S)| + |\delta(S)|$. The claim follows, and we deduce that $\nu(G,T) = \tau(G,T)$. Since G is oddly d-edge-connected, we have $\tau(G,T) = d$ and so there exist d disjoint T-joins $T_1, \ldots, T_d$, say. Each T_i must contain an odd number of edges incident with any given vertex, and so the d-regularity of G implies that each T_i is a perfect matching. Taking $T_1, \ldots, T_d$ as colour classes, we obtain a d-edge-colouring of G. □

The converse implication was proved by Guenin [45].

Theorem 4.2.6. *Conjecture 4.1.1 is true if and only if Conjecture 4.2.5 is true.*

Guenin also made the following strengthening of Conjecture 4.2.5: Let H be a graph with $V(H) = \{v_1, \ldots, v_h\}$. Let us say that a graph H is a T-minor of G if there exist disjoint subsets $V_1, \ldots, V_h$ of $V(G)$, with $G[V_i]$ connected and $|V_i \cap T|$ odd for each i, such that for each edge $v_iv_j \in E(H)$, there exists at least one edge with an end in V_i and an end in V_j.

Conjecture 4.2.7. *Let (G,T) be a graft such that the Petersen graph is not a T-minor of G. If T-cuts have the same parity, then $\nu(G,T) = \tau(G,T)$.*

Seymour (see [45]) conjectured the following implication of Conjecture 4.2.7.

Conjecture 4.2.8. *If G is a d-regular graph with no Petersen minor, then G is d-edge-colourable if and only if G is oddly d-edge-connected.*

Observe that the case $d = 3$ of the above Conjecture corresponds to Theorem 3.1.1. To our knowledge, it is not known if Conjecture 4.2.8 implies Conjecture 4.2.7 analogously to Theorem 4.2.6.

4.3 An unavoidable list of reducible configurations

We now turn to the proof of Theorem 4.1.2. The graph we wish to edge-colour has parallel edges, but it is more convenient to work with the underlying simple graph. If H is d-regular and oddly d-edge-connected, then H has no loops, because for every vertex v, v has degree d, and yet $|\delta_H(v)| \geq d$. (We write $\delta(v)$ for $\delta(\{v\})$.) Thus to recover H from the underlying simple graph G say, we just need to know the number $m(e)$ of parallel edges of H that correspond to each edge e of G. Let us say a *d-target* is a pair (G, m) with the following properties (where for $F \subseteq E(G)$, $m(F)$ denotes $\sum_{e \in F} m(e)$):

- G is a simple graph drawn in the plane;
- $m(e) \geq 0$ is an integer for each edge e;
- $m(\delta(v)) = d$ for every vertex v; and
- $m(\delta(X)) \geq d$ for every odd subset $X \subseteq V(G)$.

In this language, Conjecture 4.1.1 says that for every d-target (G, m), there is a list of d perfect matchings of G such that every edge e of G is in exactly $m(e)$ of them. (The elements of a list need not be distinct.) If there is such a list we call it a *d-edge-colouring*, and say that (G, m) is *d-edge-colourable*. For an edge $e \in E(G)$, we call $m(e)$ the *multiplicity* of e. If $X \subseteq V(G)$, $G|X$ denotes the subgraph of G induced on X. We need:

Proposition 4.3.1. *Let (G, m) be a d-target, that is not d-edge-colourable, but such that every d-target with fewer vertices is d-edge-colourable. Then*

- *$|V(G)| \geq 6$;*

- *for every $X \subseteq V(G)$ with $|X|$ odd, if $|X|, |V(G) \setminus X| \neq 1$ then $m(\delta(X)) \geq d+2$; and*

- *G is three-connected, and $m(e) \leq d-2$ for every edge e.*

Proof. If $m(e) = 0$ for some edge e, we may delete e without affecting the problem; so we may assume that $m(e) > 0$ for every edge e. It is easy to check that G is connected and $|V(G)| \geq 6$ and we omit it. For the second assertion let $X \subseteq V(G)$ with $|X|$ odd and with $|X|, |V(G) \setminus X| \neq 1$. Thus $m(\delta(X)) \geq d$ since (G, m) is a d-target; suppose that $m(\delta(X)) = d$. There is a component of $G|X$ with an odd number of vertices, with vertex set X' say; and so $m(\delta(X')) \geq d$ since (G, m) is a d-target. But $\delta(X') \subseteq \delta(X)$, and $m(e) > 0$ for every edge e; and so $\delta(X') = \delta(X)$. Since G is connected it follows that $X' = X$, and so $G|X$ is connected. Similarly $G|Y$ is connected, where $Y = V(G) \setminus X$. Replace each edge e of G by $m(e)$ parallel edges, forming H; and contract all edges of $H|Y$, forming a d-regular oddly d-edge-connected planar graph H_1 with fewer vertices than H (because $|Y| > 1$). By hypothesis it follows that H_1 is d-edge-colourable. Similarly so is the graph obtained from H by contracting all edges of $H|X$. But these colourings can be combined to give a d-edge-colouring of H, a contradiction. This proves that $m(\delta(X)) > d$. Since $m(\delta(v)) = d$ for every vertex v, it follows that $m(\delta(X))$ has the same parity as $d|X|$, and so $m(\delta(X)) \geq d+2$. This proves the second assertion.

For the third assertion, suppose that G is not three-connected. Since $|V(G)| > 3$, there is a partition (X, Y, Z) of $V(G)$ where $X, Y \neq \emptyset$ and $|Z| = 2$, such that there are no edges between X and Y. Let $Z = \{z_1, z_2\}$ say. Either both $|X|, |Y|$ are odd, or they are both even. If they are both odd, then since $\delta(X), \delta(Y)$ are disjoint subsets of $\delta(z_1) \cup \delta(z_2)$, and

$$m(\delta(X)), m(\delta(Y)) \geq d = m(\delta(z_1)), m(\delta(z_2)),$$

we have equality throughout, and in particular $m(\delta(X)), m(\delta(Y)) = d$. But then $|X| = |Y| = 1$ from the second assertion, contradicting that $|V(G)| \geq 6$. Now assume $|X|, |Y|$ are both even. Since $\delta(X \cup \{z_1\}), \delta(Y \cup \{z_2\})$ have the same union and intersection as $\delta(z_1), \delta(z_2)$, it follows that $m(\delta(X \cup \{z_1\})) = d$, contrary to the second assertion. Thus G is three-connected. Since $m(e) \geq 1$ for every edge e, and $m(\delta(v)) = d$ for every vertex v, it follows that $m(e) \leq d-2$ for every edge e.

This proves the third assertion, and hence proves Proposition 4.3.1. $\square$

A *triangle* is a region of G incident with exactly three edges. If a triangle is incident with vertices u, v, w, for convenience we refer to it as uvw, and in the same way an edge with ends u, v is called uv. Two edges are *disjoint* if they are distinct and no vertex is an end of both of them, and otherwise they *meet*. Let r be a region of G, and let $e \in E(G)$ be incident with r; let r' be the other region incident with e. We say that e is *i-heavy* (for r), where $i \geq 2$, if either $m(e) \geq i$ or r' is a triangle uvw where $e = uv$ and

$$m(uv) + \min(m(uw), m(vw)) \geq i.$$

We say e is a *door* for r if $m(e) = 1$ and there is an edge f incident with r' and disjoint from e with $m(f) = 1$. We say that r is *big* if there are at least four doors for r, and *small* otherwise. A *square* is a region with length four.

Since G is drawn in the plane and is two-connected, every region r is bounded by some cycle which we denote by C_r. In what follows we will be studying cases in which certain configurations of regions are present in G. We will give a list of regions the closure of the union of which is a disc. For convenience, for an edge e in the boundary of this disc, we call the region outside the disc incident with e the "second region" for e; and we write $m^+(e) = m(e)$ if the second region is big, and $m^+(e) = m(e) + 1$ if the second region is small. This notation thus depends not just on (G, m) but on what regions we have specified, so it is imprecise, and when there is a danger of ambiguity we will specify it more clearly.

Let us say an 8-target (G, m) is *prime* if

- $m(e) > 0$ for every edge e;
- $|V(G)| \geq 6$;
- $m(\delta(X)) \geq 10$ for every $X \subseteq V(G)$ with $|X|$ odd and $|X|, |V(G) \setminus X| \neq 1$;
- G is three-connected, and $m(e) \leq 6$ for every edge e;

and in addition (G, m) contains none of of the following:

Conf(1): A triangle uvw where u, v both have degree three.

Conf(2): A triangle uvw, where u has degree three and its third neighbour x satisfies

$$m(ux) < m(uw) + m(vw).$$

Conf(3): Two triangles uvw, uwx with $m(uv) + m(uw) + m(vw) + m(ux) \geq 8$.

Conf(4): A square $uvwx$ where $m(uv) + m(vw) + m(ux) \geq 8$ and

$$(m(uv), m(vw), m(wx), m(ux)) \neq (4, 2, 1, 2).$$

Conf(5): Two triangles uvw, uwx where $m^+(uv) + m(uw) + m^+(wx) \geq 7$.

Conf(6): A square $uvwx$ where $m^+(uv) + m^+(wx) \geq 7$.

Conf(7): A triangle uvw with $m^+(uv) + m^+(uw) \geq 7$.

Conf(8): A triangle uvw, where $m(uv) = 3$, $m(uw) = 2$, $m(vw) = 2$, and the second region for one of uv, uw, vw has no door disjoint from uw.

Conf(9): A triangle uvw with $m(uv), m(uw), m(vw) = 2$, such that u has degree at least four, and the second regions for uv, uw both have at most one door, and no door that is disjoint from uvw.

Conf(10): A square $uvwx$ and a triangle wxy, where $m(uv) = m(wx) = m(xy) = 2$, and $m(vw) = 4$.

Conf(11): A square $uvwx$ and a triangle wxy, where $m(uv) \geq 3$, $m(wy) \geq 3$, $m(wx) = 1$, $m(ux) \leq 3$, and $m^+(xy) \geq 3$.

Conf(12): A square $uvwx$ and a triangle wxy, where $m^+(uv) \geq 2$, $m(vw) \geq 2$, $m(wx) = m(wy) = 2$, $m(ux) \leq 3$, and $m^+(xy) \geq 3$.

Conf(13): A region with length five, with edges $e_1, \ldots, e_5$ in order, where $m(e_1) \geq \max(m(e_2), m(e_5))$, $m(e_1) + m(e_2) + m(e_3) \geq 8$ and $m^+(e_1) + m^+(e_4) \geq 7$.

Conf(14): A region r and an edge e of C_r, such that $m^+(e) \geq 6$ and at most six edges of C_r disjoint from e are doors for r.

Conf(15): A region r with length at least four, and an edge e of C_r, such that $m^+(e) \geq 4$ and every edge of C_r disjoint from e is 3-heavy.

Conf(16): A region r and an edge uv of C_r, and a triangle uvw, such that $m(uv) + m^+(uw) \geq 4$, and every edge of C_r not incident with u is 3-heavy; moreover, if tu denotes the second edge of C_r incident with u, then either $\max(m(vw), m(tu)) \leq m(uw)$, or r is a triangle and $m(vw) = m(uw) + 1$ and $m(tu) \leq m(tv)$.

Conf(17): A region r with length at least five, and an edge e of C_r, such that $m^+(e) \geq 5$, every edge f of C_r disjoint from e satisfies $m^+(f) \geq 2$, and at most one of them is not 3-heavy.

Conf(18): A region r with length at least four and an edge uv of C_r, and a triangle uvw, such that $m^+(uw) + m(uv) \geq 5$, and $m(vw) \leq m(uw)$, and the second edge of C_r incident with u has multiplicity at most $m(uw)$, and either

- $m(uv) = 3$ and uv is 5-heavy, and every edge f of C_r disjoint from uv satisfies $m^+(f) \geq 2$, and at most one of them is not 3-heavy, or
- $m^+(f) \geq 2$ for every edge f of C_r not incident with u, and at most one such edge is not 3-heavy.

Conf(19): A region r with length at least five and an edge e of C_r, such that $m^+(e) \geq 5$, every edge of C_r disjoint from e is 2-heavy, and at most two of them are not 3-heavy.

We will prove these restrictions are too much, that in fact no 8-target is prime (Theorem 4.4.1). To deduce Theorem 4.1.2, we will show that if there is a counterexample, then some counterexample is prime; but for this purpose, just choosing a counterexample with the minimum number of vertices is not enough, and we need a more delicate minimization. If (G, m) is a d-target, its *score sequence* is the $(d+1)$-tuple $(n_0, n_1, \ldots, n_d)$ where n_i is the number of edges e of G with $m(e) = i$. If (G, m) and (G', m') are d-targets, with score sequences $(n_0, \ldots, n_d)$ and $(n'_0, \ldots, n'_d)$ respectively, we say that (G', m') is *smaller* than (G, m) if either

- $|V(G')| < |V(G)|$, or
- $|V(G')| = |V(G)|$ and there exists i with $1 \leq i \leq d$ such that $n'_i > n_i$, and $n'_j = n_j$ for all j with $i < j \leq d$, or

- $|V(G')| = |V(G)|$, and $n'_j = n_j$ for all j with $0 < j \leq d$, and $n'_0 < n_0$.

(The anomalous treatment of n_0 is just a device to allow d-targets to have edges with $m(e) = 0$, while minimum d-counterexamples have none.) If some d-target is not d-edge-colourable, then we can choose a d-target (G, m) with the following properties:

- (G, m) is not d-edge-colourable
- every smaller d-target is d-edge-colourable.

Let us call such a pair (G, m) a *minimum d-counterexample.* To prove Theorem 4.1.2, we prove two things:

- No 8-target is prime (Theorem 4.4.1), and
- Every minimum 8-counterexample is prime (Theorem 4.5.1).

It will follow that there is no minimum 8-counterexample, and so the theorem is true.

4.4 Discharging and unavoidability

In this section we prove the following, with a discharging argument.

Theorem 4.4.1. *No 8-target is prime.*

The proof is broken into several steps, through this section. Let (G, m) be a 8-target, where G is three-connected. For every region r, we define

$$\alpha(r) = 8 - 4|E(C_r)| + \sum_{e \in E(C_r)} m(e).$$

We observe first:

Lemma 4.4.2. *The sum of $\alpha(r)$ over all regions r is positive.*

Proof. Since (G, m) is a 8-target, $m(\delta(v)) = 8$ for each vertex v, and, summing over all v, we deduce that $2m(E(G)) = 8|V(G)|$. By Euler's formula, the number R of regions of G satisfies $|V(G)| - |E(G)| + R = 2$, and so $2m(E(G)) - 8|E(G)| + 8R = 16$. But $2m(E(G))$ is the sum over all regions r, of $\sum_{e \in E(C_r)} m(e)$, and $8R - 8|E(G)|$ is the sum over all regions r of $8 - 4|E(C_r)|$. It follows that the sum of $\alpha(r)$ over all regions r equals 16. This proves Lemma 4.4.2. $\square$

Think of $\alpha(r)$ as an initial assignment of charge to each region r. Now we move some small amount of charge between neighbouring regions. Normally we pass one unit of charge from every small region to every big region with which it shares an edge; except that in some exceptional circumstances, sending one unit is too much, and we only send 1/2 or 0. More precisely, for every edge e of G, define $\beta_e(s)$ for each region s as follows. Let r, r' be the two regions incident with e.

- If $s \neq r, r'$ then $\beta_e(s) = 0$.
- If r, r' are both big or both small then $\beta_e(r), \beta_e(r') = 0$.

Henceforth we assume that r is big and r' is small; let f, f' be the edges of $C_r \setminus e$ that share an end with e.

1: If e is a door for r (and hence $m(e) = 1$) then $\beta_e(r) = \beta_e(r') = 0$.

2: If $m(e) = 2$ and $m^+(f) = m^+(f') = 6$ then $\beta_e(r) = \beta_e(r') = 0$.

3: If $m(e) = 2$ and $m^+(f) = 6$ and $m^+(f') = 5$ or vice versa then $\beta_e(r) = -\beta_e(r') = 1/2$.

4: If $m(e) = 3$ and $m^+(f) = m^+(f') = 5$ then $\beta_e(r) = \beta_e(r') = 0$.

5: If $m(e) = 3$ and exactly one of $m^+(f), m^+(f') = 5$, then $\beta_e(r) = -\beta_e(r') = 1/2$.

6: Otherwise $\beta_e(r) = -\beta_e(r') = 1$.

(Think of β_e as passing some amount of charge between the two regions incident with e.) For each region r, define $\beta(r)$ to be the sum of $\beta_e(r)$ over all edges e. We see that the sum of $\beta(r)$ over all regions r is zero.

The effect of β is passing charge from small regions to big regions with which they share an edge. We need another "discharging" function, that passes charge from triangles to small regions with which they share an edge. If r is a triangle, incident with edges e, f, g, we define its *multiplicity* $m(r) = m(e) + m(f) + m(g)$. A region r is *tough* if r is a triangle, its multiplicity is at least five, and if $r = uvw$ where $m(uv) = 1$ and $m(uw) = m(vw) = 2$, then $m^+(uw) + m^+(vw) \geq 5$. For every edge e of G, define $\gamma_e(s)$ for each region s as follows. Let r, r' be the two regions incident with e.

- If $s \neq r, r'$ then $\gamma_e(s) = 0$.

- If one of r, r' is big, or neither is tough, or they both are tough, then $\gamma_e(r) = \gamma_e(r') = 0$.

Henceforth we assume that r' is tough, and r is small and not tough. Let e, e_1, e_2 be the edges incident with r', and let r_1, r_2 be the regions different from r' incident with e_1, e_2 respectively.

1: If $m(e) = 1$ and $m(e_1), m(e_2) \geq 2$, and $m^+(e_1) + m^+(e_2) \geq 6$ then $\gamma_e(r) = -\gamma_e(r') = 1$.

2: If $m(e) = 1$ and $m^+(e_1) \geq 4$ and $m(e_2) = 1$ and r_2 is small, then $\gamma_e(r) = -\gamma_e(r') = 1/2$.

3: If $m(e) = 1$ and $m(e_1) = 3$ and $m(e_2) = 1$ and r_2 is small, and the edge f of $C_r \setminus e$ that shares an end with e, e_1 satisfies $m(f) = 4$, then $\gamma_e(r) = -\gamma_e(r') = 1/2$.

4: If $m(e) = 2$ and $m(e_1), m(e_2) \geq 2$ and $m^+(e_1) + m^+(e_2) \geq 5$, and either

- r has more than one door, or
- some door for r is disjoint from e, or
- some edge f of C_r consecutive with e has multiplicity four, and r_1, r_2 are both small,

then $\gamma_e(r) = -\gamma_e(r') = 1$.

5: If $m(e) = 2$ and $m(e_1), m(e_2) = 2$ and some end of e has degree three, incident with e_1 say, and r_1 is small and r_2 is big, then $\gamma_e(r) = -\gamma_e(r') = 1/2$.

6: If $m(e) = 3$ and $m(e_1), m(e_2) = 2$ then $\gamma_e(r) = -\gamma_e(r') = 1$.

7: Otherwise $\gamma_e(r) = \gamma_e(r') = 0$.

We observe that, immediately from the rules, we have

Lemma 4.4.3. *Let e be incident with regions r, r'. Then $\beta_e(r)$ is non-zero only if exactly one of r, r' is big; and $\gamma_e(r)$ is non-zero only if exactly one of r, r' is tough and neither is big. Thus in all cases, at most one of $\beta_e(r), \gamma_e(r)$ is non-zero. Moreover $|\beta_e(r) + \gamma_e(r)| \leq 1$.*

For each region r, define $\gamma(r)$ to be the sum of $\gamma_e(r)$ over all edges e. Again, the sum of $\gamma(r)$ over all regions r is zero. It follows that the sum over all regions r of $\alpha(r) + \beta(r) + \gamma(r)$ is positive, by Lemma 4.4.2, and so there is a region r for which $\alpha(r) + \beta(r) + \gamma(r) > 0$. By examining the possibilities for such a region r we will deduce that (G, m) is not prime. There now begins a long case analysis, and to save writing we just say "by Conf(7)" instead of "since (G, m) does not contain Conf(7)", and so on.

Lemma 4.4.4. *If r is a big region and $\alpha(r) + \beta(r) + \gamma(r) > 0$, then (G, m) is not prime.*

Proof. Suppose that (G, m) is prime. Let $C = C_r$. Since r is big it follows that $\gamma(r) = 0$, and so $\alpha(r) + \beta(r) > 0$; that is,

$$\sum_{e \in E(C)} (4 - m(e) - \beta_e(r)) < 8.$$

For $e \in E(C)$, define $\phi(e) = m(e) + \beta_e(r)$, and let us say e is *major* if $\phi(e) > 4$. If e is major, then since $\beta_e(r) \leq 1$, it follows that $m(e) \geq 4$; and so $\beta_e(r)$ is an integer, from the β-rules, and therefore $\phi(e) \geq 5$. Moreover, no two major edges are consecutive, since G has minimum degree at least three.

Let D be the set of doors for C. Let

- $\xi = 1$ if there are consecutive edges e, f in C such that $\phi(e) > 5$ and f is a door for r
- $\xi = 2$ if there is no such pair e, f.

(1) *Let e, f, g be the edges of a path of C, in order, where e, g are major. Then*

$$(4 - \phi(e)) + 2(4 - \phi(f)) + (4 - \phi(g)) \geq 2\xi|\{f\} \cap D|.$$

Let r_1, r_2, r_3 be the regions different from r incident with e, f, g respectively. Now $m(e) \leq 6$ since (G, m) is prime, and if $m(e) = 6$ then r_1 is big, by Conf(14), and so $\beta_e(r) = 0$; and so in any case, $\phi(e) \leq 6$. Similarly $\phi(g) \leq 6$. Also, $\phi(e), \phi(g) \geq 5$ since e, g are major. Thus $\phi(e) + \phi(g) \in \{10, 11, 12\}$.

Suppose that $\phi(e) + \phi(g) = 12$. We must show that $\phi(f) \leq 2 - \xi|\{f\} \cap D|$. Now $m(e) \geq 5$, and so $m(f) \leq 2$, since G is three-connected. If $m(f) = 2$ then $f \notin D$, and $\beta_f(r) = 0$ from the β-rules; and so $\phi(f) \leq 2 - \xi|\{f\} \cap D|$. If $m(f) = 1$, then $\beta_f(r) \leq 1$, so we may assume that $f \in D$; but then $\xi = 1$ and $\phi(f) = 1 \leq 2 - \xi|\{f\} \cap D|$.

Next suppose that $\phi(e) + \phi(g) = 11$. We must show that $\phi(f) \leq 5/2 - \xi|\{f\} \cap D|$. Again one of $\phi(e), \phi(g) \geq 6$, say $\phi(e) = 6$; and so $m^+(e) \geq 6$. In particular $m(e) \geq 5$, and so $m(f) \leq 2$. Since $\phi(g) \geq 5$ we have $m^+(g) \geq 5$, and so if $m(f) = 2$, then $\beta_f(r) \leq 1/2$ from the β-rules; and since $f \notin D$ we have $\phi(f) \leq 5/2 - \xi|\{f\} \cap D|$. If $m(f) = 1$, then $\phi(f) \leq 2$, and so we may assume that $f \in D$; but then $\xi = 1$ and $\phi(f) = 1$, and again $\phi(f) \leq 5/2 - \xi|\{f\} \cap D|$.

Finally, suppose that $\phi(e) + \phi(g) = 10$. We must show that $\phi(f) \le 3 - \xi|\{f\} \cap D|$. Suppose that $m(f) \ge 3$. Since $m^+(e), m^+(g) \ge 5$ (because e, g are major), it follows that $m(f) = 3$, and $m(e) = m(g) = 4$ because G is three-connected; but then $\beta_f(r) = 0$ from the β-rules, and since $f \notin D$ we have $\phi(f) \le 3 - \xi|\{f\} \cap D|$. Next suppose that $m(f) = 2$. Then $\phi(f) \le 3 = 3 - \xi|\{f\} \cap D|$ as required. Lastly if $m(f) = 1$, then $\phi(f) \le 2$, so we may assume that $f \in D$; but then $\xi \le 2$ and $\phi(f) = 1 \le 3 - \xi|\{f\} \cap D|$. This proves (1).

(2) *Let e, f be consecutive edges of C, where e is major. Then*

$$(4 - \phi(e)) + 2(4 - \phi(f)) \ge 2\xi|\{f\} \cap D|.$$

We have $\phi(e) \in \{5, 6\}$. Suppose that $\phi(e) = 6$. We must show that $\phi(f) \le 3 - \xi|\{f\} \cap D|$; but $m(f) \le 2$ since $m(e) \ge 5$, and so $\phi(f) \le 3$. We may therefore assume that $f \in D$; but then $\xi = 1$ and $\phi(f) = 1 \le 3 - \xi|\{f\} \cap D|$. Next, suppose that $\phi(e) = 5$; then we must show that $\phi(f) \le 7/2 - \xi|\{f\} \cap D|$. Since $m(e) \ge 4$, it follows that $m(f) \le 3$. If $m(f) = 3$ then $m^+(e) = 5$ and so $\beta_f(r) \le 1/2$, from the β-rules; but then $\phi(f) \le 7/2 - \xi|\{f\} \cap D|$. If $m(f) \le 2$, then $\phi(f) \le 3$, so we may assume that $f \in D$; but $\xi \le 2$, and so $\phi(f) = 1 \le 7/2 - \xi|\{f\} \cap D|$. This proves (2).

For $i = 0, 1, 2$, let E_i be the set of edges $f \in E(C)$ such that f is not major, and f meets exactly i major edges in C. Let D be the set of doors for C. By (1), for each $f \in E_2$ we have

$$\frac{1}{2}(4 - \phi(e)) + (4 - \phi(f)) + \frac{1}{2}(4 - \phi(g)) \ge \xi|\{f\} \cap D|$$

where e, g are the major edges meeting f. By (2), for each $f \in E_1$ we have

$$\frac{1}{2}(4 - \phi(e)) + (4 - \phi(f)) \ge \xi|\{f\} \cap D|$$

where e is the major edge consecutive with f. Finally, for each $f \in E_0$ we have

$$4 - \phi(f) \ge \xi|\{f\} \cap D|$$

since $\phi(f) \le 4$, and $\phi(f) = 1$ if $f \in D$. Summing these inequalities over all $f \in E_0 \cup E_1 \cup E_2$, we deduce that $\sum_{e\in E(C)}(4-\phi(e)) \geq \xi|D|$. Consequently

$$8 > \sum_{e\in E(C)} (4-m(e)-\beta_e(r)) \geq \xi|D|.$$

But $|D| \geq 4$ since r is big, and so $\xi = 1$ and $|D| \leq 7$, a contradiction by Conf(14). This proves Lemma 4.4.4. □

Lemma 4.4.5. *If r is a triangle that is not tough, and $\alpha(r)+\beta(r)+\gamma(r) > 0$, then (G, m) is not prime.*

Proof. Suppose (G, m) is prime, and let $r = uvw$. Suppose first that r has multiplicity five; and hence, since it is not tough, we may assume that $m(uv) = 1$ and $m(uw) = m(vw) = 2$, and the second regions for uw, vw are both big. Thus from the β-rules, $\beta_{uw}(r), \beta_{vw}(r) = -1$, and since $\gamma_{uw}(r), \gamma_{vw}(r) = 0$ from the γ-rules and $\beta_{uv}(r)+\gamma_{uv}(r) \leq 1$ from Lemma 4.4.3, we deduce by adding that $\beta(r)+\gamma(r) \leq -1$. But

$$\alpha(r) = -4 + m(uv) + m(vw) + m(uw) = 1,$$

contradicting that $\alpha(r)+\beta(r)+\gamma(r) > 0$. Thus r has multiplicity at most four.

Since $\alpha(r) = -4 + m(uv) + m(vw) + m(uw) \leq 0$, and $\beta(r) \leq 0$, it follows that $\gamma(r) > 0$.

(1) *$m(e) = 1$ for every edge e incident with r such that $\gamma_e(r) > 0$.*

For suppose that $m(e) > 1$ and $\gamma_e(r) > 0$, where $e = uv$. Since r has multiplicity at most four it follows that $m(e) = 2$. Since $\gamma_e(r) > 0$, there is a vertex $x \neq w$ such that uvx is a triangle, and $m(ux), m(vx) \geq 2$, and one of $m^+(ux), m^+(vx)$ is at least three, say $m^+(ux) \geq 3$; and r has two doors. By Conf(5), $m^+(vw) = 1$, and so $\beta_{vw}(r) = -1$ and $\beta_{uw}(r) \leq 0$, and hence $\beta(r) \leq -1$; yet $\gamma(r) \leq 1$, contradicting that $\alpha(r)+\beta(r)+\gamma(r) > 0$. This proves (1).

(2) *There is no edge e incident with r and with a big region such that $m(e) = 1$.*

Let r be incident with edges e, f, g, and suppose that $m(e) = 1$ and e is incident with a big region. Thus $\beta(r) \leq -1$, and so $\gamma(r) > 1$; and consequently $\gamma_f(r), \gamma_g(r) > 0$, and therefore $m(f) = m(g) = 1$ from (1). But then $\alpha(r) = -1$, and yet $\gamma(r) \leq 2$, contradicting that $\alpha(r) + \beta(r) + \gamma(r) > 0$. This proves (2).

Choose e with $\gamma_e(r) > 0$, say $e = uv$. Thus $m(uv) = 1$, and there is a tough triangle $r' = uvx$ say. By Conf(3), r' has multiplicity at most six.

(3) *We may assume that* $m^+(ux) \leq 3$ *and* $m^+(vx) \leq 3$.

For suppose that $m^+(ux) \geq 4$. By (2), $m^+(vw) \geq 2$, contrary to Conf(5). This proves (3).

Now $\gamma_{uv}(r) > 0$, and from (1), (3), it follows that $\gamma_{uv}(r)$ is determined by the first γ-rule. In particular, $m^+(ux) = 3$, and $m^+(vx) = 3$. Suppose that vw is 3-heavy. By Conf(16) it follows that $m(vx) > m(ux)$, and so $m(vx) = 3$ and $m(ux) = 2$; but then by Conf(3), $m(uw) = m(vw) = 1$, contrary to Conf(16). Thus vw and similarly uw are not 3-heavy, and so by the same argument $\gamma_{uw}(r) = 0$ and $\gamma_{vw}(r) = 0$; and so $\gamma(r) = 1$. Consequently $\alpha(r) > -1$, and so we may assume that $m(uw) = 2$. Let r_1 be the second region for uw. Now $m(ux) + m(uv) + m(uw) \leq 6$, and so there is an edge f incident with r_1 and u different from uw, ux. Moreover, $m(f) \leq 3$, since $m(ux) + m(uv) + m(uw) \geq 5$; and so if r_1 is big then $\beta_{uw}(r) = -1$, a contradiction. Thus r_1 is small, contrary to Conf(5). This proves Lemma 4.4.5. □

Lemma 4.4.6. *If* r *is a tough triangle with* $\alpha(r) + \beta(r) + \gamma(r) > 0$, *then* (G, m) *is not prime.*

Proof. Suppose (G, m) is prime, and let $r = uvw$. Now $\alpha(r) = m(uv) + m(vw) + m(uw) - 4$, so

$$m(uv) + m(vw) + m(uw) + \beta(r) + \gamma(r) > 4.$$

Let r_1, r_2, r_3 be the regions different from r incident with uv, vw, uw respectively. It follows that $\beta_e(r), \gamma_e(r) \leq 0$ for every edge e of r.

(1) *If* r_1 *is big then* $\beta_{uv}(r) = -1$.

For let us examine the β-rules. Certainly uv is not a door for r_1, since r is a triangle; so the first rule does not apply. Let f, f' be the edges incident with r_1 different from uv that are incident with u, v respectively. If the second β-rule applies then $m(uv) = 2$ and $m(f), m(f') \geq 5$, which implies that $m(uw), m(vw) = 1$, contradicting that uvw has multiplicity at least five. If the third rule applies, then $m(uv) = 2$ and $m^+(f) = 6$ and $m^+(f') = 5$ say; but then $m(uw) = 1$ and $m(vw) = 2$, contrary to Conf(1). The fourth rule does not apply, by Conf(1). Thus we assume that the fifth rule applies. Let $m(uv) = 3$, $m^+(f) = 5$, and $m^+(f') < 5$. Hence $m(f) = 4$, and so u has degree three, and $m(vw) = 1$ by Conf(2), and r_3 is small, and $\beta_{uv}(r) = -1/2$. Since

$$m(uv) + m(vw) + m(uw) + \beta(r) + \gamma(r) > 4$$

it follows that

$$\beta_{uw}(r) + \beta_{vw}(r) + \gamma_{uw}(r) + \gamma_{vw}(r) \geq 0,$$

and since all the terms on the left are non-positive it follows that they are all zero. Now r_2 is not big since $\beta_{vw}(r) = 0$, and r_3 is not a triangle by Conf(2), so the third γ-rule applies to uw, a contradiction since $\gamma_{uw}(r) = 0$. This proves (1).

Let $X = \{u, v, w\}$. Since (G, m) is prime, it follows that $|V(G) \setminus X| \geq 3$, and $m(\delta(X)) \geq 10$. But

$$m(\delta(X)) = m(\delta(u)) + m(\delta(v)) + m(\delta(w)) - 2m(uv) - 2m(uw) - 2m(vw),$$

and so $10 \leq 8+8+8-2m(uv)-2m(uw)-2m(vw)$, that is, r has multiplicity at most seven. Suppose first that r has multiplicity seven. By Conf(3), none of r_1, r_2, r_3 is a triangle. Now $\beta(r)+\gamma(r) > -3$. Consequently we may assume that $\beta_{uv}(r)+\gamma_{uv}(r) > -1$, and hence r_1 is small by (1). By Conf(7), $m(uv)+m(uw) < 6$ and hence $m(vw) \geq 2$; and similarly $m(uw) \geq 2$. Now $\gamma_{uv}(r) > -1$, and so the first, fourth and sixth γ-rules do not apply to uv. Since the first γ-rule does not apply, $m(uv) > 1$. Since the sixth γ-rule does not apply, one of $m(uw), m(vw) > 2$, say $m(uw) \geq 3$, and so $m(uv) = 2$, $m(uw) = 3$ and $m(vw) = 2$. Since the fourth γ-rule does not apply, r_1 has no door disjoint from uv, contrary to Conf(8).

Next, suppose that r has multiplicity six. Thus $\beta(r) + \gamma(r) > -2$, and so by (1), at most one of r_1, r_2, r_3 is big. Suppose that $m(uv) = 4$; then $m(vw), m(uw) = 1$. Since at most one of r_1, r_2, r_3 is big, it follows from Conf(7) that r_1 is big, and hence r_2, r_3 are small. By Conf(3), r_2, r_3 are not tough. By the second γ-rule, $\gamma_{vw}(r) = \gamma_{uw}(r) = -1/2$, and since $\beta_{uv}(r) = -1$ by (1), this contradicts $\beta(r) + \gamma(r) > -2$. Thus $m(uv) \leq 3$. Suppose next that $m(uv) = 3$; then from the symmetry we may assume that $m(uw) = 2$ and $m(vw) = 1$. Since one of r_1, r_3 is small, and r_2 is not tough by Conf(3), the first γ-rule implies that $\beta_{vw}(r) + \gamma_{vw}(r) \leq -1$. Since $\beta(r) + \gamma(r) > -2$, it follows from (1) that neither of r_1, r_3 is big, contrary to Conf(7). Thus $m(uv) \leq 2$, and similarly $m(uw), m(vw) \leq 2$, and so $m(uv), m(uw), m(vw) = 2$. Since $\beta(r) + \gamma(r) > -2$, it follows that $\beta_e(r) + \gamma_e(r) \leq -1$ for at most one edge e incident with r; and so we may assume that $\beta_{uv}(r) + \gamma_{uv}(r) > -1$ and $\beta_{uw}(r) + \gamma_{uw}(r) > -1$. By (1), r_1, r_3 are both small. By Conf(3), r_1, r_3 are not tough, and since the fourth γ-rule does not apply, it follows that r_1 has at most one door, and no door disjoint from uv, and r_3 has at most one door, and no door disjoint from uw, and u has degree at least four, contrary to Conf(9).

Finally, suppose that r has multiplicity five. Now $\beta(r)+\gamma(r) > -1$, and hence $\beta_e(r)+\gamma_e(r) > -1$ for every edge e incident with r; and so by (1) r_1, r_2, r_3 are all small. Suppose that $m(uv) = 3$, and hence $m(uw), m(vw) = 1$. If neither of r_2, r_3 is tough, then by the second γ-rule, $\gamma_{uw}(r) = \gamma_{vw}(r) = -1/2$, a contradiction. Thus we may assume that r_3 is a tough triangle uwx. By Conf(5), $m(wx) = 1$, and so $m(ux) \geq 3$ since r_3 is tough, contrary to Conf(3). Thus we may assume that $m(uv) \leq 2$; and so from the symmetry we may assume that $m(uv) = m(uw) = 2$ and $m(vw) = 1$. The first γ-rule does not apply to vw, and so r_2 is a tough triangle vwx. By Conf(3), $m(vx), m(wx) \leq 2$, and so $m(vx), m(wx) = 2$. Since r_2 is tough, one of vx, wx is incident with a small region different from uvx, contrary to Conf(5). This proves Lemma 4.4.6. $\square$

Lemma 4.4.7. *If r is a small region with length at least four and with $\alpha(r) + \beta(r) + \gamma(r) > 0$, then (G, m) is not prime.*

Proof. Suppose that (G, m) is prime. Let $C = C_r$. Note that for each $e \in E(C)$, $-1 \leq \beta_e(r) \leq 0$ and $0 \leq \gamma_e(r) \leq 1$ Since $\alpha(r) = 8 - 4|E(C)| + \sum_{e \in E(C)} m(e)$, it follows that

$$8 - 4|E(C)| + \sum_{e \in E(C)} m(e) + \sum_{e \in E(C)} (\beta_e(r) + \gamma_e(r)) > 0,$$

that is,

$$\sum_{e\in E(C)} (m(e) + \beta_e(r) + \gamma_e(r) - 4) > -8.$$

For each $e \in E(C)$, let

$$\phi(e) = m(e) + \beta_e(r) + \gamma_e(r).$$

It follows that $|\phi(e) - m(e)| \leq 1$ for each e by Lemma 4.4.3. For each integer i, let E_i be the set of edges of C such that $\phi(e) \in \{i, i - \frac{1}{2}\}$.

(1) *For every $e \in E(C)$, $\phi(e)$ is one of $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4$, and hence $E(C)$ is the union of E_0, E_1, E_2, E_3, E_4.*

For let $e \in E(C)$. Since $m(e) \geq 1$ and $\beta_e(r) \geq -1$ it follows that $\phi(e) \geq 0$. Next we show that $\phi(e) \leq 4$. Now $m(e) < 6$ by Conf(14). Suppose that $m(e) = 5$. Then the second region incident with e is big, by Conf(14); and hence $\beta_e(r) = -1$ from the β-rules, and $\gamma_e(r) = 0$ and so $\phi(e) \leq 4$. Now suppose that $m(e) = 4$. Then by the γ-rules, $\gamma_e(r) = 0$, and so $\phi(e) \leq 4$. Finally, if $m(e) \leq 3$ then $\phi(e) \leq 4$ since $\gamma_e(r) \leq 1$. Thus $\phi(e) \leq 4$ in all cases. Finally, suppose that $\phi(e) = 7/2$, and hence $m(e) = 3$ or 4. If $m(e) = 3$ then $\gamma_e(r) = 1/2$, contrary to the γ-rules; while if $m(e) = 4$ then $\beta_e(r) = -1/2$, contrary to the β-rules. This proves (1).

(2) *Let $e \in E(C)$; then $e \in E_4$ if and only if either $m^+(e) \geq 5$, or $m(e) = 3$ and e is 5-heavy. Moreover, no two edges in E_4 are consecutive in C.*

The first assertion is immediate from the β- and γ-rules. For the second, suppose that $e, f \in E_4$ share an end v. Since v has degree at least three, it follows that $m(e) + m(f) \leq 7$ and so we may assume that $m(e) = 3$. Let e have ends u, v; then from the first assertion there is a triangle uvw where $m(uw), m(vw) = 2$. Hence $m(f) = 3$, and so there is similarly a triangle containing f, with third vertex x. Consequently $w = x$; but this contradicts Conf(3) and hence proves (2).

(3) *If $e \in E_4$, and $f \in E(C)$ is disjoint from e, and every edge in $E(C) \setminus \{f\}$ disjoint from e is 3-heavy, and there is no edge of C with multiplicity one disjoint from f, then $f \in E_0$.*

For by Conf(6) if $|E(C)| = 4$ and $m^+(e) \geq 5$, or by Conf(17) or Conf(18) otherwise, it follows that $m^+(f) = 1$. Since there is no edge of C with multiplicity one disjoint from f, it follows that $\beta_f(r) = -1$ from the β-rules, and so $f \in E_0$. This proves (3).

For $0 \leq i \leq 4$, let $n_i = |E_i|$.

(4) *If $e \in E(C)$ satisfies $m(e) = 2$, and $n_4 = 0$, and r has at most one door, and no door disjoint from e, then $\phi(e) \leq 2$.*

For if not, then $\gamma_e(r) > 0$, and so from the γ-rules, there is a triangle uvw with $e = uv$, and some edge f of C consecutive with e satisfies $m^+(f) = 5$; but then $f \in E_4$, contradicting that $n_4 = 0$. This proves (4).

(5) *If u, v, w are consecutive vertices in C, and $uv \in E_4$ and $m(uv) = 3$, then $\phi(vw) \leq 2$.*

For since $uv \in E_4$, by (2) there is a triangle uvx with $m(ux) = m(vx) = 2$. From Conf(2) it follows that $m(vw) \leq 2$; and since w is not adjacent to x by Conf(3), and hence vw is not 4-heavy, the γ-rules imply that $\phi(vw) \leq 2$. This proves (5).

Let C have vertices $v_1, \ldots, v_k$ in order, and let v_{k+1} mean v_1. For $1 \leq i \leq k$ let e_i be the edge $v_i v_{i+1}$, and let r_i be the region incident with e_i different from r.

Since

$$\sum_{e \in E(C)} (\phi(e) - 4) > -8,$$

we have $4n_0 + 3n_1 + 2n_2 + n_3 \leq 7$, that is,

$$3n_0 + 2n_1 + n_2 + k - n_4 \leq 7,$$

since $n_0 + n_1 + n_2 + n_3 + n_4 = k$. But by (2), $n_4 \leq k/2$ and so

$$3n_0 + 2n_1 + n_2 + k/2 \leq 7.$$

Since $k \geq 4$ it follows that $3n_0 + 2n_1 + n_2 \leq 5$, and hence $n_0 + n_1 \leq 2$.

Case 1: $n_0 + n_1 = 2$.

Since $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$, we have $n_4 \geq n_0 + n_2 + k - 3$. Thus $n_4 > 0$. If $k = 4$, let $e \in E_4$; then by (3) the edge f of C disjoint from e belongs to E_0, and so by (2), $n_4 = 1$; but this contradicts $n_0 + n_2 + k - 3 \leq n_4$.

Thus $k \geq 5$. Since

$$3n_0 + 2n_1 + n_2 + k/2 \leq 7,$$

and $2n_0 + 2n_1 = 4$ and $k/2 \geq 5/2$, it follows that $n_0 = n_2 = 0$ and $n_1 = 2$ and $k \leq 6$.

Suppose that $k = 6$; then $n_4 = 3$ since $n_4 \geq n_0 + n_2 + k - 3$, so we may assume that $e_1, e_3, e_5 \in E_4$. By Conf(17) and Conf(18), it follows that $m^+(e_4) = 1$, and hence $e_4 \in E_0 \cup E_1$, and similarly $e_6, e_2 \in E_0 \cup E_1$, a contradiction since $n_0 + n_1 = 2$. Thus $k = 5$, and so $n_4 \geq 2$, and by (2) $n_4 = 2$ and we may assume that $e_1, e_3 \in E_4$. By Conf(17) and Conf(18), $m^+(e_4) = 1$, and similarly $m^+(e_5) = 1$. Since $n_1 = 2$, and $n_0, n_2 = 0$, it follows that $m(e_2) > 1$. But then $e_4 \in E_0$ by (3), contradicting that $n_0 = 0$.

Case 2: $k = 4$ and $n_0 + n_1 = 1$ and $n_4 > 0$.

Let $e_4 \in E_4$; by (3), $e_2 \in E_0$ and so $m(e_2) = 1$. By (2) and Conf(2) and Conf(4), it follows that $m(e_1), m(e_3) \leq 2$. Now e_2 is the only edge of C that is not 2-heavy, since $n_0 + n_1 = 1$, and in particular r has at most one door. Since $4n_0 + 3n_1 + 2n_2 + n_3 \leq 7$ and $n_0 = 1$, it follows that $n_2 \leq 1$, so we may assume that $e_1 \notin E_2$. Thus $\phi(e_1) > 2$, and hence $m(e_1) = 2$. By (2) and (5), $m^+(e_4) \geq 5$, so by Conf(4), $m(e_4) = 4$. Since $\phi(e_1) > 2$, it follows from the γ-rules that r_1 is a triangle v_1v_2w say, where $m(v_1w), m(v_2w) \geq 2$. Consequently $m(v_1w) = 2$. Since $e_3 \notin E_1$, it follows that $m^+(e_3) \geq 2$; so $m(v_2w) = m^+(v_2w) = 2$ by Conf(18) (taking v_2, v_1, w to be the vertices called u, v, w in Conf(18) respectively). From Conf(10) it follows that $m(e_3) = 1$. From the γ-rules it follows that $\phi(e_1) = 5/2$. Since $\sum_{e \in E(C)} \phi(e) > 8$ and $\phi(e_2) + \phi(e_4) \leq 4$, it follows that $\phi(e_3) \geq 2$. Since $m(e_3) = 1$, the γ-rules imply that e_3 is 3-heavy, contrary to Conf(16) (taking v_2, v_1, w to be the vertices called u, v, w in Conf(16) respectively).

Case 3: $k = 4$ and $n_0 + n_1 = 1$ and $n_4 = 0$.

Let $e_4 \in E_0 \cup E_1$, and so $m(e_4) \leq 2$. Since every edge of C that is not 2-heavy belongs to $E_0 \cup E_1$, it follows that e_1, e_2, e_3 are 2-heavy. Since $n_4 = 0$, it follows that $m^+(e_i) \leq 4$ for $i = 1, 2, 3, 4$.

Suppose that $\phi(e_1) \geq 3$, and hence $\phi(e_1) = 3$ by (1) since $n_4 = 0$. By (4) it follows that $m(e_1) \geq 3$. If $m^+(e_1) = 3$, then from the β-rules, the edge xv_2 of r_1 incident with v_2 and different from e_1 has multiplicity four and hence $m(e_2) = 1$; and since x, v_3 are non-adjacent by Conf(2), this contradicts that e_2 is 2-heavy. Thus $m^+(e_1) \geq 4$. By Conf(6), $m^+(e_3) \leq 2$, and so $\phi(e_3) \leq 2$ by (4). Since $\phi(e_2) \leq 3$, and $\phi(e_4) \leq 1$, and $\sum_{e \in E(C)} \phi(e) > 8$, it follows that $\phi(e_2) \geq 5/2$ (and so e_2 is 3-heavy), and $\phi(e_3) \geq 3/2$, and $\phi(e_4) \geq 1/2$ (and so $m^+(e_4) \geq 2$). By Conf(2), it is not the case that $m(e_3) = 2$ and the edge of r_3 consecutive with e_3 and incident with v_3 has multiplicity four; and so, since $\phi(e_3) \geq 3/2$, the β-rules imply that $m(e_3) = 1$ and r_3 is a triangle v_3v_4y say. Now by Conf(15), not both $m(v_3y), m(v_4y) \geq 2$; and $m(e_2) \leq 3$ by Conf(4), so by Conf(18), $m^+(v_3y), m^+(v_4y) \leq 3$. But then the γ-rules imply that $\phi(e_3) \leq 1$, a contradiction. This proves that $\phi(e_1) \leq 5/2$; and similarly $\phi(e_3) \leq 5/2$.

Since $\sum_{e \in E(C)} \phi(e) > 8$, and $\phi(e_2) \leq 3$ (because $n_4 = 0$) it follows that $\phi(e_1) + \phi(e_3) \geq 9/2$, and $\phi(e_4) \geq 1/2$; and from the symmetry we may assume that $\phi(e_1) = 5/2$ and $\phi(e_3) \geq 2$. The β- and γ-rules imply that $m(e_1) = 3$ (since $m^+(e_2) \leq 4$). Since $\phi(e_2) + \phi(e_3) \geq 5$, and $\phi(e_3) \leq 5/2$, it follows that $\phi(e_2) \geq 5/2$ (and hence $m(e_2) \geq 2$).

Suppose that $m(e_3) = 1$. Since $\phi(e_3) \geq 2$, the first γ-rule applies, and so r_3 is a triangle v_3v_4y, and $m(v_3y), m(v_4y) \geq 2$, and $m^+(v_3y) + m^+(v_4y) \geq 6$. By Conf(4), $m(e_2) \leq 3$, so by Conf(18), $m^+(v_3y), m^+(v_4y) \leq 3$, and hence equality holds for both. By Conf(11), $m(v_3y), m(v_4y) = 2$; but this is contrary to Conf(16).

So $m(e_3) \geq 2$, and by Conf(4), $m(e_2) = m(e_3) = 2$. If $m^+(e_3) = 2$, then from the β-rules it follows that both edges of r_3 consecutive with e_3 have multiplicity five; but this is impossible since $m(e_2) = 2$. So $m^+(e_3) = 3$. Since $\phi(e_2) \geq 5/2$ it follows that r_2 is a triangle v_2v_3x, $m(v_2x), m(v_3x) \geq 2$, and one of $m^+(v_2x), m^+(v_3x) \geq 3$, and e_4 is a door for r. Since $\phi(e_4) > 0$, we deduce that $m^+(e_4) \geq 2$. By Conf(2), $m(v_2x) = 2$. By Conf(12), $m^+(v_3x) = 2$ and $m^+(v_2x) = 2$, a contradiction.

Case 4: $k = 4$ and $n_0 + n_1 = 0$.

Since $n_0, n_1 = 0$, it follows that $\phi(e_i) \geq 3/2$ and hence e_i is 2-heavy, for $1 \leq i \leq 4$. Consequently

$n_4 = 0$, from (3). Since $\sum_{e \in E(C)} \phi(e) > 8$, we may assume because of the symmetries of the square that $\phi(e_1) + \phi(e_3) \geq 9/2$, and $\phi(e_1) \geq \phi(e_3)$, and therefore $\phi(e_1) \geq 5/2$. Thus $m(e_1) \geq 3$ from (4). If some edge f of the boundary of r_1 consecutive with e_1 satisfies $m(f) = 4$, say $f = v_1 x$, then $m(e_4) = 1$ and v_1 has degree three; but since e_4 is 2-heavy, it follows that x, v_4 are adjacent, contrary to Conf(2). Thus there is no such f, and so by the β-rules, $m^+(e_1) \geq 4$.

Suppose that $m(e_3) \geq 2$. By Conf(6) it follows that $m^+(e_3) = 2$, and in particular r_3 is big. Since $\phi(e_3) \geq 3/2$, the β-rules imply that some edge f of the boundary of r_3 consecutive with e_3 satisfies $m(f) = 5$, say $f = v_4 x$; and since x, v_1 are nonadjacent by Conf(2) it follows that $e_4 \in E_0 \cup E_1$, a contradiction. Thus $m(e_3) = 1$. Since e_3 is 2-heavy it follows that r_3 is a triangle $v_3 v_4 x$ say.

By Conf(4), $m(e_2), m(e_4) \leq 3$. By Conf(15), we may assume that $m(v_3 x) = 1$; and by Conf(18), $m^+(v_4 x) \leq 3$. Since $m(e_4) \leq 3$, the γ-rules imply that $\phi(e_3) \leq 1$, a contradiction.

Case 5: $k \geq 5$ and $n_0 + n_1 = 1$.

Since $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$, we have $n_4 \geq n_0 + n_2 + k - 5$. Let $E_0 \cup E_1 = \{e_k\}$.

Suppose that $n_4 = 0$. Then since $n_4 \geq n_0 + n_2 + k - 5$ it follows that $k = 5$. Since

$$\sum_{e \in E(C)} \phi(e) > 4k - 8 = 12,$$

and $\phi(e_5) \leq 1$, and $\phi(e_i) \leq 3$ for $i = 1, 2, 3, 4$ (by (1), since $n_4 = 0$) it follows that $\phi(e_i) \geq 5/2$ for $i = 1, 2, 3, 4$, and hence $e_1, \ldots, e_4$ are 3-heavy. If $m(e_1) \leq 2$, then since $\phi(e_1) \geq 5/2$ it follows from the γ-rules that $m(e_2) = 4$ and r_2 is small; but then $e_2 \in E_4$, a contradiction. Thus $m(e_1) \geq 3$; so $m(e_1) = m^+(e_1) = 3$ by Conf(15). Since $m(e_2) \geq 2$, it follows that not both edges of r_1 consecutive with e_1 have multiplicity four, and so from the β-rules, $\phi(e_1) \leq 5/2$. Similarly $\phi(e_4) \leq 5/2$, contradicting that $\sum_{e \in E(C)} \phi(e) > 12$. This proves that $n_4 > 0$.

Suppose that $n_2 = 0$. Thus $e_1, \ldots, e_4$ are 3-heavy. Since $n_4 > 0$, (3) implies that $n_0 = 1$. Since $\phi(e_1) > 2$, the β- and γ-rules imply that either:

- $m(e_1) = 2$ and r_1 is a triangle $v_1 v_2 w$ say; and $m(v_1 w), m(v_2 w) \geq 2$, and $m(e_2) = 4$. Consequently $m(v_2 w) = 2$, contrary to Conf(16).

- $m(e_1) = 3$ and r_1 is big, and, if u_1-v_1-v_2-u_2 is the three-edge path of C_{r_1} with middle edge e_1, then one of $m(u_1v_1), m(u_2v_2) = 4$ and is incident with a small region. But if $m(u_1v_1) = 4$ then the second region incident with it is r_k, and this is not small since $n_0 = 1$; and if $m(u_2v_2) = 4$ then v_2 has degree three and $m(e_2) = 1$, and since e_2 is 3-heavy it follows that u_2, v_3 are adjacent, and $m(u_2v_3) \geq 2$, contrary to Conf(2).

- $m^+(e_1) \geq 4$; but this is contrary to Conf(15).

This proves that $n_2 \geq 1$.

Since $3n_0 + 2n_1 + n_2 + k/2 \leq 7$, we have $n_0 + n_2 + k/2 \leq 5$, and in particular $n_2 \leq 2$. If $e \in E(C)$ is not 3-heavy, then $\phi(e) \leq 2$ from the γ-rules, and so at most two edges of $E(C)$ not in $E_0 \cup E_1$ are not 3-heavy. By Conf(8) and Conf(19) it follows that $e_1, e_{k-1} \notin E_4$, so every edge in E_4 is disjoint from e_k. Since there are three consecutive edges of C not in E_4, and no two edges in E_4 are consecutive by (2), it follows that $n_4 \leq k/2 - 1$; and since $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$, it follows that $n_0 + n_2 + k/2 \leq 4$, and so $n_2 = 1$, and $n_0 = 0$, and $k \leq 6$. In particular, from (5) every edge $e \in E_4$ has $m(e) \geq 4$.

Suppose that $k = 6$. Since $n_4 \geq n_0 + n_2 + k - 5$ and $n_4 \leq k/2 - 1$, it follows that $n_4 = 2$; and so $E_4 = \{e_2, e_4\}$, since the members of E_4 are disjoint from e_6 and from each other. Since $e_2 \in E_4$, (3) implies that e_5 is not 3-heavy, and so $e_5 \in E_2$; and similarly $e_1 \in E_2$, a contradiction since $n_2 = 1$.

Thus $k = 5$. Since $n_4 \leq k/2 - 1$ it follows that $n_4 = 1$, so we may assume that $E_4 = \{e_2\}$. By (3), e_4 is not 3-heavy, and so $\phi(e_4) \leq 2$. Consequently $E_2 = \{e_4\}$, and $\phi(e_1) + \phi(e_3) \geq 11/2$. Since $\phi(e_4), \phi(e_5) > 0$, it follows that $m^+(e_4), m^+(e_5) \geq 2$, and since $m^+(e_2) \geq 5$, two applications of Conf(13) imply that $m(e_3) + m(e_4) \leq 3$ and $m(e_1) + m(e_5) \leq 3$. Since $m(e_1), m(e_3) \geq 2$ (because $\phi(e_1), \phi(e_3) > 2$) it follows that $m(e_1), m(e_3) = 2$ and e_1, e_3 are 4-heavy; and $m(e_4), m(e_5) = 1$. Since $\phi(e_4) > 1$, r_4 is a triangle v_4v_5x say. Since e_4 is not 3-heavy, one of $m(v_4x), m(v_5x) = 1$. If $m(v_4x) = 1$ then by Conf(16), $m(xv_5) \leq 2$; but then $\phi(e_4) = 1$ from the γ-rules, a contradiction. So $m(v_5x) = 1$. Since $\phi(e_4) > 1$, the γ-rules imply that $m^+(v_4x) \geq 4$. But this contradicts Conf(18).

Case 6: $k \geq 5$ and $n_0 + n_1 = 0$.

Since $n_0, n_1 = 0$, it follows that $\phi(e_i) \geq 3/2$ and hence e_i is 2-heavy, for $1 \leq i \leq k$. Since $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$, we have $n_4 \geq n_2 + k - 7$.

Suppose first that $n_4 > 0$. By (2) and Conf(8) and Conf(19), every edge in E_4 is disjoint from at least three edges that are not 3-heavy and that therefore belong to E_2. In particular $n_2 \geq 3$. Let $e \in E_4$; then e is disjoint from all the other edges in E_4, and from at least three edges in E_2, so $k - 3 \geq n_4 - 1 + 3$, that is, $k \geq n_4 + 5$. But $n_4 \geq n_2 + k - 7 \geq k - 4$, a contradiction.

This proves that $n_4 = 0$, and so $E(C) = E_2 \cup E_3$. Since $n_4 \geq n_2 + k - 7$, it follows that $n_2 + k \leq 7$. In particular, $k \in \{5, 6, 7\}$. From (4), every edge $e \in E(C)$ with $m(e) = 2$ belongs to E_2, since $n_4 = 0$ and there are no doors for r. Consequently every $e \in E_3$ satisfies $m(e) \geq 3$. Suppose that $m^+(e) = 3$ for some $e \in E_3$, say $e = e_1$. Thus r_1 is big, and $\beta_e(r) > -1$ since $\phi(e) > 2$. Hence from the β-rules, some edge of C_{r_1} consecutive with e_1 has multiplicity four, say $v_1 x$. Hence $m(e_k) = 1$, and since $n_0, n_1 = 0$, it follows that r_k is a triangle, and therefore x, v_k are adjacent, contrary to Conf(2). This proves that $m^+(e) \geq 4$ for every $e \in E_3$.

By Conf(15), every edge in E_3 is disjoint from some edge in E_2, and in particular $n_2 \geq 2$. Since $n_2 + k \leq 7$, we have $k = 5$ and $n_2 = 2$. Every edge in E_3 is disjoint from one of the edges in E_2, so we may assume that $e_1, e_2 \in E_2$, and $e_3, e_4, e_5 \in E_3$. Since $m^+(e_3), m^+(e_4), m^+(e_5) \geq 4$, Conf(13) implies that $m^+(e_1) \leq 2$; and by Conf(15), e_1 is not 3-heavy. From the γ-rules, $\phi(e_1) \leq 3/2$, and similarly $\phi(e_2) \leq 3/2$. But for $i = 3, 4, 5$, $\phi(e_i) \leq 3$ since $n_4 = 0$; and so $\sum_{e \in E(C)} \phi(e) \leq 12$, contradicting our initial assumption that

$$\sum_{e \in E(C)} (\phi(e) - 4) > -8.$$

This completes the proof of Lemma 4.4.7. □

Proof of Theorem 4.4.1. Suppose that (G, m) is a prime 8-target, and let α, β, γ be as before. Since the sum over all regions r of $\alpha(r) + \beta(r) + \gamma(r)$ is positive, there is a region r with $\alpha(r) + \beta(r) + \gamma(r) > 0$. But this is contrary to one of Lemmas 4.4.4, 4.4.5, 4.4.6, or 4.4.7. This proves Theorem 4.4.1. □

4.5 Reducibility

Now we begin the second half of the paper, devoted to proving the following.

Theorem 4.5.1. *Every minimum 8-counterexample is prime.*

Again, the proof is broken into several steps. Clearly no minimum 8-counterexample (G, m) has an edge e with $m(e) = 0$, because deleting e would give a smaller 8-counterexample; and by Proposition 4.3.1, every minimum 8-counterexample satisfies the conclusions of Proposition 4.3.1. Thus, it remains to check that (G, m) contains none of Conf(1)–Conf(19). Sometimes it is just as easy to prove a result for general d instead of $d = 8$, and so we do so.

Lemma 4.5.2. *If (G, m) is a minimum d-counterexample, then every triangle has multiplicity less than d.*

Proof. Let uvw be a triangle of G, and let $X = \{u, v, w\}$. Since $|V(G)| \geq 6$, Proposition 4.3.1 implies that $m(\delta(X)) \geq d + 2$. But

$$m(\delta(X)) = m(\delta(u)) + m(\delta(v)) + m(\delta(w)) - 2m(uv) - 2m(uw) - 2m(vw),$$

and so $d + 2 \leq d + d + d - 2m(uv) - 2m(uw) - 2m(vw)$, that is, $m(uv) + m(uw) + m(vw) \leq d - 1$. This proves 4.5.2. □

If C is a cycle of length four in G, say with vertices u, v, w, x in order, let m' be defined as follows: $m'(uv) = m(uv) - 1$, $m'(vw) = m(vw) + 1$, $m'(wx) = m(wx) - 1$, $m'(ux) = m(ux) + 1$, and $m'(e) = m(e)$ for all other edges e. If (G, m) is a minimum d-counterexample, then because of the second statement of Proposition 4.3.1, it follows that (G, m') is a d-target. (Note that possibly $m'(uv), m'(wx)$ are zero; this is the reason to permit $m(e) = 0$ in a d-target.) We say that (G, m') is obtained from (G, m) by *switching on the sequence u-v-w-x-u*. If (G, m') is smaller than (G, m), we say that the sequence u-v-w-x-u is *switchable.*

Lemma 4.5.3. *No minimum d-counterexample contains Conf(1).*

Proof. Suppose that (G, m) is a minimum d-counterexample, with a triangle uvw, where u, v have degree three. Let the neighbours of u, v not in $\{u, v, w\}$ be x, y respectively. Let H be a simple graph obtained from G by adding new edges if necessary to make w, x, y pairwise adjacent, and extend m to $E(H)$ by setting $m(e) = 0$ for every new edge. Thus (H, m) is not d-edge-colourable, and although it may not be a minimum d-counterexample, no d-counterexample has fewer vertices.

Define $f(w) = m(uw) + m(vw)$, $f(x) = m(ux)$, and $f(y) = m(vy)$. Since $m(\delta(\{u, v\}))$ is even, it follows that $f(w) + f(x) + f(y)$ is even. Define

$$\begin{aligned} n(wx) &= \frac{1}{2}(f(x) + f(w) - f(y)) \\ n(wy) &= \frac{1}{2}(f(y) + f(w) - f(x)) \\ n(xy) &= \frac{1}{2}(f(x) + f(y) - f(w)). \end{aligned}$$

It follows that $n(wx), n(wy), n(xy)$ are integers. Since $m(\delta(\{u, v, w\})) \geq d$ and $m(\delta(w)) = d$, it follows that $m(ux)+m(vy) \geq m(uw)+m(vw)$ and hence $n(xy) \geq 0$. Similarly, since $m(\delta(\{u, v, x\})) \geq d$ and $m(\delta(x)) = d$, it follows that $n(wy) \geq 0$, and similarly $n(wx) \geq 0$.

Let $G' = H \setminus \{u, v\}$. For each edge e of G', define $m'(e)$ as follows. If e is incident with a vertex different from x, y, w let $m'(e) = m(e)$. For $e = xy, wx, wy$ let $m'(e) = m(e) + n(e)$. We claim that (G', m') is a d-target. To show this, let $X \subseteq V(G')$ with $|X|$ odd; we must show that $m'(\delta_{G'}(X)) \geq d$. By replacing X by its complement if necessary (which also is odd, since $|V(G)|$ is even), we may assume that X contains at most one of w, x, y. But then from the choice of $f(w), f(x), f(y)$, it follows that $m'(\delta_{G'}(X)) = m(\delta_G(X)) \geq d$ as required. Thus (G', m') is a d-target. Since $|V(G')| < |V(G)|$, there are d perfect matchings $F'_1, \ldots, F'_d$ of G' such that every edge $e \in E(G')$ is in exactly $m'(e)$ of them. Now each of $F'_1, \ldots, F'_d$ contains at most one of the edges wx, wy, xy. Let I_1, I_2, I_3, I_0 be the sets of $i \in \{1, \ldots, d\}$ such that F'_i contains wx, wy, xy or none of the three, respectively. Thus $|I_1| = m'(wx) = m(wx) + n(wx)$. For $n(wx)$ values of $i \in I_1$ let $F_i = (F'_i \setminus \{wx\}) \cup \{ux, vw\}$, and for the remaining $m(wx)$ values let $F_i = F'_i \cup \{uv\}$. Thus F_i is a perfect matching of G for each $i \in I_1$. Define F_i $(i \in I_2)$ similarly. For $n(xy)$ values of $i \in I_3$ let $F_i = (F'_i \setminus \{xy\}) \cup \{ux, vy\}$, and for the others let $F_i = F_i \cup \{uv\}$. For $i \in I_0$ let $F_i = F'_i \cup \{uv\}$. Then $F_1, \ldots, F_d$ are perfect matchings of G, and we claim that every edge e is in exactly $m(e)$ of them. This is clear if e has an end different from u, v, w, x, y; and true from the construction if both ends of e are in $\{w, x, y\}$. From the symmetry we may therefore assume that e is incident with u. If $e = ux$, then e belongs to $n(wx) + n(xy)$ of $F_1, \ldots, F_d$; but

$$n(wx) + n(xy) = \frac{1}{2}(f(x) + f(w) - f(y)) + \frac{1}{2}(f(x) + f(y) - f(w)) = f(x) = m(ux)$$

as required. The other two cases are similar. This is a contradiction, since (G, m) is a minimum

d-counterexample, and so there is no such triangle uvw. This proves Lemma 4.5.3. $\square$

Incidentally, a similar proof would show that G is four-connected except for cutsets of size three that cut off just one vertex, but we do not need this.

If (G, m) is a d-target, and x, y are distinct vertices both incident with some common region r, we define $(G, m) + xy$ to be the d-target (G', m') obtained as follows:

- If x, y are adjacent in G, let $(G', m') = (G, m)$.
- If x, y are non-adjacent in G, let G' be obtained from G by adding a new edge xy, extending the drawing of G to one of G' and setting $m'(e) = m(e)$ for every $e \in E(G)$ and $m'(xy) = 0$.

Lemma 4.5.4. *No minimum d-counterexample contains Conf(2).*

Proof. Let (G, m) be a minimum d-counterexample, with a triangle uvw, and suppose that u has only one other neighbour x, and $m(ux) < m(uw) + m(vw)$. Let $(G', m'') = ((G, m) + vx) + wx$. For each $e \in E(G')$, define $m'(e)$ as follows. If $e \neq ux, uw, vw, vx$ let $m'(e) = m(e)$. Let

$$\begin{aligned} m'(vx) &= m''(vx) + m(vw) \\ m'(vw) &= 0 \\ m'(ux) &= m(ux) - m(vw) \\ m'(uw) &= m(uw) + m(vw). \end{aligned}$$

Since $m(uv) + m(uw) + m(ux) = d$ and $m(uv) + m(uw) + m(vw) \leq d$ since $m(\delta(\{u, v, w\})) \geq d$, it follows that $m(ux) \geq m(vw)$, and so $m'(e) \geq 0$ for every edge e. Moreover, $m'(\delta(z)) = d$ for every vertex z, from the construction. We claim that (G', m') is a d-target. For let $X \subseteq V(G')$ with $|X|$ odd; and we may assume that $u \notin X$. We must show that $m'(\delta(X)) \geq d$. If X contains at most one of v, w, x then $m'(\delta(X)) = m(\delta(X)) \geq d$ as required, so we may assume that X contains at least two of v, w, x. If $v, w, x \in X$ then $m'(\delta(X)) \geq m'(\delta(u)) = d$ as required. If $X \cap \{v, w, x\} = \{v, w\}$ then $m'(\delta(X)) = m(\delta(X)) + 2m(vw) \geq d$, and if $X \cap \{v, w, x\} = \{w, x\}$ then $m'(\delta(X)) = m(\delta(X)) \geq d$, so we may assume that $X \cap \{v, w, x\} = \{v, x\}$, and hence $m'(\delta(X)) = m(\delta(X)) - 2m(vw)$. We must therefore show that in this case, $m(\delta(X)) \geq 2m(vw) + d$. To see this, note that

$$\begin{aligned} m(\delta(X \cup \{u, w\})) &\leq m(\delta(X)) - m(ux) - m(uv) - m(vw) - m''(xw) \\ &\quad +(d - m(uw) - m(vw) - m''(xw)) \leq m(\delta(X)) - 2m(vw) \end{aligned}$$

since $m''(xw) \geq 0$ and $m(ux) + m(uv) + m(uw) = d$. Since $m(\delta(X \cup \{u, w\})) \geq d$, it follows that $m(\delta(X)) \geq 2m(vw) + d$ as required. This proves that (G', m') is a d-target. Since $m'(uw) > m(ux), m(vw)$ (the first from the hypothesis), it follows that (G', m') is smaller than (G, m), and so is d-edge-colourable; let $F_1', \ldots, F_d'$ be a d-edge-colouring. Now every perfect matching containing vx also contains uw, since vx is not disjoint from any other edge incident with u. Hence there are at least $m(vw)$ of $F_1', \ldots, F_d'$ that contain both vx and uw. Choose $m(vw)$ of them, say $F_1', \ldots, F_{m(vw)}'$; and for $1 \leq i \leq m(vw)$ define $F_i = (F_i' \backslash \{vx, uw\}) \cup \{vw, ux\}$. Define $F_i = F_i'$ for $m(vw)+1 \leq i \leq d$. Then every edge e of G is in $m(e)$ of $F_1, \ldots, F_d$, a contradiction. Thus there is no such triangle uvw. This proves Lemma 4.5.4. □

Lemma 4.5.5. *No minimum 8-counterexample contains Conf(3) or Conf(4).*

Proof. To handle both cases at once, let us assume that (G, m) is an 8-target, and uvw, uwx are triangles with $m(uv)+m(uw)+m(vw)+m(ux) \geq 8$, (where possibly $m(uw) = 0$); and either (G, m) is a minimum 8-counterexample, or $m(uw) = 0$ and deleting uw gives a minimum 8-counterexample (G_0, m_0) say. We must show that $m(uw) = 0$ and $(m(uv), m(vw), m(wx), m(ux)) = (4, 2, 1, 2)$. Let (G, m') be obtained by switching (G, m) on u-v-w-x-u.

(1) (G, m') *is not smaller than* (G, m).

Because suppose it is. Then it admits an 8-edge-colouring; because if (G, m) is a minimum 8-counterexample this is clear, and otherwise $m(uw) = 0$, and (G', m') is smaller than (G_0, m_0). Let $F_1', \ldots, F_8'$ be an 8-edge-colouring of (G', m'). Since

$$m'(uv) + m'(uw) + m'(vw) + m'(ux) \geq 9,$$

one of $F_1', \dots, F_8'$, say F_1', contains two of uv, uw, vw, ux and hence contains vw, ux. Then

$$(F_1' \setminus \{vw, ux\}) \cup \{uv, wx\}$$

is a perfect matching, and it together with $F_2', \dots, F_8'$ provide an 8-edge-colouring of (G, m), a contradiction. This proves (1).

From (1) we deduce that $\max(m(ux), m(vw)) < \max(m(uv), m(wx))$. It follows that

$$m(uv) + m(uw) + m(vw) + m(wx) \leq 7,$$

by (1) applied with u, w exchanged; and

$$m(uv) + m(ux) + m(wx) + m(uw) \leq 7,$$

by (1) applied with v, x exchanged. Consequently $m(ux) > m(wx)$, and hence $m(ux) \geq 2$; and $m(vw) > m(wx)$, and so $m(vw) \geq 2$. Suppose that $m(uv) \leq 3$. Since

$$\max(m(ux), m(vw)) < \max(m(uv), m(wx)),$$

it follows that $m(uv) = 3$ and $m(vw) = m(ux) = 2$; and therefore $m(wx) = 1$, since $m(ux) > m(wx)$. But this is contrary to (1).

We deduce that $m(uv) \geq 4$. Since $m(vw) \geq 2$ and $m(uv) + m(uw) + m(vw) + m(wx) \leq 7$, it follows that $m(uw) + m(wx) \leq 1$; so $m(uw) = 0$ and $m(wx) = 1$. But then

$$(m(uv), m(vw), m(wx), m(ux)) = (4, 2, 1, 2).$$

This proves Lemma 4.5.5. $\square$

4.6 Guenin's cuts

We still have many configurations to handle to finish the proof of Theorem 4.5.1, but all the others are handled by a method of Guenin [45], which we introduce in this section. In particular, nothing so far has assumed the truth of Conjecture 4.1.1 for $d = 7$, but now we will need to use that.

Let (G, m) be a d-target, and let x-u-v-y be a three-edge path of G, where x, y are incident with a common region. Let (G', m') be obtained from $(G, m) + xy$ by switching on the cycle x-u-v-y-x. We say that (G', m') is obtained from (G, m) by *switching on* x-u-v-y. If (G', m') is smaller than (G, m), we say that the path x-u-v-y is *switchable.*

Let G be a three-connected graph drawn in the plane, and let G^* be its dual graph; let us identify $E(G^*)$ with $E(G)$ in the natural way. A *cocycle* means the edge-set of a cycle of the dual graph; thus, $Q \subseteq E(G)$ is a cocycle of G if and only if Q can be numbered $\{e_1, \ldots, e_k\}$ for some $k \geq 3$ and there are distinct regions $r_1, \ldots, r_k$ of G such that for $1 \leq i \leq k$, e_i is incident with r_i and with r_{i+1} (where r_{k+1} means r_1).

Guenin's method is the use of the following:

Lemma 4.6.1. *Let $d \geq 1$ be an integer such that every $(d-1)$-regular oddly $(d-1)$-edge-connected planar graph is $(d-1)$-edge-colourable. Let (G, m) be a minimum d-counterexample, and let x-u-v-y be a path of G with x, y on a common region. Let (G', m') be obtained by switching on x-u-v-y, and let $F_1, \ldots, F_d$ be a d-edge-colouring of (G', m'), where $xy \in F_k$. Let $I = \{1, \ldots, d\} \setminus \{k\}$ if $xy \notin E(G)$, and $I = \{1, \ldots, d\}$ if $xy \in E(G)$. Then for each $i \in I$, there is a cocycle Q_i of G' with the following properties:*

- *for $1 \leq j \leq d$ with $j \neq i$, $|F_j \cap Q_i| = 1$;*
- *$|F_i \cap Q_i| \geq 5$;*
- *there is a set $X \subseteq V(G)$ with $|X|$ odd such that $\delta_{G'}(X) = Q_i$; and*
- *$uv, xy \in Q_i$ and $ux, vy \notin Q_i$.*

Proof. Let $i \in I$. If $i \neq k$ and $xy \in F_i$, it follows that $m'(xy) \geq 2$ since $xy \in F_k$; and so $xy \in E(G)$. Thus in either case F_i is a perfect matching of G. For each edge e of G', let $p(e) = 1$ if $e \in F_i$, and $p(e) = 0$ otherwise; and for each edge e of G, let $n(e) = m(e) - p(e)$. Thus (G, n) has the property that for each vertex z, $n(\delta_G(z)) = d - 1$. If there is a list of $d - 1$ perfect matchings of G such that every edge e is in $n(e)$ of them, then adding F_i to this list gives a d-edge-colouring of (G, m), a contradiction. Thus by hypothesis, there exists $Y \subseteq V(G)$ with $|Y|$ odd and with $n(\delta_G(Y)) < d-1$. Since $|Y|$ and $n(\delta_G(Y))$ have the same parity, it follows that $n(\delta_G(Y)) \leq d - 3$. Since $\delta_G(Y)$ is an edge-cut of the connected graph G, it can be partitioned into "bonds" (edge-cuts $\delta_G(X)$ such that $G|X$, $G \setminus X$ are both connected), and hence one of these bonds $\delta_G(X)$ has $n(\delta_G(X))$ odd, and consequently $|X|$ also odd. Since $\delta_G(X)$ is a bond of G and hence $\delta_{G'}(X)$ is a bond of G', there is a cocycle Q_i of G' with $Q_i = \delta_{G'}(X)$. We claim that Q_i satisfies the theorem. For we have seen the third assertion; we must check the other three.

From the choice of X we have $n(\delta_G(X)) \leq d-3$. Since $|X|, |V(G) \setminus X| \geq 3$ (because $n(\delta_G(z)) = d-1$ for each vertex z), it follows from Proposition 4.3.1 that $m(\delta_G(X)) \geq d+2$, and so $p(\delta_G(X)) \geq 5$, that is, $|F_i \cap Q_i| \geq 5$. This proves the second assertion. We recall that $F_1, \ldots, F_d$ is a d-edge-colouring of (G', m'); and so for $1 \leq j \leq d$ with $j \neq i$, some edge of $\delta_{G'}(X)$ belongs to F_j, and so

$$\sum_{1 \leq j \leq d, j \neq i} |F_j \cap Q_i| \geq d-1.$$

On the other hand, every edge e of G' belongs to $m'(e)$ of $F_1, \ldots, F_d$, and hence to $m'(e) - p(e)$ of the $d-1$ perfect matchings in this list without F_i. Consequently

$$\sum_{1 \leq j \leq d, j \neq i} |F_j \cap Q_i| = \sum_{e \in Q_i} m'(e) - p(e).$$

It follows that $\sum_{e \in Q_i} m'(e) - p(e) \geq d-1$; but $m'(e) - p(e) = n(e)$ for all edges of G' except xu, uv, vy, xy, and so

$$|\{uv, xy\} \cap Q_i| - |\{ux, vy\} \cap Q_i| + \sum_{e \in Q_i} n(e) \geq d-1.$$

Since $\sum_{e \in Q_i} n(e) \leq d-3$, it follows that $uv, xy \in Q_i$ and $ux, vy \notin Q_i$. This proves the fourth assertion. Moreover, since

$$\sum_{1 \leq j \leq d, j \neq i} |F_j \cap Q_i| = d-1,$$

it follows that $|F_j \cap Q_i| = 1$ for all $j \in \{1, \ldots, d\}$ with $j \neq i$. This proves the first assertion, and so proves Lemma 4.6.1. $\square$

By the result of [19], every 7-regular oddly 7-edge-connected planar graph is 7-edge-colourable, so we can apply Lemma 4.6.1 when $d = 8$.

Lemma 4.6.2. *No minimum 8-counterexample contains Conf(5) or Conf(6).*

Proof. To handle both at once, let us assume that (G, m) is an 8-target, and uvw, uwx are two triangles with $m^+(uv) + m(uw) + m^+(wx) \geq 7$; and either (G, m) is a minimum 8-counterexample, or $m(uw) = 0$ and deleting uw gives a minimum 8-counterexample. We claim that

$$m(uv) + m(uw) + m(vw) + m(wx) \leq 7.$$

If $m(uw) > 0$ this follows from Lemma 4.5.5 since we do not have Conf(3); and if $m(uw) = 0$ then one of $m(uv), m(wx) \geq 3$, and since Lemma 4.5.5 implies that we do not have Conf(4), again the claim holds. This proves that $m(uv) + m(uw) + m(vw) + m(wx) \leq 7$. Since $m^+(uv) + m(uw) + m^+(wx) \geq 7$ and hence $m(uv) + m(uw) + m(wx) \geq 5$, it follows that $m(vw) \leq 2$ and similarly $m(ux) \leq 2$.

We claim that u-x-w-v-u is switchable. For suppose not; then we may assume that $m(vw) > \max(m(uv), m(wx))$ and $m(vw) \geq m(ux)$. Yet $m(vw) \leq 2$, and so $m(uv), m(wx) = 1$, and $m(ux) \leq 2$. Since u-x-w-v-u is not switchable, it follows that $m(ux) = 2$; and since $m^+(uv) + m(uw) + m^+(wx) \geq 7$, it follows that $m(uw) \geq 3$, giving Conf(3), contrary to Lemma 4.5.5. This proves that u-x-w-v-u is switchable.

Let r_1, r_2 be the second regions incident with uv, wx respectively, and for $i = 1, 2$ let D_i be the set of doors for r_i. Let $k = m(uv) + m(uw) + m(wx) + 2$. Let (G, m') be obtained by switching, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of (G, m'), where F_i contains one of uv, uw, wx for $1 \leq i \leq k$. For $1 \leq i \leq 8$, let Q_i be as in Lemma 4.6.1.

(1) *For $1 \leq i \leq 8$, either $F_i \cap Q_i \cap D_1 \neq \emptyset$, or $F_i \cap Q_i \cap D_2 \neq \emptyset$; and both are nonempty if either $k = 8$ or $i = 8$.*

For let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = wx$, $e_2 = uw$, and $e_3 = uv$. Since F_j contains one of e_1, e_2, e_3 for $1 \leq j \leq k$, it follows that none of $e_4, \ldots, e_n$ belongs to any F_j with $j \leq k$ and $j \neq i$, and, if $k = 7$ and $i \neq 8$, that only one of them is in F_8. But since at most one of e_1, e_2, e_3 is in F_i and $|F_i \cap Q_i| \geq 5$, it follows that $n \geq 7$; so either e_4, e_5 belong only to F_i, or e_n, e_{n-1} belong only to F_i, and both if $k = 8$ or $i = 8$. But if e_4, e_5 are only contained in F_i, then they both have multiplicity one, and are disjoint, so e_4 is a door for r_1 and hence $e_4 \in F_i \cap Q_i \cap D_1$. Similarly if e_n, e_{n-1} are only contained in F_i then $e_n \in F_i \cap Q_i \cap D_2$. This proves (1).

Now $k \le 8$, so one of r_1, r_2 is small since $m^+(uv) + m(uw) + m^+(wx) \ge 7$; and if $k = 8$ then by (1) $|D_1|, |D_2| \ge 8$, a contradiction. Thus $k = 7$, so both r_1, r_2 are small, but from (1) $|D_1| + |D_2| \ge 9$, again a contradiction. This proves Lemma 4.6.2. $\square$

Lemma 4.6.3. *No minimum* 8*-counterexample contains Conf(7).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that uvw is a triangle with $m^+(uv) + m^+(uw) \ge 7$. Let r_1, r_2 be the second regions for uv, uw respectively, and for $i = 1, 2$ let D_i be the set of doors for r_i. By Lemma 4.6.2, we do not have Conf(5), so neither of r_1, r_2 is a triangle. Since $m(uv) + m(uw) \ge 5$, one of $m(uv), m(uw) \ge 3$, so we may assume that $m(uv) \ge 3$. Let tu be the edge incident with r_2 different from uw. Since $m(uv) + m(uw) \ge 5$, it follows that $m(tu) \le 3$, and by 4.5.2, $m(vw) \le 2$. Thus the path t-u-v-w is switchable. Note that t, w are non-adjacent in G, since r_2 is not a triangle. Let (G', m') be obtained by switching on this path, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of it. Let $k = m(uv) + m(uw) + 2$; thus $k \ge 7$, since $m(uv) + m(uw) \ge 5$, and we may assume that for $1 \le j < k$, F_j contains one of uv, uw, and $tw \in F_k$.

Let $I = \{1, \ldots, 8\} \setminus \{k\}$, and for each $i \in I$, let Q_i be as in Lemma 4.6.1. Now let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$, $e_2 = uw$, and $e_3 = tw$. Since F_j contains one of e_1, e_2, e_3 for $1 \le j \le k$ it follows that none of $e_4, \ldots, e_n$ belongs to any F_j with $j \le k$ and $j \ne i$; and if $k = 7$ and $i \ne 8$, only one of them belongs to F_8. Since F_i contains at most one of e_1, e_2, e_3 and $|F_i \cap Q_i| \ge 5$, it follows that $n \ge 7$, and so either e_4, e_5 are only contained in F_i, or e_n, e_{n-1} are only contained in F_i; and both if either $k = 8$ or $i = 8$. Thus either $e_4 \in F_i \cap Q_i \cap D_2$ or $e_n \in F_i \cap Q_i \cap D_1$, and both if $k = 8$ or $i = 8$. Since $k \le 8$, one of r_1, r_2 is small since $m^+(uv) + m^+(uw) \ge 7$; and yet if $k = 8$ then $|D_1|, |D_2| \ge |I| = 7$, a contradiction. Thus $k = 7$, so r_1, r_2 are both small, and yet $|D_1| + |D_2| \ge 8$, a contradiction. This proves Lemma 4.6.3. $\square$

Lemma 4.6.4. *No minimum* 8*-counterexample contains Conf(8).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that uvw is a triangle, and its edges have multiplicities $3, 2, 2$ (in some order). We will show that the second region r for uw has a door disjoint from uw. By Lemma 4.5.5, we do not have Conf(3), so r is not a triangle. By exchanging u, w if necessary we may assume that $m(vw) = 2$. Let tu be the edge incident with r different from uw. We claim that the path t-u-v-w is switchable. For certainly $m(uv) \geq m(vw)$, so it suffices to check that $m(uv) \geq m(tu)$. If not, then since $m(uv) \geq 2$ and $m(uv) + m(uw) \geq 5$, it follows that $m(uv) = 2$, $m(tu) = 3$ and $m(uw) = 3$, and we have Conf(2), contrary to 4.5.4. Thus t-u-v-w is switchable. Let (G', m') be obtained by switching, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of (G', m'). Since $m'(uv) + m'(uw) = 6$, we may assume that $F_1, \ldots, F_6$ each contain one of uv, uw; and $tw \in F_7$, and therefore $vw \in F_8$. Let $I = \{1, \ldots, 6, 8\}$; and for $i \in I$, let Q_i be as in Lemma 4.6.1. Since Q_8 contains uv, uw, tw and $F_1, \ldots, F_7$ each contain one of uv, uw, tw, it follows that no other edge of Q_8 belongs to any of $F_1, \ldots, F_7$, and so $Q_8 \cap F_8$ contains a door for r, say e. Moreover $e \neq tu$ since $tu \notin Q_8$; and e is not incident with w since $vw \in F_8$. Consequently e is disjoint from uw. This proves Lemma 4.6.4. $\square$

Lemma 4.6.5. *No minimum* 8*-counterexample contains Conf(9).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that uv_1v_2 is a triangle, with $m(uv_1), m(uv_2), m(v_1v_2) = 2$, such that the second regions r_1, r_2 for uv_1, uv_2 respectively both have at most one door, and no door that is disjoint from uv_1v_2. For $i = 1, 2$, let D_i be the set of doors for r_i. For $i = 1, 2$, let ux_i and v_iy_i be edges incident with r_i different from uv_i.

Now $x_1 \neq x_2$ since u has degree at least four; and so $m(ux_1) + m(ux_2) \leq 4$ and we may assume that $m(ux_1) \leq 2$. Consequently the path x_1-u-v_2-v_1 is switchable. Note that v_1, x_1 may be adjacent, but if so then $m(v_1x_1) = 1$ from Lemma 4.5.5. Let (G', m') be obtained by switching, and let $F_1, \ldots, F_8$ be an 8-edge-colouring, where $uv_2 \in F_1, F_2, F_3$, and $uv_1 \in F_4, F_5$ and $v_1x_1 \in F_6$, and $v_1x_1 \in F_7$ if $v_1x_1 \in E(G)$. Since v_1v_2 belongs to some F_i, and v_1v_2 meets all of uv_2, uv_1, v_1x_1, we may assume that $v_1v_2 \in F_8$. Let $I = \{1, \ldots, 5, 7, 8\}$ if $x_1v_1 \notin E(G)$, and $I = \{1, \ldots, 8\}$ otherwise. For $i \in I$, let Q_i be as in Lemma 4.6.1.

We claim that $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$ for $i = 7, 8$. First suppose that $v_1x_1 \notin E(G)$. Then for $1 \leq j \leq 6$ and for $i = 7, 8$, $F_j \cap Q_i \cap \{uv_2, uv_1, v_1x_1\} \neq \emptyset$, and so no other edges of Q_i belong to any F_j with $j \in \{1, \ldots, 6\}$. Since only one edge of $Q_i \setminus \{uv_2, uv_1, v_1x_1\}$ belongs to the F_j with $j \in \{7, 8\} \setminus \{i\}$, it follows that $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$ as required. Now suppose that $v_1x_1 \in E(G)$. Then for $1 \leq j \leq 7$ and for $i = 7, 8$, $F_j \cap Q_i \cap \{uv_2, uv_1, v_1x_1\} \neq \emptyset$, and so no other edges of Q_i belong to any F_j with $j \in \{1, \dots, 7\}$ and $j \neq i$. For $i = 7$, as before it follows that $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$; for $i = 8$ we find that $F_i \cap Q_i \cap D_1, F_i \cap Q_i \cap D_2 \neq \emptyset$. Thus in any case, we have $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$ for $j = 7, 8$.

Now by hypothesis, $D_1 \cup D_2 \subseteq \{ux_1, ux_2, v_1y_1, v_2y_2\}$; and $ux_1 \notin Q_7, Q_8$ from the choice of switchable path, and $v_1y_1, v_2y_2 \notin F_8$ since $v_1v_2 \in F_8$. Thus $ux_2 \in F_8 \cap D_2$. Since $|D_2| \leq 1$ by hypothesis, it follows that $v_2y_2 \notin D_2$, and $ux_2 \notin F_7$ since $ux_2 \in F_8$ and $m(ux_2) = 1$. Thus $v_1y_1 \in D_1$. Now $m(ux_2) = 1$, and so the path x_2-u-v_1-v_2 is switchable; so by the same argument with v_1, v_2 exchanged, it follows that $ux_1 \in D_1$ and $v_2y_2 \in D_2$, contrary to the hypothesis. This proves Lemma 4.6.5. $\square$

Lemma 4.6.6. *No minimum 8-counterexample contains Conf(10).*

Proof. For suppose that (G, m) is a minimum counterexample, with a square $uvwx$ and a triangle wxy, where $m(uv) = m(wx) = m(xy) = 2$, and $m(vw) = 4$. By Lemma 4.5.5, we do not have Conf(4), and it follows that $m(ux) = 1$. Since $m(\delta(w)) = 8$ it follows that $m(wy) \leq 2$, and so u-x-y-w is switchable. Let (G', m') be obtained by switching on this path, and let $F_1, \dots, F_8$ be an 8-edge-colouring of it. We may assume that $xy \in F_1, F_2, F_3$, and $xw \in F_4, F_5$, and $uw \in F_6$. Let $I = \{1, \dots, 8\} \setminus \{6\}$, and let Q_i $(i \in I)$ be as in Lemma 4.6.1. Now $vw \notin F_4, F_5, F_6$, so there are four values of $i \in \{1, 2, 3, 7, 8\}$ such that $vw \in F_i$, and from the symmetry we may assume that F_1, F_2, F_7 contain vw (and so does one of F_3, F_8). It follows that $vw \notin Q_i$ for $i \in I$, and so $uv \in Q_i$ for each $i \in I$. Since uv belongs to two of $F_1, \dots, F_8$, there exists $j \neq 8$ with $uv \in F_j$. Moreover, F_j does not contain vw, and so $j \neq 1, 2, 7$; so $j \in \{3, 4, 5, 6\}$. But $|Q_1 \cap F_j| \geq 2$, since one of $xy, xw, vw \in Q_1 \cap F_j$, a contradiction. This proves Lemma 4.6.6. $\square$

Lemma 4.6.7. *No minimum 8-counterexample contains Conf(11), Conf(12) or Conf(13).*

Proof. To handle all these cases simultaneously, let us assume that (G, m) is a 8-target, and v_1-v_2-v_3-v_4-v_5-v_1 are the vertices in order of some cycle of G, and this cycle bounds a disc which is the union of three triangles of G, namely $v_1v_2v_3$, $v_1v_3v_5$ and $v_3v_4v_5$. Moreover, there is a subset $Z \subseteq \{v_1v_3, v_3v_5\}$ such that $m(e) = 0$ for all $e \in Z$ and deleting the edges in Z gives a minimum 8-counterexample. Finally, we assume that

$$m(v_1v_2) + m(v_1v_3) + m(v_2v_3) + m(v_3v_4) + m(v_3v_5) \geq 8,$$

and

$$m^+(v_1v_2) + m(v_1v_3) + m(v_3v_5) + m^+(v_4v_5) \geq 7.$$

To obtain the subcases Conf(11), Conf(12) and Conf(13), we set, respectively,

- $Z = \{v_1v_3\}$, $m(v_1v_2) \geq 3$, $m(v_3v_4) \geq 3$, $m(v_3v_5) = 1$, $m^+(v_4v_5) \geq 3$, and $m(v_1v_5) \leq 3$
- $Z = \{v_3v_5\}$, $m^+(v_1v_2) \geq 3$, $m(v_2v_3) = 2$, $m(v_3v_4) \geq 2$, $m(v_1v_3) = 2$, $m(v_1v_5) \leq 3$ and $m^+(v_4v_5) \geq 2$
- $Z = \{v_1v_3, v_3v_5\}$, $m(v_1v_2) \geq \max(m(v_2v_3), m(v_1v_5))$.

(Edges not mentioned are unrestricted.) Let (G, m') be obtained by switching on the sequence v_2-v_3-v_5-v_1-v_2. (We postpone for the moment the question of whether this sequence is switchable.) Let us suppose (for a contradiction) that (G, m') admits an 8-edge-colouring $F_1, \ldots, F_8$. Let $k = m(v_1v_2) + m(v_1v_3) + m(v_3v_5) + 2$; then we may assume that $F_1, \ldots, F_k$ each contain exactly one of v_1v_2, v_1v_3, v_3v_5, and $v_3v_5 \in F_k$. Hence $k \leq 8$. Let $I = \{1, \ldots, 8\}$ if $m(v_3v_5) \geq 1$, and $I = \{1, \ldots, 8\} \setminus \{k\}$ otherwise. Since v_2v_3 meets all the edges v_1v_2, v_1v_3, v_3v_5, it follows that none of $F_1, \ldots, F_k$ contain v_2v_3, and so $k + m(v_2v_3) - 1 \leq 8$ and we may assume that $v_2v_3 \in F_j$ for $k + 1 \leq j \leq k + m(v_2v_3) - 1$. Thus there are exactly $9 - k - m(v_2v_3)$ values of $j \in \{1, \ldots, 8\}$ such that F_j contains none of $v_1v_2, v_1v_3, v_3v_5, v_2v_3$. Since by hypothesis

$$m(v_1v_2) + m(v_1v_3) + m(v_2v_3) + m(v_3v_4) + m(v_3v_5) \geq 8,$$

and so $m(v_3v_4) > 9 - k - m(v_2v_3)$, there exists $h \leq k + m(v_2v_3) - 1$ such that $v_3v_4 \in F_h$; since v_3v_4 meets each of v_1v_3, v_2v_3 and v_3v_5, it follows that $v_1v_2 \in F_h$, and so $h < k$; and from the symmetry we may assume that $h = 1$.

For each $i \in I$ let Q_i as in Lemma 4.6.1. Now $|F_j \cap Q_i| = 1$ for $1 \leq j \leq 8$ with $j \neq i$; and since F_1 contains v_1v_2, v_3v_4 it follows that for $i \neq 1$ $v_3v_4 \notin Q_i$. Consequently $v_4v_5 \in Q_i$ for all $i \in I \setminus \{1\}$. Let r_1, r_2 be the second regions for v_1v_2, v_4v_5 respectively, and let their sets of doors be D_1, D_2. Hence for each $j \in \{1, \ldots, 8\}$, since there exists $i \in I \setminus \{1\}$ with $i \neq j$, it follows that F_j contains at most one of $v_1v_2, v_1v_3, v_3v_5, v_4v_5$, and so we may assume that $v_4v_5 \in F_j$ for $k + 1 \leq j \leq k'$ where $k' = k + m(v_4v_5)$, and in particular $k' \leq 8$. From the hypothesis, $k' \geq 7$.

(1) *For $i \in I \setminus \{1\}$, one of $F_i \cap D_1$, $F_i \cap D_2$ is non-empty, and both if $k' = 8$ or $i = 8$.*

Let $e_1, \dots, e_n, e_1$ be the edges of Q_i in order, where $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_3v_5$ and $e_4 = v_4v_5$. Thus for $1 \le j \le k'$, F_j contains one of e_1, e_2, e_3, e_4, and hence contains none of $e_5, \dots, e_n$ if $j \ne i$. Now since F_i contains at most one of e_1, e_2, e_3, e_4 and $|F_i \cap Q_i| \ge 5$, it follows that $n \ge 8$. Hence $e_5, \dots, e_n$ belong only to F_i, except that one belongs to F_8 if $i, k' < 8$. This proves (1) as usual.

Since $k' \le 8$, one of r_1, r_2 is small since $m^+(v_1v_2) + m(v_1v_3) + m(v_3v_5) + m^+(v_4v_5) \ge 7$. Consequently, (1) implies that $k' = 7$; and so r_1, r_2 are both small, again a contradiction to (1).

This proves that (G, m') is not 8-edge-colourable, and in particular the sequence v_2-v_3-v_5-v_1-v_2 is not switchable. Let us look at the subcases for Conf(11), Conf(12), Conf(13) listed above. In the Conf(11) subcase, $m(v_1v_2) \ge 3 \ge m(v_1v_5)$, so we only need to check that $m(v_1v_2) \ge m(v_2v_3)$. If not, then $m(v_2v_3) = 4$, contrary to Conf(2). In the Conf(13) subcase, the condition that $m(v_1v_2) \ge \max(m(v_2v_3), m(v_1v_5))$ is explicitly given. In the Conf(12) subcase, $m(v_1v_2) \ge 2 \ge m(v_2v_3)$, so we only need to check that $m(v_1v_2) \ge m(v_1v_5)$. Suppose not; then $m(v_1v_5) = 3$ and $m(v_1v_2) = 2$. In this case the sequence v_2-v_3-v_5-v_1-v_2 is not switchable, so we need a different approach.

Since (G, m') given above is not 8-colourable, it follows from Proposition 4.3.1 that $m'(\delta(X)) \ge 10$ for every subset $X \subseteq V(G)$ with $|X|$ odd and $|X|, |V(G) \setminus X| \ge 3$. Let (G, m'') be obtained from (G, m') by switching again on the same sequence. Now (G, m'') is a 8-target, since $m(v_2v_3), m(v_1v_5) \ge 2$; and it is smaller than (G, m), and therefore admits an 8-edge-colouring, say $F_1, \dots, F_8$. Since $m''(v_1v_2) + m''(v_1v_3) + m''(v_3v_5) + m''(v_1v_5) > 8$, some F_i contains two of $v_1v_2, v_1v_3, v_3v_5, v_1v_5$, and therefore contains v_1v_2 and v_3v_5. By replacing F_i by $(F_i \setminus \{v_1v_2, v_3v_5\}) \cup \{v_2v_3, v_1v_5\}$ we therefore obtain an 8-edge-colouring of (G, m'), a contradiction. This proves Lemma 4.6.7. □

Lemma 4.6.8. *No minimum 8-counterexample contains Conf(14).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that some edge uv is incident with regions r_1, r_2 where r_1 has at most six doors disjoint from uv, and $m(uv) \ge 5$, and either $m(uv) \ge 6$ or r_2 is small. By exchanging r_1, r_2 if necessary, we may assume that if r_1, r_2 are both small, then the length of r_1 is at least the length of r_2. By Lemma 4.5.5, we do not have Conf(3), so not both r_1, r_2 are triangles, and by 4.5.2, if $m(uv) \geq 6$ then neither of r_1, r_2 is a triangle; so r_1 is not a triangle. Let x-u-v-y be a path of C_{r_1}. Since $m(e) \geq 5$, this path is switchable; let (G', m') be obtained from (G, m) by switching on it, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of (G', m'). Let $k = m'(uv) + m'(xy) \geq 7$. Let $I = \{1, \ldots, 8\} \setminus \{k\}$ if x, y are non-adjacent in G, and $I = \{1, \ldots, 8\}$ if $xy \in E(G)$. For $i \in I$, let Q_i be as in Lemma 4.6.1. Since Q_i contains both uv, xy for each $i \in I$, it follows that for $1 \leq j \leq 8$, F_j contains at most one of uv, xy. Thus we may assume that $uv \in F_i$ for $1 \leq i \leq m'(uv)$, and $xy \in F_i$ for $m'(uv) < i \leq k$. Thus $k \leq 8$. Let D_1 be the set of doors for r_1 that are disjoint from e, and let D_2 be the set of doors for r_2.

(1) *For each $i \in I$, one of $F_i \cap Q_i \cap D_1, F_i \cap Q_i \cap D_2$ is nonempty, and if $k = 8$ or $i > k$ then both are nonempty.*

Let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$ and $e_2 = xy$. Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e_1, e_2, it follows that $n \geq 6$. Suppose that $k = 8$. Then for $1 \leq j \leq 8$, F_j contains one of e_1, e_2; and hence for all $j \in \{1, \ldots, 8\}$ with $j \neq i$, $e_3, \ldots, e_n \notin F_j$. It follows that e_n, e_{n-1} belong only to F_i and hence $e_n \in F_i \cap Q_i \cap D_2$. Since this holds for all $i \in I$, it follows that $|D_2| \geq |I| \geq 7$. Hence r_2 is big, and so by hypothesis, $m(uv) \geq 6$. Since $k = 8$ it follows that $xy \notin E(G)$. Consequently e_3 is an edge of C_{r_1}, and since e_3, e_4 belong only to F_i, it follows that e_3 is a door for r_1. But $e_3 \neq ux, vy$ from the choice of the switchable path, and so $e_3 \in F_i \cap Q_i \cap D_1$. Hence in this case (1) holds.

Thus we may assume that $k = 7$; and so $m(e) = 5$, and r_2 is small, and $xy \notin E(G)$, and $uv \in F_1, \ldots, F_6$, and $xy \in F_7$. Thus $I = \{1, \ldots, 6, 8\}$. If $i = 8$, then since $uv, xy \in Q_i$ and F_j contains one of e_1, e_2 for all $j \in \{1, \ldots, 7\}$, it follows as before that $e_3 \in F_i \cap Q_i \cap D_1$ and $e_n \in F_i \cap Q_i \cap D_2$. Thus we may assume that $i \leq 6$. For $1 \leq j \leq 8$ with $j \neq i$, $|F_j \cap Q_i| = 1$, and for $1 \leq j \leq 7$, F_j contains one of e_1, e_2. Hence $e_3, \ldots, e_n$ belong only to F_i and to F_8, and only one of them belongs to F_8. If neither of e_n, e_{n-1} belong to F_8 then $e_n \in F_i \cap Q_i \cap D_2$ as required; so we assume that F_8 contains one of e_n, e_{n-1}; and so $e_3, \ldots, e_{n-2}$ belong only to F_i. Since $n \geq 6$, it follows that $e_3 \in F_i \cap Q_i \cap D_1$ as required. This proves (1).

If $k = 8$, then (1) implies that $|D_1| \geq 7$ as required. So we may assume that $k = 7$ and hence

$m(e) = 5$ and $xy \notin E(G)$; and r_2 is small. Suppose that there are three values of $i \in \{1, \ldots, 6\}$ such that $|F_i \cap D_1| = 1$ and $F_i \cap D_2 = \emptyset$, say $i = 1, 2, 3$. Let $f_i \in F_i \cap D_1$ for $i = 1, 2, 3$, and we may assume that f_3 is between f_1 and f_2 in the path $C_{r_1} \setminus \{uv\}$. Choose $X \subseteq V(G')$ such that $\delta_{G'}(X) = Q_3$. Since only one edge of $C_{r_1} \setminus \{e\}$ belongs to Q_3, one of f_1, f_2 has both ends in X and the other has both ends in $V(G') \setminus X$; say f_1 has both ends in X. Let Z be the set of edges of G' with both ends in X. Thus $(F_1 \cap Z) \cup (F_2 \setminus Z)$ is a perfect matching, since $e \in F_1 \cap F_2$, and no other edge of $\delta_{G'}(X)$ belongs to $F_1 \cup F_2$; and similarly $(F_2 \cap Z) \cup (F_1 \setminus Z)$ is a perfect matching. Call them F_1', F_2' respectively. Then $F_1', F_2', F_3, F_4, \ldots, F_8$ form an 8-edge-colouring of (G', m'), yet f_1, f_2 are the only edges of $D_1 \cup D_2$ included in $F_1' \cup F_2'$, and neither of them is in F_2', contrary to (1). Thus there are no three such values of i; and similarly there are at most two such that $|F_i \cap D_2| = 1$ and $F_i \cap D_1 = \emptyset$. Thus there are at least three values of $i \in I$ such that $|F_i \cap D_1| + |F_i \cap D_2| \geq 2$ (counting $i = 8$), and so $|D_1| + |D_2| \geq 10$. But $|D_1| \leq 6$ by hypothesis and $|D_2| \leq 3$ since r_2 is small, a contradiction. This proves Lemma 4.6.8. $\square$

Lemma 4.6.9. *No minimum* 8*-counterexample contains Conf(15) or Conf(16).*

Proof. To handle both at once, we assume that (G, m) is an 8-target with a region r, and $uv \in E(C_r)$, and uvw is another region, satisfying:

- either (G, m) is a minimum 8-counterexample, or $m(uv) = 0$ and deleting uv gives a minimum 8-counterexample
- $m(uv) + m^+(uw) \geq 4$
- every edge of C_r not incident with u is 3-heavy
- let tu be the second edge of C_r incident with u; then the path t-u-w-v is switchable.

Note that while Conf(16) fits these conditions, some instances of Conf(15) may not, and we will handle them later. Let (G', m') be obtained by switching on the path t-u-w-v, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of it. Let $k = m(uw) + m(uv) + 2 \geq 5$; then we may assume that $F_1, \ldots, F_{k-1}$ contain one of uw, uv, and $tv \in F_k$. Let $I = \{1, \ldots, 8\}$ if $tv \in E(G)$, and $I = \{1, \ldots, 8\} \setminus \{k\}$ otherwise. For each $i \in I$ let Q_i be as in Lemma 4.6.1. Thus each Q_i contains all of uw, uv, tv, and so no edge of $Q_i \setminus \{uw, uv, tv\}$ belongs to F_j for any $j \neq i$ with $j \leq k$.

(1) $k = 5$.

For suppose that $k \geq 6$. Choose $i \in I \cap \{7, 8\}$. Since Q_i contains uv, uw, tv, it follows that $F_1, \ldots, F_6$ all contain an edge in $\{uv, uw, tv\} \cap Q_i$; and hence no edge of $Q_i \setminus \{uv, uw, tv\}$ belongs to any of $F_1, \ldots, F_6$. Choose an edge f of $C_r \setminus \{u, v\}$ with $f \in Q_i$. Now $f \neq tu$ by the choice of switchable path, and so f is 3-heavy (with respect to (G, m)), and if $f = tv$ then $m'(f) > m(f)$. Consequently there are three values of $j \in \{1, \ldots, 8\} \setminus \{k\}$ such that $F_j \cap Q_i$ contains an edge different from uv, uw, and hence some such j belongs to $\{1, \ldots, 5\}$, a contradiction. This proves (1).

Let r_1 be the second region for uw, and let D_1 be the set of doors for r_1. From (1) it follows that r_1 is small, and so $|D_1| \leq 3$.

(2) *For $i = 6, 7, 8$, $|F_i \cap D_1| = 1$; and the edges of F_6 and F_8 in Q_7 have a common end (they may be the same).*

For let $i \in \{6, 7, 8\}$; then $i \in I$. Let the edges of Q_i be $e_1, \ldots, e_n, e_1$ in order, where $e_1 = uw$, $e_2 = uv$ and $e_3 = tv$. Then $n \geq 7$, since $|F_i \cap Q_i| \geq 5$. Let $h = 3$ if $tv \in E(G)$, and $h = 4$ otherwise. Then e_h is an edge of C_r not incident with u, and so it is 3-heavy; and hence either $m(e_h) \geq 3$, or the second region for e_h is a triangle and e_{h+1} is an edge of it, and $m(e_h) + m(e_{h+1}) \geq 3$. Moreover, if $e_h = tv$ then $m'(e_h) > m(e_h)$. Thus in all cases it follows that there are three values of $j \neq 5$ with $1 \leq j \leq 8$ such that $F_j \cap Q_i$ contains one of e_h, e_{h+1}. We deduce that these three values of j are $6, 7, 8$, since $F_j \cap Q_i \subseteq \{uv, uw\}$ for $1 \leq j \leq 4$. Consequently for $1 \leq j \leq 8$, $F_j \cap Q_i$ includes one of e_1, e_2, e_3, e_4, e_5. It follows that only F_i contains e_n, e_{n-1}, and consequently $e_n \in F_i \cap D_1$. Since $|D_1| \leq 3$, this proves the first assertion of (2). The second follows since, taking $i = 7$ and defining e_h as before, F_6 and F_8 each contain one of e_h, e_{h+1}, and these edges have a common end. This proves (2).

Let $F_i \cap D_1 = \{f_i\}$ for $i = 6, 7, 8$. Thus f_6, f_7, f_8 are distinct, and we may assume that f_6, f_7, f_8 are in order in the path $C_{r_1} \setminus \{uw\}$. Choose $X \subseteq V(G)$ with $\delta_{G'}(X) = Q_7$. Let H be the subgraph of G' with vertex set $V(G)$ and edge set $(F_6 \setminus F_8) \cup (F_8 \setminus F_6)$. Thus each component of H is either a single vertex or a cycle of even length. Now there are either no edges, or two edges, of H that belong to $\delta_{G'}(X)$; and if there are two then they have a common end by (2). It follows that the component of H, say C, that contains f_6 does not contain f_8. Let $F'_6 = (F_8 \cap E(C)) \cup (F_6 \setminus E(C))$ and $F'_8 = (F_6 \cap E(C)) \cup (F_8 \setminus E(C))$; then F'_6, F'_8 are perfect matchings of G', and $F_1, \ldots, F_5, F'_6, F_7, F'_8$ is an 8-edge-colouring of (G, m'). On the other hand both f_6, f_8 belong to F'_8, so this 8-edge-colouring does not satisfy (2), a contradiction.

This completes the argument for Conf(16), and also for Conf(15) when (with notation as in the definition of Conf(15)) the two edges of C_r consecutive with e both have multiplicity at most $m(e)$ (to see this, let u-w-v be a subpath of C_r where $e = uw$, and add a new edge uv with multiplicty zero). Now we handle the remaining case of Conf(15); we assume that

- (G, m) is a minimum 8-counterexample
- r is a region of length at least four, and e is an edge of C_r
- $m^+(e) \geq 4$, and every edge of C_r disjoint from e is 3-heavy
- one of the edges of C_r incident with e has multiplicity more than $m(e)$.

Let C_r have vertices $v_1, \ldots, v_p$ in order, where $p \geq 4$, $e = v_1v_2$, and $m(v_2v_3) > m(e)$. It follows that $m(v_1v_2) = 3$ and $m(v_2v_3) = 4$. From Lemma 4.5.5, we do not have Conf(4) so $p \geq 5$. The path v_1-v_2-v_3-v_4 is switchable; let (G, m') be obtained by switching on it. We may assume that $v_2v_3 \in F_i$ for $1 \leq i \leq 5$ and $v_1v_4 \in F_6$. Since $m'(v_1v_2) = 2$ and v_1v_2 meets both v_2v_3 and v_1v_4, it follows that $v_1v_2 \in F_7, F_8$. Consequently $v_pv_1 \in F_h$ for some h with $1 \leq h \leq 5$. Let $I = \{1, \ldots, 8\} \setminus \{6\}$. For each $i \in I$ let Q_i be as in Lemma 4.6.1. Now Q_7 contains v_2v_3, v_1v_4, and so for $1 \leq j \leq 6$, $F_j \cap Q_7 \subseteq \{v_2v_3, v_1v_4\}$. In particular $v_pv_1 \notin Q_7$. But Q_7 contains an edge f of C_r, different from v_1v_2, and this edge is 3-heavy, since it is different from v_pv_1 and hence disjoint from e; and so $F_j \cap Q_i \setminus \{v_2v_3, v_1v_4\} \neq \emptyset$ for three values of $j \in \{1, \ldots, 8\}$, a contradiction. This proves Lemma 4.6.9. $\square$

Lemma 4.6.10. *No minimum 8-counterexample contains Conf(17) or Conf(18).*

Proof. To handle both at once, we assume that (G, m) is an 8-target with a region r with length at least four, and $uv \in E(C_r)$, and uvw is another region, satisfying:

- either (G, m) is a minimum 8-counterexample, or $m(uv) = 0$ and deleting uv gives a minimum 8-counterexample
- $m(uv) + m^+(uw) \geq 5$
- let t, x be the second neighbours of u, v in C_r respectively; if $m(uv) = 3$ and uv is 5-heavy let $P = C_r \setminus \{u, v\}$, and otherwise let $P = C_r \setminus \{u\}$; then every edge f of P satisfies $m^+(f) \geq 2$, and at most one edge of P is not 3-heavy
- $m(tu), m(vw) \leq m(uw)$.

The path t-u-w-v is switchable; let (G', m') be obtained by switching on it, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of (G', m'). Since r has length at least four, $tv \notin E(G)$. Let $k = m(uw) + (uv) + 2 \geq 6$; we may assume that F_i contains one of uv, uw for $1 \leq i < k$, and F_k contains tv. Let $I = \{1, \ldots, 8\} \setminus \{k\}$; and for each $i \in I$ let Q_i be as in Lemma 4.6.1.

(1) *There is at most one value of $i \in I$ such that $Q_i \cap E(P) = \emptyset$, and if i is such a value then $k = 7$ and $m(uv) = 3$ and $m(uw), m(vw) = 2$ and $uw \in F_i$.*

For suppose that $i \in I$ and $Q_i \cap E(P) = \emptyset$. It follows that $P = C_r \setminus \{u, v\}$, and so $m(uv) = 3$ and uv is 5-heavy. Hence $m(uw), m(vw) \geq 2$, and so $m(uw), m(vw) = 2$ by 4.5.2, and $k = 7$. Now for $1 \leq i \leq 7$, F_i contains one of uw, uv, tv, and since vw meets all of these edges it follows that $vw \in F_8$. But vx belongs to some F_j such that F_j contains none of tv, uv, vw, and so $uw \in F_j$. Then $|F_j \cap Q_i| \geq 2$, so $j = i$ and hence $uw \in F_i$. This proves (1).

Let I' be the set of $i \in I$ such that $Q_i \cap E(P) \neq \emptyset$. By (1), $|I'| \geq 6$. Let r_1 be the second region for uw, and let its set of doors be D_1. Thus $|D_1| \leq 3$ if $k = 6$, since $m(uv) + m^+(uw) \geq 5$. Let I'' be the set of $i \in I'$ such that the edge in $Q_i \cap E(P)$ is not 3-heavy.

(2) *There is a unique edge $f \in E(P)$ that is not 3-heavy, and it belongs to none of $F_1, \ldots, F_k$.*

Moreover, if $i \in I' \setminus I''$ then $k = 6$ and $i \leq 5$ and $F_i \cap Q_i \cap D_1 \neq \emptyset$.

Suppose that $i \in I' \setminus I''$. There are therefore three values of $j \in \{1, \dots, 8\}$ such that $F_j \cap Q_i \not\subseteq \{uw, uv, tv\}$, and so at least two that are also different from i. Consequently, for those two values of j, it follows that $uw, uv, yv \notin F_j$ and hence $k = 6$ and $j \in \{7, 8\}$. Thus $i \leq 5$. Let the edges of Q_i in order be $e_1, \dots, e_n, e_1$, where $e_1 = uw$, $e_2 = uv$ and $e_3 = tv$; then $n \geq 7$, since $|F_i \cap Q_i| \geq 5$. But $F_1, \dots, F_8$ each contain one of $e_1, \dots, e_5$, so

$e_n \in F_i \cap Q_i \cap D_1$. This proves the second assertion of (2). For the first assertion, since $|D_1| \leq 3$, it follows that $|I' \setminus I''| \leq 3$. Since $|I'| \geq 6$, it follows that $|I''| \geq 3$. But by hypothesis, there is at most one edge in P that is not 3-heavy, and so this edge exists, say f. It follows that $f \in Q_i$, for all $i \in I''$. Now let $j \in \{1, \dots, k\}$. Choose $i \in I''$ with $i \neq j$; then $F_j \cap Q_i \subseteq \{uw, uv, tv\}$, and so F_j does not contain f. This proves (2).

By (2) we may assume that $f \in F_{k+1}$. Let r_2 be the second region at f, and let D_2 be its set of doors. By hypothesis, if $m(f) = 1$ then $|D_2| \leq 3$.

Suppose that $k \geq 7$. By (2), $I'' = I'$ and $m(f) = 1$. Let $i \in I'$, and let the edges of Q_i in order be $e_1, \dots, e_n$, where $e_1 = uw$, $e_2 = uv$, $e_3 = tv$, and $e_4 = f$. Since only one of $e_1, \dots, e_4$ belongs to F_i, and $|F_i \cap Q_i| \geq 5$, it follows that $n \geq 8$. But $F_1, \dots, F_8$ each contain one of $e_1, \dots, e_4$, and so $e_5, \dots, e_n$ only belong to F_i; and hence $e_5 \in F_i \cap Q_i \cap D_2$. Consequently $|D_2| \geq |I'| \geq 6$, a contradiction.

This proves that $k = 6$, and hence $|D_1| \leq 3$, and $I' = I$ by (1), and $7, 8 \in I''$ by (2). Now let $i \in I''$. Let the edges of Q_i in order be $e_1, \dots, e_n, e_1$, where $e_1 = uw$, $e_2 = uv$, $e_3 = tv$, and $e_4 = f$. Again $n \geq 8$.

Suppose that $m(f) \geq 2$; then $m(f) = 2$ by (2), and $f \in F_7, F_8$, and so $F_1, \dots, F_8$ each contain one of $e_1, \dots, e_4$, and therefore $e_5, \dots, e_n$ belong to no F_j with $j \neq i$. Since $n \geq 8$, it follows that $e_n \in D_1$, and so $F_i \cap Q_i \cap D_1 \neq \emptyset$. By (2), it follows that $F_i \cap Q_i \cap D_1 \neq \emptyset$ for all $i \in I'$, and so $|D_1| \geq |I'| = 7$, a contradiction. Thus $m(f) = 1$, and so $|D_2| \leq 3$.

Again, let $i \in I''$, and let $e_1, \dots, e_n, e_1$ be as before. Now $F_1, \dots, F_7$ each contain one of $e_1, \dots, e_4$, and so $e_5, \dots, e_n$ belong to no F_j with $1 \leq j \leq 7$ and $j \neq i$, and only one of them belongs to F_8 if $i \neq 8$. We assume first that $i \neq 8$. Since $n \geq 8$, either $e_5, e_6 \notin F_8$, or $e_n, e_{n-1} \notin F_8$, and so either $e_5 \in D_2$ or $e_n \in D_1$. Now we assume $i = 8$. Then $e_5, \ldots, e_n$ belong to no F_j with $1 \leq j \leq 7$, and so $e_5 \in D_2$ and $e_n \in D_1$.

In summary, we have shown that for each $i \in I''$, either $F_i \cap D_1 \neq \emptyset$, or $F_i \cap D_2 \neq \emptyset$ (both if $i = 8$); and $8 \in I''$. By (2), if $i \in I' \setminus I''$ then either $F_i \cap D_1 \neq \emptyset$, or $F_i \cap D_2 \neq \emptyset$; and so $|D_1| + |D_2| \geq |I'| + 1 \geq 7$, a contradiction. This proves Lemma 4.6.10. $\square$

Lemma 4.6.11. *No minimum 8-counterexample contains Conf(19).*

Proof. Let (G, m) be a minimum 8-counterexample, and suppose that r is a region with length at least five, and e is an edge of C_r, such that $m^+(e) \geq 5$, and every edge of C_r disjoint from e is 2-heavy, and at most two of them are not 3-heavy. By Lemma 4.6.10, we do not have Conf(17), so there are at least two edges in C_r disjoint from e that are not 3-heavy, and so by hypothesis, there are exactly two, say g_1, g_2. Thus $m(g_1), m(g_2) \leq 2$. By hypothesis, g_1, g_2 are 2-heavy.

Let $e = uv$, and let the second neighbours of u, v in C_r be t, w respectively. Since $m(e) \geq 4$, it follows that $m(tu), m(vw) \leq m(uv)$ and so the path t-u-v-w is switchable. Let (G', m') be obtained by switching on this path, and let $F_1, \ldots, F_8$ be an 8-edge-colouring of it. Let $k = m(e) + 2$. We may assume that $tw \in F_k$. Let $I = \{1, \ldots, 8\} \setminus \{k\}$, and for each $i \in I$ let Q_i be as in Lemma 4.6.1. Let I_1, I_2, I_3 be the sets of $i \in I$ such that $g_1 \in Q_i$, $g_2 \in Q_i$, and $g_1, g_2 \notin Q_i$ respectively.

(1) *$k = 6$.*

For suppose that $k > 6$. Let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$ and $e_2 = tw$. Thus e_3 is an edge of C_r disjoint from e. Since $|F_i \cap Q_i| \geq 5$ and $|F_i \cap \{e_1, e_2\}| \leq 1$, it follows that $n \geq 6$. Now there are $k \geq 7$ values of $j \in \{1, \ldots, 8\}$ such that F_j contains one of e_1, e_2; and so there is at most value of $j \neq i$ such that F_j contains one of e_3, e_4. It follows that e_3 is not 3-heavy and so $i \in I_1 \cup I_2$. Since this holds for all $i \in I$, we may assume that $|I_1| \geq 4$. Let $i \in I_1$; as before, there is at most one value of $j \neq i$ such that F_j contains one of e_3, e_4. Now $m(g_1) \leq 2$. If $m(g_1) = 2$, then $g_1 \in F_i$, and since this holds for all $i \in I_1$ it follows that g_1 is contained in F_i for four different values of i, a contradiction. Thus $m(g_1) = 1$. Since g_1 is 2-heavy, the second region for g_1 is a triangle with edge set $\{g_1, p, q\}$ say, where $e_4 = p$. Hence one of g_1, p_q has multiplicity one and is contained in F_i. Since this holds for all $i \in I_1$ and $|I_1| \geq 4$, this is impossible. This proves (1).

We may therefore assume that $uv \in F_i$ for $1 \le i \le 5$ and $tw \in F_6$. Since $k = 6$, it follows that $m(e) = 4$ and since $m^+(e) \ge 5$, the second region r_1 for uv is small. Let D_1 be its set of doors.

(2) *If $i \in I_3$ then $i \le 5$ and $F_i \cap Q_i \cap D_1 \neq \emptyset$.*

For let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$ and $e_2 = tw$. Then $F_1, \ldots, F_6$ each contain an edge in $\{e_1, e_2\}$, and so for $1 \le j \le 6$ with $j \neq i$, none of $e_3, \ldots, e_n$ belongs to F_j. Now e_3 is 3-heavy, and so there are three values of j such that F_j contains one of e_3, e_4; and so these three values are $i, 7, 8$, and $i \neq 7, 8$. (Thus $i \le 5$ since $6 \notin I$.) Hence for $1 \le j \le 8$, F_j contains one of $e_1, \ldots, e_4$; and so e_n, e_{n-1} belong only to F_i. Hence $e_n \in D_1$. This proves (2).

For $h = 1, 2$, let I'_h be the set of all $i \in I_h$ such that $F_i \cap Q_i \cap D_1 = \emptyset$.

(3) *For $h = 1, 2$, $|I'_h| \le 2$, and $7, 8 \notin I'_h$, and if $|I'_h| = 2$ then $7, 8 \notin I_h$.*

For let $h = 1$ say. Suppose first that $m(g_1) = 2$, and let $g_1 \in F_a, F_b$ where $1 \le a < b \le 8$. Let $i \in I'_1$, and let $e_1, \ldots, e_n$ be as before; then $e_3 = g_1$. Again, for $1 \le j \le 6$ with $j \neq i$, none of $e_3, \ldots, e_n$ belongs to F_j, and consequently $a, b \in \{i, 7, 8\}$. In particular, $b \ge 7$, and $a \in \{i, 7\}$. Thus if $a \le 6$ then $i = a$ and so $|I'_1| = 1$ and the claim holds. We assume then that $(a, b) = (7, 8)$. But then $F_1, \ldots, F_8$ each contain one of e_1, e_2, e_3, and so $e_n \in D_1$, contradicting that $i \in I'_1$. So the claim holds if $m(g_1) = 2$.

Next we assume that $m(g_1) = 1$. Since g_1 is 2-heavy, the second region at g_1 is a triangle with edge set $\{g_1, p, q\}$ say. Let $g_1 \in F_a$. Let $i \in I'_1$, and let $e_1, \ldots, e_n$ be as before; then $e_3 = g_1$. Again, for $1 \le j \le 6$ with $j \neq i$, none of $e_3, \ldots, e_n$ belong to F_j, and consequently $a \in \{i, 7, 8\}$. Thus if $a \neq 7, 8$ then $i = a$ and $|I'_1| = 1$ and the claim holds. We assume then that $a = 7$. Thus each of $F_1, \ldots, F_7$ contains one of e_1, e_2, e_3, and for $1 \le j \le 7$ with $j \neq i$, F_j contains none of $e_4, \ldots, e_n$. Since $F_i \cap Q_i \cap D_1 = \emptyset$, there exists $j \in \{1, \ldots, 8\}$ with $j \neq i$ such that F_j contains one of e_n, e_{n-1}; and hence $j = 8$, and so $i \neq 8$. (Also, $i \neq 7$ since $g_1 \in F_7$ and g_1 meets e_4. Consequently, $7, 8 \notin I'_1$.) Thus $F_1, \ldots, F_8$ each contain one of $e_1, e_2, e_3, e_{n-1}, e_n$, and so e_4 is only contained in

F_i. Consequently, i has the property that one of p, q has multiplicity one, and F_i contains it. Thus there are at most two such values of i, and so $|I_1'| \leq 2$. Moreover, if there are two such values, say c, d, then $c, d \leq 5$ and F_c contains one of p, q and F_d contains the other. Consequently if $7 \in I_1$, then one of F_c, F_d contains two edges of Q_7, a contradiction. So if $|I_1'| = 2$ then $7, 8 \notin I_1$. This proves (3).

From (2), we may assume that $7 \in I_1$, and so $|I_1'| + |I_2'| \leq 3$ by (3). Consequently there are at least four values of $i \in I$ such that $F_i \cap Q_i \cap D_1 \neq \emptyset$, and so $|D_1| \geq 4$, a contradiction. This proves Lemma 4.6.11. $\square$

This completes the proof of Theorem 4.5.1 and hence of Theorem 4.1.2. Perhaps despite appearances, there was some system to our choice of the β- and γ-rules. We started with the idea that we would normally pass a charge of one from each small region to each big region sharing an edge with it, and made the minimum modifications we could to the β-rules so that the proof of Lemma 4.4.4 worked. Then we experimented with the γ-rules to make Lemmas 4.4.5, 4.4.6, and 4.4.7 work out.

It is to be hoped that solving these special cases of the main conjecture 4.1.1 will lead us to a proof of the general case, but that seems far away at the moment. The same approach does indeed work (more simply) for seven-regular planar graphs, and this gives an alternative proof of the result of [32], to appear in [19]. We tried the same again for nine-regular graphs, but there appeared to be some serious difficulties. Maybe more perseverance will bring it through, but it seems much harder than the eight-regular case.

Chapter 5

Fractionally colouring graphs without large cliques

5.1 Introduction

In this chapter, we give some upper bounds for the fractional chromatic number of graphs which don't contain large cliques. We consider only simple, undirected graphs and work completely within the rational numbers. This work is joint with Andrew King and has been published in [35] and [34].

The idea of bounding the chromatic number χ based on the clique number ω and maximum degree Δ goes all the way back to Brooks' Theorem.

Theorem 5.1.1 (Brooks' Theorem [9])**.** *Let G be a graph with maximum degree $\Delta \geq 3$. If $\omega(G) \leq \Delta$ then $\chi(G) \leq \Delta$.*

In other words, when $\Delta \geq 3$, as long as $G \neq K_{\Delta+1}$ its chromatic number is at most Δ. More recently, Borodin and Kostochka conjectured that if $\Delta \geq 9$, then a similar statement holds for K_Δ-free graphs.

Conjecture 5.1.2 (Borodin, Kostochka [7])**.** *Let G be a graph with $\Delta \geq 9$. If $\omega(G) \leq \Delta - 1$ then $\chi(G) \leq \Delta - 1$.*

The example of $C_5 \boxtimes K_3$ (see Figure 5.2) tells us that the condition that $\Delta \geq 9$ cannot be improved. Conjecture 5.1.2 remains open, but has seen some progress. Kostochka showed that

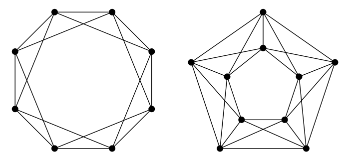

Figure 5.1: C_8^2 (left) and $C_5 \boxtimes K_2$ (right).

every $K_{\Delta-28}$-free graph satisfies $\chi < \Delta$ [60] and subsequently Mozhan showed that every $K_{\Delta-3}$-free graph with $\Delta \geq 31$ satisfies $\chi < \Delta$ [66]. Later Reed [68] proved a weakening of Conjecture 5.1.2 that had been conjectured independently by Beutelspacher and Hering [5]:

Theorem 5.1.3 (Reed [68])**.** *For graphs with $\Delta \geq 10^{14}$, if $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.*

In the paper, Reed claims that a more careful analysis could replace 10^{14} with 10^3.

This is the state of the art on the chromatic number of K_Δ-free graphs, but what about the *fractional* chromatic number χ_f (we will define it soon) of K_Δ-free graphs? Albertson, Bollobás, and Tucker noted in the 1970s that even when $\Delta \geq 3$, there are at least two K_Δ-free graphs with $\chi_f = \Delta$, namely C_8^2 and $C_5 \boxtimes K_2$ [2] (see Figure 5.1). It turns out that these are the only such graphs. For $\Delta \geq 3$ we define $f(\Delta)$ as:

$$f(\Delta) \;=\; \min_G \left\{ \; \Delta - \chi_f(G) \;\mid\; \Delta(G) \leq \Delta; \;\; \omega(G) < \Delta; \;\; G \notin \{C_8^2, C_5 \boxtimes K_2\} \; \right\}.$$

From Brooks' Theorem we know that $f(\Delta)$ is always nonnegative. Considering Theorem 5.1.3, one may be inclined to believe that $f(\Delta)$ increases with Δ. As proven by King, Lu, and Peng, this is indeed the case for $\Delta \geq 4$ [59]. [1] In Table 5.1 we show the known and conjectured bounds for various values of Δ. Figure 5.2 shows graphs demonstrating the best known (and conjectured) upper bounds on $f(\Delta)$ for $3 \leq \Delta \leq 8$.

When $\Delta = 3$ a best-possible bound is known. Evidently from the definition of the fractional chromatic number, a graph on n vertices with $\chi_f = k$ must contain a stable set of size at least $\frac{n}{k}$. Heckman and Thomas proved in 2001 that every triangle-free graph with $\Delta \leq 3$ contains a stable set of size at least $\frac{5n}{14}$ [48]. In the same paper, they conjectured that every such graph has fractional chromatic number at most $\frac{14}{5}$. In other words, they asked whether $f(3) = \frac{1}{5}$. The graph $P(7, 2)$

[1] For $\Delta \geq 6$, this is a consequence of the fact that when $\omega > \frac{2}{3}(\Delta + 1)$, there is a stable set hitting every maximum clique [58]. For $\Delta \in \{4, 5\}$ more work is required.

Δ	$f(\Delta)$ lower bounds			$f(\Delta)$ upper bound			conjectured value	
3	3/64	0.0468	[47]					
3	3/43	0.0697	[63]					
3	1/11	0.0909	[42]					
3	2/15	0.1333	[62]					
3	1/5	0.2	[29]	1/5	$P(7,2)$	[41]	1/5	[48]
4	2/67	0.0298	[59]	1/3	C_{11}^2		1/3	[59]
5	2/67	0.0298	[59]	1/3	$C_7 \boxtimes K_2$		1/3	[59]
6	1/22.5	**0.0445**		1/2	$(C_5 \boxtimes K_3) - 4v$		**1/2**	
7	1/11.2	**0.0899**		1/2	$(C_5 \boxtimes K_3) - 2v$		**1/2**	
8	1/8.9	**0.1135**		1/2	$C_5 \boxtimes K_3$	[10]	**1/2**	
9	1/7.7	**0.1307**		1	K_8		1	[7]
10	1/7.1	**0.1423**		1	K_9		1	[7]
1000	1	1	[68]	1	K_{999}		1	[5]

Table 5.1: The state of the art. New results and conjectures are in boldface. For $\Delta \leq 5$, the fractional bound is the proven bound. For $\Delta \geq 6$, the decimal bound approximates the proven bound, and the fractional expression approximates the decimal bound for ease of comparison.

(see Figure 5.2) shows that their result and conjecture are tight. The conjecture received a fair bit of attention; a succession of bounds were given by various authors before it was proved by Dvořák, Sereni and Volec in 2013 (see Table 5.1).

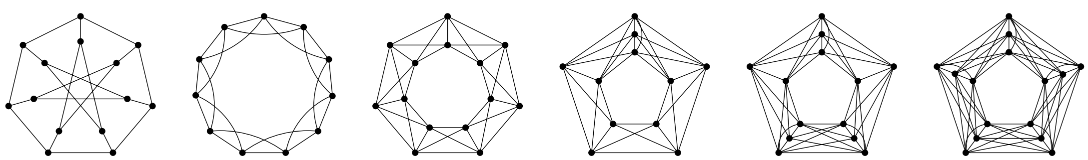

Figure 5.2: From left to right, the graphs $P(7,2)$, C_{11}^2, $C_7 \boxtimes K_2$, $(C_5 \boxtimes K_3) - 4v$, $(C_5 \boxtimes K_3) - 2v$, and $C_5 \boxtimes K_3$.

In this chapter we give improved bounds on $f(\Delta)$ for $\Delta \geq 6$ up until whenever Theorem 5.1.3 takes effect, which we assume to be $\Delta = 1000$. We also conjecture that the upper bound of $f(\Delta) \leq \frac{1}{2}$ is tight for $\Delta \in \{6,7,8\}$:

Conjecture 5.1.4. *For $\Delta \in \{6,7,8\}$, let G be a graph with maximum degree Δ and clique number at most $\Delta - 1$. Then the fractional chromatic number of G is at most $\Delta - \frac{1}{2}$.*

One of the major questions in this area, as is evident from Table 5.1, is the following:

Conjecture 5.1.5. *For $\Delta \geq 3$, $f(\Delta) \leq f(\Delta+1)$.*

Conjecture 5.1.4 and the conjecture of King, Lu and Peng [59] imply the following bounds on the stability number which may be easier to prove.

Conjecture 5.1.6. *For $\Delta \in \{4, 5\}$, let G be a graph on n vertices with maximum degree Δ and $\omega(G) \le \Delta - 1$. If G is not isomorphic to C_8^2 or $C_5 \boxtimes K_2$, then G has a stable set of size at least $\frac{n}{\Delta - \frac{1}{3}}$.*

Conjecture 5.1.7. *For $\Delta \in \{6, 7, 8\}$, let G be a graph on n vertices with maximum degree Δ and $\omega(G) \le \Delta - 1$. Then G has a stable set of size at least $\frac{n}{\Delta - \frac{1}{2}}$.*

To our knowledge, the analogous consequence of Conjecture 5.1.2 is not known either.

The rest of the chapter is organized as follows. In Sections 5.2 and 5.2.1 we give the necessary definitions and some results about fractional colourings. In Section 5.3 we give an overview of our approach which consists of characterizing a minimum counterexample, and constructing a fractional colouring of the supposed minimum counterexample. Sections 5.4 to 5.6 are dedicated to the colouring argument, and the structural reductions are found in Section 5.7. Finally, in Section 5.8, we provide a different upper bound on the fractional chromatic number: a 'superlocal' strengthening of the fractional relaxation of Reed's conjecture (more details about this in Section 5.2.1).

5.2 Fractionally colouring weighted and unweighted graphs

We must consider fractional colourings of both vertex-weighted and unweighted graphs, because we will begin to fractionally colour an unweighted graph G in one way that does very well on particularly tricky vertices, then finish the colouring in another way that does fairly well on all vertices. The second step requires a weighted generalization of a known result; the weight on a vertex reflects how much colour we have yet to assign to the vertex.

Let $G = (V, E)$ be a graph, let $\mathcal{S} = \mathcal{S}(G)$ be the set of stable sets of G, and let k be a nonnegative rational. Now let $\kappa : \mathcal{S} \to \mathcal{P}([0, k))$ be a function assigning each stable set S of G a subset of $[0, k)$ such that for every $S \in \mathcal{S}$, $\kappa(S)$ is the union of disjoint half-open intervals[2] with rational endpoints between 0 and k, and for any distinct S, S' in $\mathcal{S}$, $\kappa(S) \cap \kappa(S') = \emptyset$. For a set $\mathcal{S}' \subseteq \mathcal{S}$ of stable sets, define $\kappa(\mathcal{S}')$ as $\cup_{S \in \mathcal{S}'} \kappa(S)$. For each $v \in V$, define $\kappa[v]$ as $\cup_{S \ni v} \kappa(S)$. For a set $X \subseteq V$, define $\kappa[X]$ as $\cup_{S \cap X \neq \emptyset} \kappa(S) = \cup_{v \in X} \kappa[v]$.

[2](containing their lower endpoint but not their upper)

Now consider a nonnegative vertex weight function $w : V \to [0, \infty)$; in this case we say that G is a *w-weighted graph.* (Recall that w, like all numbers considered in this chapter, is rational.) If for every vertex $v \in V$ we have $|\kappa[v]| \geq w(v)$, then κ is a *fractional $\wr w$-colouring* of G with *weight k*; in other words it is a *fractional $k \wr w$-colouring* of G. The minimum weight of a fractional $\wr w$-colouring G is denoted $\chi_f^w(G)$, or simply χ_f^w when the context is clear. If $w = \mathbf{1}$ (i.e. the weight function uniformly equal to 1), then we may omit it from the notation, i.e. we define fractional colourings and the fractional chromatic number of unweighted graphs. If some vertex v has $|\kappa[v]| < w(v)$, we say that we have a *partial fractional $k \wr w$-colouring* of G.

In both settings, $\kappa[v]$ is the *colour set* assigned to v. We denote the colours *available to v* (i.e. not appearing on the neighbourhood of v) by $\alpha(v)$, that is, $\alpha(v) = [0, k) \setminus \kappa[N(v)]$.

This is just one of several ways to think about fractional colourings; we hold the following proposition to be self-evident[3]:

Proposition 5.2.1. *Let G be a w-weighted graph. The following are equivalent:*

(1) G has a fractional $k \wr w$-colouring.

(2) There is an integer c and a multiset of ck stable sets of G such that every vertex v is contained in at least $c \cdot w(v)$ of them.

(3) There is a probability distribution on $\mathcal{S}$ such that for each $v \in V$, given a stable set S drawn from the distribution, $\Pr(v \in S) \geq w(v)/k$.

For more background on fractional colourings we refer the reader to [78]. At this point it is convenient to prove a useful consequence of Hall's Theorem that we will use repeatedly in Section 5.7:

Lemma 5.2.2. *Let κ be a partial fractional $k \wr w$-colouring of G, and let X be the set of vertices v with $|\kappa[v]| < w(v)$. Suppose for every $X' \subseteq X$ we have*

$$\left| \bigcup_{v \in X'} \alpha(v) \right| \geq \sum_{v \in X'} w(v). \tag{5.1}$$

Then there is a fractional $k \wr w$-colouring of G.

[3] The unweighted version is described as folklore in [42] and was used earlier in [53], and probably elsewhere.

Proof. We may assume (by uncolouring X) that for every $v \in X$, $\kappa[v] = \emptyset$. Thus we have a fractional $k \wr w$-colouring of $G - X$. By Proposition 5.2.1 there is an integer c and a multiset of ck stable sets $S_1, \ldots, S_{ck}$ of $G - X$ such that every vertex $v \notin X$ is in at least $c \cdot w(v)$ of them.

We now set up Hall's Theorem by constructing a bipartite graph H with vertex set $A \cup B$. Let A consist of, for every $v \in X$, $c \cdot w(v)$ copies of v. Let B consist of vertices $b_1, \ldots b_{ck}$. For every vertex a of A, let a be adjacent to b_i if and only if the vertex v in X corresponding to a has no neighbour in S_i. Equation (5.1) guarantees that for every $A' \subseteq A$, $|N(A')| \geq |A'|$, so by Hall's Theorem we have a matching in H saturating A. This matching corresponds to a partial mapping $m : [ck] \to X$ such that

- for every $i \in [ck]$ in the domain of m, $S_i \cup m(i)$ is a stable set, and
- for every $v \in X$, at least $c \cdot w(v)$ elements of $[ck]$ map to v.

Thus we can extend the stable sets S_i appropriately; by Proposition 5.2.1, this gives the desired fractional $k \wr w$-colouring of G. $\square$

We remark that this lemma is most sensibly applied when X is a clique.

5.2.1 Reed's Conjecture and fractional colourings

Our approach to fractionally colouring K_Δ-free graphs is inspired by the following result of Reed ([65], §21.3):

Theorem 5.2.3. *Every graph G satisfies $\chi_f(G) \leq \frac{1}{2}(\Delta(G) + 1 + \omega(G))$.*

This is the fractional relaxation of Reed's ω, Δ, χ conjecture [69], which proposes that every graph satisfies $\chi \leq \lceil \frac{1}{2}(\Delta+1+\omega) \rceil$. However, we do not consider the conjecture, or even the fractional relaxation, but rather a weighted version of a local strengthening observed by McDiarmid ([65], p.246). For a vertex v let $\omega(v)$ be the size of the largest clique containing v. Then:

Theorem 5.2.4. *Every graph G satisfies $\chi_f(G) \leq \max_v \frac{1}{2}(d(v) + 1 + \omega(v))$.*

The proof of this theorem was never published, but appears in Section 2.2 of [57] and is almost identical to the proof of Theorem 5.2.3. What we need is a new weighted version of this theorem, which we prove here. First we need some notation. For a vertex v let $\tilde{N}(v)$ denote the closed neighbourhood of v. Given a w-weighted graph G and a vertex $v \in V(G)$, we define:

- The *degree weight* $w_d(v)$ of v, defined as $\sum_{u\in\tilde{N}(v)} w(u)$.

- The *clique weight* $w_c(v)$ of v, defined as the maximum over all cliques C containing v of $\sum_{u\in C} w(u)$.

- The *Reed weight* $\rho_w(v)$ of v, defined as $\frac{1}{2}(w_d(v)+w_c(v))$ (we sometimes denote $\rho_{\mathbf{1}}$ by ρ). For a graph G, we define $\rho_w(G)$ as $\max_{v\in V(G)}\rho_w(v)$.

Our result is a natural generalization of McDiarmid's:

Theorem 5.2.5. *Every graph G satisfies $\chi_f^w(G) \le \rho_w(G)$.*

Proof. Let c be a positive integer such that for every v, $cw(v)$ is an integer; c exists since the weights are rational. Let G_w be the graph constructed from G by replicating each vertex v into a clique C_v of size $cw(v)$. [4] Applying Theorem 5.2.4 to G_w tells us that there is a fractional $c\rho_w(G)$-colouring κ_w of G_w. From this we construct a cw-fractional $c\rho_w(G)$-colouring κ of G by setting, for each $v\in V(G)$,

$$\kappa[v] = \kappa_w[C_v].$$

The result follows from Proposition 5.2.1 (3). □

As an aside, we mention the following strengthening of Theorem 5.2.4 that we proved in [34].

Theorem 5.2.6. *Every graph G satisfies $\chi_f(G) \le \max_{uv\in E(G)} \frac{1}{4}(d(u)+d(v)+\omega(u)+\omega(v)+2)$*

While McDiarmid's Theorem says that Theorem 5.2.3 can be strengthened so as to consider only the possible bounds achieved in the closed neighbourhood of a vertex, Theorem 5.2.6 says that the bounds can be taken in the closed neighbourhood of an edge. The idea is that a graph should be easy to colour if no two vertices with high ρ are adjacent. While this strengthening of Theorem 5.2.4 does not improve the analysis in this chapter, it may be of use in future work. We provide the proof of Theorem 5.2.6 in Section 5.8.

5.3 The general approach

Fix some $\Delta \ge 6$ and $0 < \epsilon \le \frac{1}{2}$, and suppose we wish to prove that $f(\Delta) \ge \epsilon$. Let G be a graph with maximum degree Δ and clique number $\omega \le \Delta - 1$; by Theorem 5.2.3 we know that $\chi_f(G) \le \Delta - \frac{1}{2}$

[4] That is, $x \in C_u$ and $y \in C_v$ are adjacent precisely if u, v are adjacent or if $u = v$ and x, y are distinct.

if $\omega \leq \Delta - 2$, so we assume G has clique number $\omega = \Delta - 1$. We define V_ω as the set of vertices in ω-cliques, and V'_ω as the set of vertices in V_ω with degree Δ. Let G_ω and G'_ω denote the subgraphs of G induced on V_ω and V'_ω respectively. Notice that a vertex v will have $\rho_{\mathbf{1}}(v) > \Delta - \frac{1}{2}$ if and only if v is in V'_ω. In plain language, our approach is:

1. Prove that in a minimum counterexample, G_ω has a nice structure.

2. Spend a little bit of weight on a fractional colouring that lowers the Reed weight for vertices in V'_ω at a rate of $(1 + \epsilon')$ per weight spent, i.e. we spend y weight and $(1 + \epsilon')y = y + \epsilon$. If y is sufficiently small, this lowers the maximum Reed weight over all vertices of G by $y + \epsilon$.

3. Having already "won" by ϵ, i.e. having lowered $\rho(G)$ by $y + \epsilon$ using only y colour weight, we can finish the colouring using Theorem 5.2.5.

More specifically, we find a vertex weighting w such that we have a fractional $y \wr w$-colouring of G, and such that $\rho_{(\mathbf{1}-w)}(G) \leq \Delta - y - \epsilon$. We then apply Theorem 5.2.5 to find a fractional $(\Delta - y - \epsilon) \wr (\mathbf{1} - w)$-colouring of G. Combining these two partial fractional colourings gives us a fractional $(\Delta - \epsilon)$-colouring of G.

Since any $v \notin V'_\omega$ satisfies $\rho_{\mathbf{1}}(v) \leq \Delta - \frac{1}{2}$, if $(1 + \epsilon')y \leq \frac{1}{2}$ we only need to ensure that ρ drops by $(1+\epsilon')y$ for vertices with $\rho_{\mathbf{1}}(v) = \Delta$. Actually we can ensure that while we do this, ρ also drops at a decent rate (easily at least $\frac{1}{2}y$) for vertices with $\rho < \Delta$. This means that we can spend more weight (i.e. increase y), thereby improving ϵ. It is in our interests to first worry about maximizing ϵ', then worry about maximizing y.

This method depends heavily on properly understanding the structure of vertices with $\rho_{\mathbf{1}}(v) = \Delta$. We simplify this structure through reductions, or if you prefer, the structural characterization of a minimum counterexample:

Lemma 5.3.1. *Fix some $\Delta \geq 5$ and some $\epsilon \leq \frac{1}{2}$, with the further restriction that $\epsilon \leq \frac{1}{3}$ if $\Delta = 5$. Let G be a graph with maximum degree Δ and clique number at most $\Delta - 1$ such that*

- *if $\Delta = 5$, no component of G is isomorphic to $C_5 \boxtimes K_2$,*
- *G has fractional chromatic number greater than $\Delta - \epsilon$, and*
- *no graph on fewer vertices has these properties.*

Then

(i) the maximum cliques of G are pairwise disjoint, and

(ii) there is no vertex v outside a maximum clique C such that $|N(v) \cap C| > 1$.

Together, these properties allow us to apply the following result of Aharoni, Berger, and Ziv [1]:

Theorem 5.3.2. *Let k be a positive integer and let G be a graph whose vertices are partitioned into cliques of size $\omega \geq 2k$. If G has maximum degree at most $\omega + k - 1$, then $\chi_f(G) = \omega$.*

Applying this theorem to an induced subgraph of G_ω is the key to proving that we can lower ρ quickly for any vertex v with $\rho_{\mathbf{1}}(v) = \Delta$. The proof of Lemma 5.3.1 is technical, independent of the main proof, and does not give insight to our approach, so we defer it to Section 5.7. We now consider the probability distribution on stable sets that, via Proposition 5.2.1, characterizes our initial colouring phase.

From now until Section 5.7, we consider G to be a graph with maximum degree $\Delta \geq 6$, clique number $\omega = \Delta - 1$, and satisfying properties *(i)* and *(ii)* of Lemma 5.3.1. We remark that Lemma 5.3.1 gives a characterization of minimum counterexamples with $\Delta = 5$; although we do not make use of the characterization here, it is likely to be useful in the future.

5.4 A probability distribution

For every vertex v of G, let $N_\omega(v)$ denote $N(v) \cap V_\omega$ and let $d_\omega(v)$ denote $|N_\omega(v)|$. The initial phase of our colouring involves choosing a random stable set S_w of G_w, then extending it randomly to a stable set S of G such that S_w and S have the following desirable properties:

1. For every $v \in V_\omega$,

$$\Pr(v \in S_\omega) = \tfrac{1}{\omega}. \tag{5.2}$$

2. For every $v \notin V_\omega$,

$$\begin{aligned}\Pr(N_\omega(v) \cap S_\omega = \emptyset) &\geq \sum_{i=0}^{3} \tfrac{1}{4} \Pr\left(\mathrm{Bin}(d_\omega(v), \tfrac{4}{\omega}) \leq i\right) \\ &= \sum_{i=0}^{3} \tfrac{4-i}{4} \Pr\left(\mathrm{Bin}(d_\omega(v), \tfrac{4}{\omega}) = i\right).\end{aligned} \tag{5.3}$$

3. For every $v \notin V_\omega$,

$$\Pr(v \in S) \geq \frac{\Pr(N_\omega(v) \cap S_\omega = \emptyset)}{(d(v) - d_\omega(v)) + 1} \geq \frac{\sum_{i=0}^{3} \frac{4-i}{4} \Pr\left(\mathrm{Bin}(d_\omega(v), \frac{4}{\omega}) = i\right)}{(d(v) - d_\omega(v)) + 1}. \tag{5.4}$$

4. S is maximal.

We will put weight on stable sets according to this distribution until we can no longer guarantee that ρ is dropping quickly. We discuss this stopping condition in Section 5.5.1.

5.4.1 Choosing S_ω

Denote the maximum cliques of G by $B_1, \ldots, B_\ell$, bearing in mind that they are vertex-disjoint. To choose S_ω we first select, for each $1 \leq i \leq \ell$, a subset B'_i of B_i of size 4, uniformly at random and independently for each i. Setting $\tilde{G}_\omega$ to be the subgraph of G induced on $\cup_i B'_i$, note that every vertex in B_i has at most two neighbours outside B_i and therefore $\Delta(\tilde{G}_\omega) \leq 5$. Thus Theorem 5.3.2 tells us that $\tilde{G}_\omega$ is fractionally 4-colourable. It follows from Proposition 5.2.1 that there is a probability distribution on the stable sets of $\tilde{G}_\omega$ such that given a stable set $\tilde{S}$ chosen from this distribution, for any $v \in \tilde{G}_\omega$, $\Pr(v \in \tilde{S}) = \frac{1}{4}$.

We therefore choose S_ω from this distribution, subject to our random choice of $\tilde{G}_\omega$. Since every $v \in G_\omega$ satisfies $\Pr(v \in \tilde{G}_\omega) = \frac{4}{\omega}$, for any $v \in G_\omega$ we clearly have $\Pr(v \in S_\omega) = \frac{1}{\omega}$, i.e. (5.2) holds. We must now prove that (5.3) holds (the reader may have noticed that any old fractional ω-colouring of G_ω would have given us S_ω satisfying (5.2)).

The first step is to observe that for $v \notin G_\omega$ and $0 \leq i \leq 3$,

$$\Pr\Big((N_\omega(v) \cap S_\omega = \emptyset) \mid (|N_\omega(v) \cap \tilde{G}_\omega| = i)\Big) \geq \frac{4-i}{4}. \tag{5.5}$$

This is because every neighbour of v in $\tilde{G}_\omega$ is in S_w with probability $\frac{1}{4}$, and in the worst case these events may be disjoint for all i such neighbours (we later conjecture that it is possible to avoid this worst case; this would improve our bounds substantially for $\Delta \in \{5, 6\}$).

The second step is to observe that for $v \notin G_\omega$ and $0 \leq i \leq d_\omega(v)$,

$$\Pr\left(|N_\omega(v) \cap \tilde{G}_\omega| = i\right) = \Pr\left(\mathrm{Bin}(d_\omega(v), \tfrac{4}{\omega}) = i)\right). \tag{5.6}$$

To see this, note that Lemma 5.3.1 tells us that any two neighbours $x, y \in G_\omega$ of v are in different blocks B_i, and therefore the events of x being in $\tilde{G}_\omega$ and y being in $\tilde{G}_\omega$ are independent. Equation (5.3) follows immediately from Equations (5.5) and (5.6).

5.4.2 Choosing S

Given a choice of S_ω, we randomly extend to S as follows:

1. Choose an ordering π of $V(G) \setminus V_\omega$ uniformly at random, and label the vertices of $V(G) \setminus V_\omega$ as $v_1, \ldots, v_r$ in the order in which they appear in π.

2. Set $S = S_\omega$.

3. For each of $i = 1, \ldots, r$ in order, put v_i in S if and only if it currently has no neighbour in S.

Since every vertex in V_ω is in S_ω or has a neighbour in S_ω, and every vertex not in V_ω is in S or has a neighbour in S, we can see that S is always a maximal stable set. A vertex $v_i \in V(G) \setminus V_\omega$ is in S if it has no neighbours in S_ω, and it is not adjacent to any $v_j \in V(G) \setminus V_\omega$ for $j < i$. Since we choose π uniformly at random, any vertex $v \in V(G) \setminus V_\omega$ satisfies

$$\Pr\left((v \in S) \mid (N_\omega(v) \cap S_\omega = \emptyset)\right) \geq \frac{1}{|N(v) \setminus V_\omega| + 1}. \tag{5.7}$$

Equation (5.4) follows immediately from Equation (5.7).

5.4.3 Bounding the rate at which ρ initially decreases

Suppose we spend weight y to colour G according to the probability distribution on S that we just described. That is, for $S' \in \mathcal{S}(G)$, we place weight $q(S')$ on S', where

$$q(S') = y \cdot \Pr(S = S').$$

Then we wish to argue that $\rho(G)$ drops by $(1 + \epsilon')y$ for some positive ϵ'. For now, to avoid consideration of stopping conditions[5], suppose that y is very small ($y = \frac{1}{10}$ will do for now).

For a fixed Δ and $0 \leq d \leq \Delta$ we define $p(\Delta, d)$ as

$$p(\Delta, d) = \frac{\sum_{i=0}^{3} \frac{1}{4} \Pr\left(\mathrm{Bin}(d, \frac{4}{\omega}) \leq i\right)}{(\Delta - d) + 1}, \tag{5.8}$$

noting that a vertex $v \notin G_\omega$ with $d_\omega(v) = d$ is in S with probability at least $p(\Delta, d)$. Following this, we define

$$\mu_k(\Delta) = \min_{0 \leq d \leq k} p(\Delta, d) \qquad \text{and} \qquad \mu(\Delta) = \mu_\Delta(\Delta) = \min_{0 \leq d \leq \Delta} p(\Delta, d),$$

noting that any vertex $v \notin G_\omega$ is in S with probability at least $\mu(\Delta)$.

Lemma 5.4.1. *For every vertex $v \in V(G)$, $\Pr(v \in S) \geq \mu(\Delta)$.*

Proof. To see this we only need to prove that $v \in G_\omega$ is in S with probability at least $\mu(\Delta)$. This is clearly the case since v is in S with probability $\frac{1}{\omega} > \frac{1}{\Delta+1} = p(\Delta, 0) \geq \mu(\Delta)$. □

We now set ϵ' to be $\mu(\Delta)$. Table 5.2 gives some computed values of $\mu(\Delta)$, and Figure 5.3 shows some values of $p(\Delta, d)$. (We will define and consider $\tilde{y}(\Delta)$ in the next section.) These numbers were computed using Sage; the code is available at [31].

Lemma 5.4.2. *For any vertex v in V'_ω, $E(|S \cap \tilde{N}(v)|) \geq 1 + 2\epsilon'$.*

Proof. Since v is in some B_i and has degree $\Delta = 1 + \omega$, v has exactly two neighbours outside B_i. Each is in S with probability at least ϵ', and S contains a vertex in B_i with probability 1. Therefore the lemma follows from linearity of expectation. □

[5] i.e. when y is large enough to make our model fail

Δ	$\mu(\Delta)$	$\mu(\Delta)(\Delta+1)$	d for which $\mu(\Delta) = p(\Delta, d)$	$\tilde{y}(\Delta)$	$\tilde{y}(\Delta)\mu(\Delta)$
6	.029376	.205	6	1.518	0.04459
7	.054869	.439	6	1.640	0.08999
8	.062947	.567	7	1.804	0.11353
9	.066406	.664	7	1.969	0.13077
10	.066328	.730	8	2.146	0.14234
100	.009843	.994	29	20.003	0.19691
1000	.000998	.999	135	199.979	0.19973

Table 5.2: Some values of $\mu(\Delta)$, where they are achieved, and corresponding values of $\tilde{y}$, which we discuss later. Note that $p(\Delta, 0) = 1/(\Delta+1)$ is an upper bound for $\mu(\Delta)$. These values are calculated in [31].

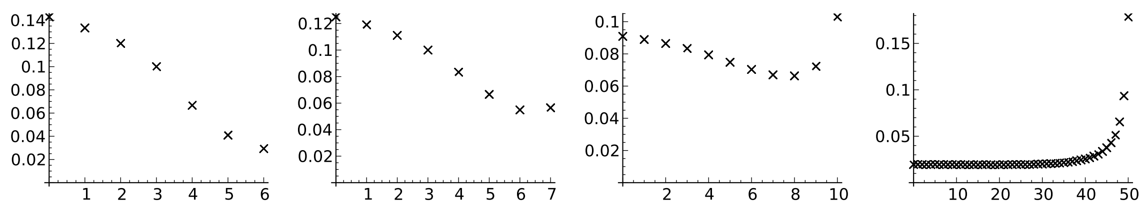

Figure 5.3: Values of d versus $p(\Delta, d)$ for $\Delta \in \{6, 7, 10, 50\}$.

Let v be a vertex in $V'_\omega \cap B_i$. Since $E(|S \cap B_i|) = 1$, and B_i is the unique maximum clique containing v, we know that at the outset, when we spend weight y, $\rho(v)$ will drop by $\frac{1}{2}(1+1+2\epsilon')y = (1+\epsilon')y$.

For $k \leq \omega$, let V_k be the set of vertices in a clique of size k but not a clique of size $k+1$, noting that these vertex sets partition $V(G)$. We note the following.

Lemma 5.4.3. *If $4 \leq k \leq \omega - 1$ and v is a vertex in V_k, then v has at most $\Delta + 1 - k$ neighbours in V_ω.*

Proof. It suffices to prove that if X is a k-clique containing v, then X does not intersect an ω-clique. Suppose it does intersect some B_i, and note that it may only intersect B_i once by Lemma 5.3.1. Since any vertex in B_i has at most two neighbours outside B_i, $|X|$ must be at most 3, a contradiction. □

Corollary 5.4.4. *If $v \in V_k$ for some $4 \leq k \leq \omega - 1$, then $\Pr(v \in S) \geq \mu_{\Delta+1-k}(\Delta)$.*

5.5 The initial colouring

The probability distribution described in the previous section tells us what to do in the initial colouring phase: we choose colour classes according to the distribution. The only thing we need to worry about is giving a vertex more than colour weight 1. To avoid this, when a vertex is full we simply delete it and continue as though it never existed. This is the same approach taken in the proof of Theorems 5.2.3 and 5.2.4. Vertices in V_ω will never be full before the end of our process.

Lemma 5.5.1. *For any $y \in [0, \omega]$ there exists a vertex weighting w and a fractional $y \wr w$-colouring of G such that w satisfies the following conditions:*

(a) Every vertex v in V_ω has $w(v) = y/\omega$.

(b) For $0 \leq \ell \leq \Delta$, every vertex $v \notin V_\omega$ with exactly ℓ neighbours in V_ω has $w(v) \geq \min\{p(\Delta, \ell)y, 1\}$.

(c) For $1 \leq k < \omega$, every clique X of size k has $w(X) \geq k \min\{\mu(\Delta)y, 1\}$.

(d) For $4 \leq k < \omega$, every clique X of size k has $w(X) \geq k \min\{\mu_{\Delta+1-k}(\Delta)y, 1\}$.

(e) Every vertex v with $w(v) < 1$ has $w(\tilde{N}(v)) \geq y$.

Note that $\mu(\Delta)y$ and $\mu_{\Delta+1-k}(\Delta)y$ are less than 1.

Proof. We proceed using the following algorithm.
Initially, set $H_0 = G$, set $\mathit{leftover}_0 = y$, and set $\mathit{capacity}_0(v) = 1$ for every vertex in H_0. For $i = 0, 1, \ldots$ do the following.

1. Let R_i be a random stable set drawn from the distribution giving S described in Section 5.4. For every vertex v we set $\mathit{prob}_i(v)$ as $\Pr(v \in R_i)$.

2. Set y'_i to be $\min_{v \in V(H_i)}(\mathit{capacity}_i(v)/\mathit{prob}_i(v))$, and set y_i to be $\min\{\mathit{leftover}_i, y'_i\}$.

3. For every $v \in V(H_i)$, set $w_i(v)$ to be $\mathit{prob}_i(v)y_i$.

4. For every $v \in V(H_i)$, set $\mathit{capacity}_{i+1}(v)$ to be $\mathit{capacity}_i(v) - w_i(v)$.

5. Set $\mathit{leftover}_{i+1}$ to be $\mathit{leftover}_i - y_i$.

6. If $\mathit{leftover}_{i+1} = 0$, we terminate the process. Otherwise, let U_i be the vertex set $\{v \in V(H_i) \mid \mathit{capacity}_{i+1}(v) = 0\}$, and set H_{i+1} to be $H_i - U_i$.

Let ν denote the value of i for which $\mathit{leftover}_{i+1} = 0$. For every vertex v, let $w(v) = \sum_{i=0}^{\nu} w_i(v)$. Observe that $y = \sum_{i=0}^{\nu} y_i$.

We first prove that this process must terminate. Our choice of each y_i implies that either $\mathit{leftover}_{i+1} = 0$, or $|U_{i+1}| < |U_i|$. Thus we terminate after at most $|V(G)|$ iterations. Now observe that every vertex $v \in G_\omega$ has $\mathit{prob}_i(v) = 1/\omega$ throughout the process, and therefore $\mathit{capacity}_\nu(v) > 0$ since $\mathit{leftover}_0 = y \leq \omega$ (this can easily be proved by induction on i). Note that (a) also follows from this observation. As a further consequence, we can see that G_ω is a subgraph of every H_i.

We claim that we actually have a collection of fractional $y_i \wr w_i$-colourings for $0 \leq i \leq \nu$. To see this we simply appeal to Proposition 5.2.1 (3), noting that $\Pr(v \in R_i) = w_i(v)/y_i$. Since $w = \sum_{i=1}^{\nu} w_i$ and $y = \sum_{i=0}^{\nu} y_i$, it follows immediately that these colourings together give us a fractional $y \wr w$-colouring of G.

To prove (b), we take $v \notin V_\omega$ with ℓ neighbours in V_ω, and assume that $w(v) < 1$, otherwise we are done. Since every H_i contains G_ω, we can see that

$$\Pr(v \in R_i) \geq \frac{\sum_{i=0}^{3} \frac{4-i}{4} \Pr\left(\mathrm{Bin}(d_\omega(v), \frac{4}{\omega}) = i\right)}{|N(v) \cap V(H_i)| - d_\omega(v) + 1} \geq p(\Delta, \ell). \tag{5.9}$$

Consequently $\mathit{prob}_i(v) \geq p(\Delta, \ell)$ for all i, and (b) follows. Note that (c) follows immediately from (b). Similarly, (d) follows from (b) and Lemma 5.4.3.

To see that (e) holds, simply note that R_i is always a maximal stable set in H_i. Therefore if $w(v) < 1$, then $\mathit{capacity}_\nu(v) > 0$, thus $v \in H_i$ for every i, meaning that R_i intersects $\tilde{N}(v)$ with probability 1. □

5.5.1 Maximizing the expenditure

Here we consider the best possible choice of y in Lemma 5.5.1. The optimal value of y will be the largest possible such that the upper bound on $\rho_{\mathbf{1}-w}(G)$ is still achieved by some vertex in G_ω. If we increase y beyond this point, we will find that $\rho_{\mathbf{1}-w}(G)$ is no longer guaranteed to drop as fast as y increases.

In light of this goal, for $1 \leq k \leq 3$ we let $\tilde{y}_k(\Delta)$ denote the maximum value of y such that

$$(1+\mu(\Delta))y \leq \tfrac{1}{2}(\Delta - 1 - k) + \left(\tfrac{1}{2} + \tfrac{1}{2}k\mu(\Delta)\right) y. \tag{5.10}$$

For $4 \leq k \leq \Delta - 2$ we let $\tilde{y}_k(\Delta)$ denote the maximum value of y such that

$$(1+\mu(\Delta))y \leq \tfrac{1}{2}(\Delta - 1 - k) + \left(\tfrac{1}{2} + \tfrac{1}{2}k\mu_{\Delta+1-k}(\Delta)\right) y. \tag{5.11}$$

Now let $\tilde{y}(\Delta)$ denote $\min\{\min_k \tilde{y}_k(\Delta), \omega, \frac{\omega-3}{1-3\mu(\Delta)}\}$ (the latter two bounds are for convenience of proof, and do not affect our results). Our initial colouring phase culminates in the following consequence of Lemma 5.5.1.

Theorem 5.5.2. *For any $0 \leq y \leq \tilde{y}(\Delta)$, there is a vertex weighting w and fractional $y \wr w$-colouring of G such that $\rho_{\mathbf{1}-w}(G) \leq \Delta - (1+\mu(\Delta))y$.*

Proof. Let v be any vertex in G; it suffices to prove that $\rho_{\mathbf{1}-w}(v) \leq \Delta - (1+\mu(\Delta))y$. We take the fractional $y \wr w$-colouring guaranteed by Lemma 5.5.1.

First suppose $v \in G_\omega$, and assume without loss of generality that $v \in B_1$. We know that $w(B_1) = y$ by Lemma 5.5.1(a), and that for any $u \in \tilde{N}(v) \setminus B_1$, $w(u) \geq y\mu(\Delta)$ (by Lemma 5.5.1(b)). Therefore $|\tilde{N}(v)| - w(\tilde{N}(v)) \leq \omega - y + 2(1 - y\mu(\Delta)) = \Delta + 1 - y - 2y\mu(\Delta)$. We now claim that for any clique C containing v, $|C| - w(C) \leq \omega - y$. Clearly $w(B_1) = y$. For C not equal to B_1, Lemma 5.3.1 tells us that $|C| \leq 3$. Therefore $|C| - w(C) \leq 3 - 3y\mu(\Delta)$. If $\omega - y < 3 - 3y\mu(\Delta)$, then $\omega - 3 < y(1 - 3\mu(\Delta))$, contradicting the fact that $y \leq \tilde{y}(\Delta) \leq \frac{\omega-3}{1-3\mu(\Delta)}\}$. Therefore $|C| - w(C) \leq \omega - y = \Delta - 1 - y$. Thus

$$\rho_{\mathbf{1}-w}(v) \leq \tfrac{1}{2}(\Delta - 1 - y) + \tfrac{1}{2}(\Delta + 1 - y - 2y\mu(\Delta)) = \Delta - (1+\mu(\Delta))y. \tag{5.12}$$

Now suppose that v is not in V_ω, and let C be a clique containing v such that $|C| - w(C)$ is maximum. Denote the size of C by k. By Lemma 5.5.1(e), we know that $w(\tilde{N}(v)) \geq y$, so

$$|\tilde{N}(v)| - w(\tilde{N}(v)) \leq \Delta + 1 - y. \tag{5.13}$$

Therefore to prove that $\rho_{\mathbf{1}-w}(v) \leq \Delta - (1+\mu(\Delta))y$, it is sufficient to prove that

$$k - w(C) \leq \Delta - 1 - y - 2y\mu(\Delta), \tag{5.14}$$

i.e.

$$(\mu(\Delta) + \tfrac{1}{2})y \leq \tfrac{1}{2}(\Delta - 1 - k) + \tfrac{1}{2}w(C). \tag{5.15}$$

By Lemma 5.5.1(c) we know that $w(C) \geq k\mu(\Delta)y$. If $k \geq 4$, by Lemma 5.5.1(d) we know that $w(C) \geq k\mu_{\Delta+1-k}(\Delta)y$. We also know that $y \leq \tilde{y}(\Delta) \leq \tilde{y}_k(\Delta)$, so if $k \leq 3$ then

$$(1+\mu(\Delta))y \leq \tfrac{1}{2}(\Delta - 1 - k) + \left(\tfrac{1}{2} + \tfrac{1}{2}k\mu(\Delta)\right)y, \tag{5.16}$$

and if $k \geq 4$ then

$$(1+\mu(\Delta))y \leq \tfrac{1}{2}(\Delta - 1 - k) + \left(\tfrac{1}{2} + \tfrac{1}{2}k\mu_{\Delta+1-k}(\Delta)\right)y. \tag{5.17}$$

In either case,

$$(1+\mu(\Delta))y \leq \tfrac{1}{2}(\Delta - 1 - k) + \left(\tfrac{1}{2}y + \tfrac{1}{2}w(C)\right), \tag{5.18}$$

so

$$(\mu(\Delta) + \tfrac{1}{2})y \leq \tfrac{1}{2}(\Delta - 1 - k) + \tfrac{1}{2}w(C), \tag{5.19}$$

as desired. Thus $\rho_{\mathbf{1}-w}(v) \leq \Delta - (1+\mu(\Delta))y$. □

Since equations 5.10 and 5.11 are linear, we can easily find the optimal values of $\tilde{y}_k(\Delta)$ by solving for

$$\tilde{y}_k(\Delta) = \frac{\frac{1}{2}(\Delta - 1 - k)}{\frac{1}{2} + \mu(\Delta) - \frac{1}{2}k\mu(\Delta)} \tag{5.20}$$

when $k \leq 3$ and for

$$\tilde{y}_k(\Delta) = \frac{\frac{1}{2}(\Delta - 1 - k)}{\frac{1}{2} + \mu(\Delta) - \frac{1}{2}k\mu_{\Delta+1-k}(\Delta)} \tag{5.21}$$

when $\Delta - 2 \geq k \geq 4$. See [31] and Table 5.2 for numerical values.

5.6 Proving the main result

We now have enough results in hand to prove the main result easily.

Theorem 5.6.1. *For $\Delta \geq 6$, let G be a graph with maximum degree Δ and clique number at most $\Delta - 1$. Then G has fractional chromatic number at most $\Delta - \min\{\frac{1}{2}, \tilde{y}(\Delta)\mu(\Delta)\}$.*

Proof. Let G be a minimum counterexample; Theorem 5.2.4 tells us that G has maximum degree Δ and clique number $\omega = \Delta - 1$. Lemma 5.3.1 tells us that all ω-cliques of G are disjoint, and that no vertex v has two neighbours in an ω-clique not containing v.

We may therefore set $y = \tilde{y}(\Delta)$ and apply Theorem 5.5.2. This gives us a vertex weighting w and fractional $y \wr w$-colouring of G such that $\rho_{\mathbf{1}-w}(G) \leq \Delta - (1 + \mu(\Delta))y$. By Theorem 5.2.5, $\chi_f^{\mathbf{1}-w} \leq \rho_{\mathbf{1}-w}(G) \leq \Delta - (1 + \mu(\Delta))y$. That is, there is a fractional $(\Delta - (1 + \mu(\Delta))y) \wr (\mathbf{1} - w)$-colouring of G. Combining this colouring with the initial fractional $y \wr w$-colouring gives us a fractional $(\Delta - \tilde{y}(\Delta)\mu(\Delta))$-colouring, which tells us that $\chi_f(G) \leq \Delta - \tilde{y}(\Delta)\mu(\Delta)$. $\square$

For all values of Δ we have investigated, $\tilde{y}(\Delta)\mu(\Delta) < \frac{1}{5}$. We believe that this is always the case.

5.7 The structural reduction

In this section we prove Lemma 5.3.1, which tells us that we need only consider graphs whose maximum cliques behave nicely. First observe that every proper induced subgraph of G is fractionally $(\Delta - \epsilon)$-colourable, since deleting vertices from a graph with $\Delta = 5$ cannot create a copy of $C_5 \boxtimes K_2$. We prove the lemma in two parts:

Lemma 5.7.1. *Part (i) of Lemma 5.3.1 holds.*

Lemma 5.7.2. *Part (ii) of Lemma 5.3.1 holds.*

5.7.1 Part (i)

We actually split the proof of Lemma 5.7.1 into three parts. Suppose C and C' are two intersecting ω-cliques. Since $\omega = \Delta - 1$, we can immediately observe that $|C \cap C'| \geq \omega - 2$. Therefore Lemma 5.7.1 follows as an easy corollary of the next three Lemmas 5.7.3, 5.7.4, 5.7.5. Throughout this section we will make implicit use of the fact that every vertex in G has at least $\Delta - 1$ neighbours, as is trivially implied by the minimality of G. Furthermore note that whenever we reduce G to a graph G', no component of which is 5-regular, no component of G' can be isomorphic to $C_5 \boxtimes K_2$.

Lemma 5.7.3. *G does not contain three ω-cliques mutually intersecting in $\omega - 1$ vertices.*

Proof. Suppose that G contains an $(\omega-1)$-clique X and vertices x_1, x_2, x_3 each of which is complete to X. Because there is no $(\omega+1)$-clique, $\{x_1, x_2, x_3\}$ is a stable set. Let $G' = G \setminus (X \cup \{x_1, x_2, x_3\})$; as previously observed, since G' is a proper induced subgraph of G, there is a fractional $(\Delta - \epsilon)$-colouring κ of G'. We extend κ to a fractional $(\Delta - \epsilon)$-colouring of G to obtain a contradiction, beginning by colouring $\{x_1, x_2, x_3\}$ using weight at most $2 - \epsilon$.

First suppose $\Delta = 5$, so $\epsilon \leq \frac{1}{3}$. Since each x_i has at most two neighbours in G', we have $|\alpha(x_i)| \geq \Delta - \epsilon - 2$. Note that $|\alpha(x_i) \cup \alpha(x_j)| \leq \Delta - \epsilon$, so for any $\{i, j\} \subseteq \{1, 2, 3\}$ we have $|\alpha(x_i) \cap \alpha(x_j)| \geq \Delta - \epsilon - 4 \geq 1 - \epsilon \geq \frac{2}{3}$. We extend κ to $\{x_1, x_2, x_3\}$ such that

- $|\kappa[x_1] \cap \kappa[x_2]| \geq \frac{2}{3}$, and
- There exist disjoint subsets s_1 and s_2 of $\kappa[x_3]$, each of size $\frac{1}{3}$, such that $s_1 \subset \kappa[x_1]$ and $s_2 \subset \kappa[x_2]$.

To do this, we first give x_1 and x_2 weight $\frac{2}{3}$ of colour in common, then give x_1 and x_3 weight $\frac{1}{3}$ of colour each such that all the colour on x_3 is in $\kappa[x_1]$, then give x_2 and x_3 weight $\frac{1}{3}$ of colour each such that all the new colour on x_3 is in $\kappa[x_2]$. Finally we complete the colouring of x_3 arbitrarily. Confirming that this is possible is straightforward given the pairwise intersections of $\alpha(x_i)$. Furthermore since $|\kappa[\{x_1, x_2\}]| \leq \frac{4}{3}$ and at least $\frac{2}{3}$ of the colour in $\kappa[x_3]$ is in $\kappa[\{x_1, x_2\}]$, we use weight at most $2 - \epsilon$ on $\{x_1, x_2, x_3\}$.

Now suppose $\Delta \geq 6$, so $\epsilon \leq \frac{1}{2}$. Our approach is the same as before, except now for any $\{i, j\} \subseteq \{1, 2, 3\}$ we have $|\alpha(x_i) \cap \alpha(x_j)| \geq \Delta - \epsilon - 4 \geq \frac{3}{2}$. Thus we can proceed by giving x_1 and x_2 weight $\frac{1}{2}$ of colour in common, then assign s_1 and s_2 as before, but with size $\frac{1}{2}$ each. Again we use weight at most $2 - \epsilon$ on $\{x_1, x_2, x_3\}$.

We now have $\{v \in V(G) : |\kappa(v)| < 1\} = V(X)$. For every $v \in V(X)$, we have $|\alpha(v)| \geq \Delta - \epsilon - (2 - \epsilon) = \omega - 1 = |V(X)|$. We may therefore apply Lemma 5.2.2 and extend κ to a fractional $(\Delta - \epsilon)$-colouring of G. □

Lemma 5.7.4. *G does not contain two ω-cliques intersecting in $\omega - 1$ vertices.*

Proof. Suppose C and C' are two ω-cliques intersecting in $\omega - 1$ vertices. Let $v_1, \ldots, v_{\omega-1}$ be the vertices in $C \cap C'$, let x be the vertex in $C \setminus C'$, and let y be the vertex in $C' \setminus C$, noting that x

and y are nonadjacent. For $1 \leq i \leq \omega - 1$, if v_i has a neighbour outside $C \cup C'$ call it u_i.

Claim 1. *There exists a fractional $(\Delta - \epsilon)$-colouring κ of $G \setminus (C \cap C')$ satisfying the following:*

(1) If $\Delta = 5$, then $|\kappa[\{x, y\}]| \leq 1 + \epsilon$.

(2) If $\Delta \geq 6$, then $|\kappa[\{x, y\}]| = 1$.

(3) $|\bigcap_{i \leq \omega-1} \kappa[u_i]| \leq \epsilon$.

We first show how the claim implies the lemma. For each $v_i \in C \cap C'$, $|\alpha(v_i)| \geq \Delta - \epsilon - |\kappa[\{x, y, u_i\}]| \geq \Delta - \epsilon - 1 - |\kappa[\{x, y\}]| \geq \omega - 2$. Thus to apply Lemma 5.2.2 and extend κ to G it is enough to show that $|\bigcup_{i \leq \omega-1} \alpha(v_i)| \geq \omega - 1$. Indeed, the set of colours available to at least some of the vertices in $C \cap C'$ are those which are not forbidden to all of them: If $\Delta \geq 6$, then

$$\left|\bigcup_{i \leq \omega-1} \alpha(v_i)\right| \geq \Delta - \epsilon - |\kappa[\{x, y\}]| - \left|\bigcap_{i \leq \omega-1} \kappa[u_i]\right| \geq \omega - 2\epsilon \geq \omega - 1$$

and if $\Delta = 5$,

$$\left|\bigcup_{i \leq \omega-1} \alpha(v_i)\right| \geq \omega - 3\epsilon \geq \omega - 1.$$

Lemma 5.2.2 then guarantees a fractional $(\Delta - \epsilon)$-colouring of G, a contradiction.

Proof of Claim 1. There are two cases. Note that by Lemma 5.7.3, if u_i exists for each i then $|\{u_i : 1 \leq i \leq \omega - 1\}| \geq 2$.

Case 1: $2 \leq |\{u_i : 1 \leq i \leq \omega - 1\}| < \omega - 1$ and u_i exists for each i.
Without loss of generality suppose that $u_1 = u_2$ and consider $G' = G \setminus (C \cup C' \cup \{u_1\})$. Again, since G' is a proper induced subgraph of G, there exists a fractional $(\Delta - \epsilon)$-colouring κ of G'. We extend κ to a fractional colouring of $G \setminus (C \cap C')$, first colouring x and y, then u_1.

Each of x and y has at most two neighbours in G' so we have $|\alpha(x)|, |\alpha(y)| \geq \Delta - \epsilon - 2$. Since $|\alpha(x) \cup \alpha(y)| \leq \Delta - \epsilon$ it follows that $|\alpha(x) \cap \alpha(y)| \geq \Delta - \epsilon - 4 \geq 1$ when $\Delta \geq 6$, and $|\alpha(x) \cap \alpha(y)| \geq 1 - \epsilon$ when $\Delta = 5$. We extend κ in the obvious way so that if $\Delta \geq 6$ then $\kappa[x] = \kappa[y]$, and if $\Delta = 5$ then $|\kappa[x] \cap \kappa[y]| \geq 1 - \epsilon$, satisfying (1) and (2). It remains to colour u_1. Note that u_1 has degree at most $\omega - 1$ in $G \setminus (C \cap C')$ so $|\alpha(u_1)| \geq 2 - \epsilon$. Because $|\bigcap_{3 \leq i \leq \omega-1} \kappa[u_i]| \leq 1$, we can choose $\kappa[u_1]$ from $\alpha(u_1)$ in such a way that $|\kappa[u_1] \cap \bigcap_{3 \leq i \leq \omega-1} \kappa[u_i]| \leq \epsilon$, satisfying (3).

Case 2: $|\{u_i : 1 \leq i \leq \omega - 1\}| = \omega - 1$ or u_i does not exist for some i.

If there exists an edge u_iu_j in G for some $i \neq j$, then let $G' = G \setminus (C \cup C')$. Otherwise choose $i \neq j$ such that adding the edge u_iu_j to $G \setminus (C \cup C')$ yields a graph with $\omega < \Delta$ and let $G' = G \setminus (C \cup C') \cup u_iu_j$. To see that such i and j exist, consider u_1, u_2 and u_3 and suppose that each pair of these has an $(\omega - 1)$-clique in the common neighbourhood. Because $\Delta = \omega + 1$ there must be a vertex contained in each of these three cliques, but Lemma 5.7.3 forbids the existence of three pairwise intersecting ω-cliques.

By the minimality of G, there exists a fractional $(\Delta - \epsilon)$-colouring κ of G'. We need to extend κ to x and y. Because each of x and y has at most two neighbours in G' we have $|\alpha(x)|, |\alpha(y)| \geq \Delta - \epsilon - 2$. It follows that $|\alpha(x) \cap \alpha(y)| \geq 1$ if $\Delta \geq 6$ and $|\alpha(x) \cap \alpha(y)| \geq 1 - \epsilon$ if $\Delta = 5$ so we can extend κ in the obvious way to satisfy (1) and (2). Requirement (3) is guaranteed by the existence of the edge u_iu_j. This proves the claim. $\square$

As we have shown, the claim implies the lemma. $\square$

Lemma 5.7.5. *G does not contain two ω-cliques intersecting in $\omega - 2$ vertices.*

Proof. Suppose C and C' are two ω-cliques intersecting in $\omega - 2$ vertices. Let x, x' be the vertices in $C \setminus C'$ and let y, y' be those in $C' \setminus C$. Suppose that x is adjacent to y. Then C and $(C \setminus \{x'\}) \cup \{y\}$ are two ω-cliques intersecting in $\omega - 1$ vertices, contradicting Lemma 5.7.4. By symmetry we may therefore assume there is no edge between $\{x, x'\}$ and $\{y, y'\}$. The case $\Delta = 5$ gives us the most difficulty by far, so we deal with it separately.

Case 1: $\Delta \geq 6$.

We construct the graph G' from G by identifying x, y and x', y' into two new vertices z and z', respectively, and deleting $C \cap C'$. Clearly $\Delta(G') \leq \Delta(G)$. If G' contains a Δ-clique, then since z and z' have degree at most 5, we have $\Delta = 6$, and furthermore the Δ-clique must contain both z and z'. Thus there is a set of four vertices C'' forming a 6-clique with z and z'. This means there must be eight edges between $\{x, x', y, y'\}$ and C'' in G.

If any vertex in C'' has a neighbour outside of $\{x, x', y, y'\}$ then C'' is a clique cutset in G, contradicting the fact that every proper induced subgraph of G is fractionally $(\Delta - \epsilon)$-colourable. Thus $V(G) = V(C) \cup V(C') \cup V(C'')$. Further, $(N(x) \cup N(y)) \cap V(C'') = V(C'')$ and $(N(x') \cup$

$N(y')) \cap V(C'') = V(C'')$. If x and x' have the same two neighbours in C'' then G is the graph $(C_5 \boxtimes K_3) - 4v$ shown in Figure 5.2, contradicting the assumption that $\chi_f(G) > \Delta - \frac{1}{2}$. Thus x and y' have a common neighbour in C''. We may safely switch the roles of y and y' in this case to ensure that $\omega(G') \leq \omega(G)$.

It now follows from the minimality of G that there exists a fractional $(\Delta - \epsilon)$-colouring κ of G'. By unidentifying x, y and x', y', we may think of κ as a fractional colouring of $G \setminus (C \cap C')$ where $\kappa[x] = \kappa[y]$ and $\kappa[x'] = \kappa[y']$. We now extend κ to a $(\Delta - \epsilon)$-colouring of G. We have $\{v \in V(G) : |\kappa(v)| < 1\} = V(C \cap C')$. Further, for each $v \in V(C \cap C')$, $|\alpha(v)| \geq \Delta - \epsilon - 2 \geq \omega - 2$. Thus applying Lemma 5.2.2 gives the extension of κ to G, a contradiction.

Case 2: $\Delta = 5$.

We construct G' as in the previous case. If G' has a fractional $(\Delta - \epsilon)$-colouring, we reach a contradiction as before. Otherwise, it must be the case that G' contains a Δ-clique or $C_5 \boxtimes K_2$. To deal with these cases we prove four claims.

Our first claim is that no vertex in $G \setminus (C \cup C')$ has a neighbour in both $\{x, x'\}$ and $\{y, y'\}$. To prove this, assume for a contradiction that x and y have a common neighbour $w \notin C \cup C'$. Let $G'' = G \setminus (C \cup C')$. By the minimality of G there exists a fractional $(\Delta - \epsilon)$-colouring κ of G'' that we now extend to a fractional colouring of G. We do so in two steps, first colouring $\{x, y, x', y'\}$.

Since x and y have a common neighbour plus at most one other coloured neighbour each, we have $|\alpha(x) \cap \alpha(y)| \geq \Delta - \epsilon - 3$. On the other hand, each of x' and y' has at most two coloured neighbours, so $|\kappa[N(x') \cup N(y')]| \leq 4$. We choose $\kappa[x] = \kappa[y]$ from $\alpha(x) \cap \alpha(y)$ maximizing its intersection with $\kappa[N(x') \cup N(y')]$, so that after colouring x and y we still have $|\kappa[N(x') \cup N(y')]| \leq 4$. This means that $|\alpha(x') \cap \alpha(y')| \geq 1 - \epsilon$ so we may choose colours for x' and y' so that $|\kappa[x'] \cap \kappa[y']| \geq 1 - \epsilon$. This ensures that $|\kappa[\{x, y, x', y'\}]| \leq 2 + \epsilon$.

It remains to extend the colouring to the vertices in $C \cap C'$. For each vertex $v \in V(C \cap C')$, $|\alpha(v)| \geq \Delta - \epsilon - (2 + \epsilon) \geq \omega - 2$. Applying Lemma 5.2.2, we find a fractional $(\Delta - \epsilon)$-colouring of G, a contradiction. This proves the first claim, so we may henceforth assume no vertex in $G \setminus (C \cup C')$ has a neighbour in both $\{x, x'\}$ and $\{y, y'\}$.

Our second claim is that G does not contain an edge cut of size at most two. For if it does, we can take a fractional $(\Delta - \epsilon)$-colouring of either side of this cut. The edges of the cut have

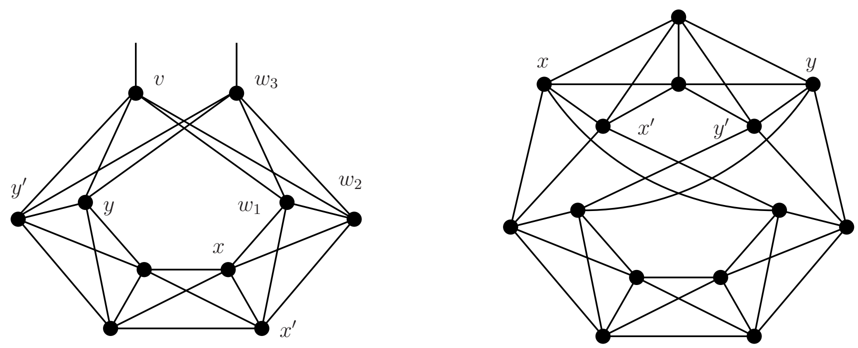

Figure 5.4: Left: If G contains $(C_5 \boxtimes K_2) - e$, we can easily reduce. Right: Reducing on the six top vertices renders G' isomorphic to $C_5 \boxtimes K_2$.

colour weight at most four on their endpoints, and since $\Delta - \epsilon > 2 \cdot 2$, we can safely merge the $(\Delta - \epsilon)$-colouring of either side of the cut into a fractional $(\Delta - \epsilon)$-colouring of G, a contradiction. This proves the second claim.

Our third claim is that G' does not contain a Δ-clique. Suppose it does; we now investigate the structure of G. In $G \setminus (C \cup C')$ there is an $\omega - 1$ clique C'', each vertex of which is complete (in G) to either $\{x, x'\}$ or $\{y, y'\}$, since no vertex has neighbours in both $\{x, x'\}$ and $\{y, y'\}$ (by the first claim). Since $|C''| = 3$, we may assume that x and x' have two common neighbours w_1 and w_2 in C'', and y and y' have a common neighbour w_3 in $C'' \setminus \{w_1, w_2\}$. Call the neighbours of y and y' in $G \setminus (C' \cup C'')$ v and v' respectively, if these vertices exist. We assume v and v' exist, as adding them as pendant vertices does not affect the proof adversely. Let G'' be the graph obtained from $G \setminus (C \cup C' \cup C'')$ by adding the edge vv' if possible (v and v' may not be two distinct vertices, or may already be adjacent). This construction does not create a Δ-clique in G'' since no pair of cliques in G intersects in $\omega - 1$ vertices by Lemma 5.7.4. Bearing in mind that $\Delta = 5$, G'' cannot contain a copy of $C_5 \boxtimes K_2$, since the existence of $(C_5 \boxtimes K_2) - e$ in G would violate the second claim. Therefore the minimality of G guarantees that G'' has a fractional $(\Delta - \epsilon)$-colouring κ. We extend in two cases based on whether or not $|\{v, v'\}| = 2$.

Note that if $|\{v, v'\}| = 1$, we may assume one of w_1, w_2 is nonadjacent to v, say w_1 is nonadjacent to v, otherwise G contains a copy of $(C_5 \boxtimes K_2) - e$, violating the second claim (see Figure 5.4 (left)). Now assume $|\{v, v'\}| \leq 1$. We recolour v (if it exists) such that $|\kappa[v] \cap (\alpha(w_1) \cup \alpha(w_2))| \geq 1 - \epsilon$. This is possible because $|\alpha(v)| \geq 2 - \epsilon$ and $|\alpha(w_1)| \geq 4 - \epsilon$, so the intersection of these two sets is at least $(6 - 2\epsilon) - (5 - \epsilon) = 1 - \epsilon$. Now we may easily extend κ by colouring w_1 such that

$|\kappa[v] \cap \kappa[w_1]| \geq 1 - \epsilon$. Next we extend κ by colouring w_2 and w_3, which we can do greedily since each of these vertices has at most three neighbours in $G \setminus (C \cup C')$. Now it remains to colour $C \cup C'$. Since $|\kappa[\{v, w_1, w_2, w_3\}]| \leq 4 - (1 - \epsilon)$, there is weight $\frac{4}{3}$ of colour we can use on both $\{x, x'\}$ and $\{y, y'\}$. Since each vertex in $\{x, x', y, y'\}$ has only three neighbours in $G \setminus (C \cap C')$, we can extend κ to a colouring of $G \setminus (C \cap C')$ such that $|\kappa[\{x, x'\}] \cap \kappa[\{y, y'\}]| \geq \frac{4}{3}$. After doing this we can easily extend κ to a fractional $(5 - \epsilon)$-colouring of G by applying Lemma 5.2.2, a contradiction.

Now we handle the case $|\{v, v'\}| = 2$, starting with a fractional $(5 - \epsilon)$-colouring of G'' which we take as a partial coloring of G. We begin by extending κ by colouring w_3 such that $\kappa[w_3] \subset \kappa[\{v, v'\}]$, which is possible because $\kappa[v]$ and $\kappa[v']$ are disjoint (and w_3 is adjacent to at most one of v and v', since it is adjacent to w_1, w_2, y and y'). We now extend κ by colouring w_1 and w_2 in any way, which we can do greedily. At this point, we have $|\alpha(y)| \geq \frac{8}{3}$, $|\alpha(y')| \geq \frac{8}{3}$, and $|\alpha(y) \cup \alpha(y')| \geq \frac{11}{3}$. Therefore $|\alpha(y) \setminus \kappa[\{w_1, w_2\}]| \geq \frac{2}{3}$, $|\alpha(y') \setminus \kappa[\{w_1, w_2\}]| \geq \frac{2}{3}$, and $|(\alpha(y) \cup \alpha(y')) \setminus \kappa[\{w_1, w_2\}]| \geq \frac{5}{3}$. We may therefore give y weight $\frac{2}{3}$ of colour not in $\kappa[\{w_1, w_2\}]$, and give y' weight $\frac{2}{3}$ of colour not in $\kappa[\{w_1, w_2\}]$, then finish colouring y and y' greedily, since each has at most three neighbours in $G \setminus C$. It follows that $|\kappa[\{w_1, w_2\}] \cap \kappa[\{y, y'\}]| \leq \frac{2}{3}$, so we can extend κ by colouring $\{x, x'\}$ such that $|\kappa[\{w_1, w_2\}] \cap \kappa[\{x, x'\}]| \geq \frac{4}{3}$. We can now extend κ to a fractional $(\Delta - \epsilon)$-colouring of G by applying Lemma 5.2.2 as in the previous case. This contradiction proves the third claim.

Our fourth claim, which is sufficient to complete the proof, is that G' does not contain $C_5 \boxtimes K_2$. If it does, there must be four vertices w, w', v, and v' such that in G', $\{w, w', z, z'\}$ and $\{v, v', z, z'\}$ are cliques. Each of w, w', v, and v' therefore has two neighbours in $\{x, x', y, y'\}$. By the first claim, there are two cases, by symmetry: w and w' are adjacent to both x and x', or w and v are adjacent to both x and x'. In the first case, the component of G containing C is isomorphic to $C_7 \boxtimes K_2$, a contradiction since $\chi_f(C_7 \boxtimes K_2) = \frac{14}{3}$. In the second case, the component of G containing G is isomorphic to the graph shown in Figure 5.4 (right). Observe that the outer seven vertices induce C_7, as do the inner seven vertices. Therefore $\chi_f(G) \leq 2\chi_f(C_7) = 5 - \frac{1}{3}$, a contradiction. This completes the proof of the lemma. $\square$

5.7.2 Part (ii)

Our approach to proving Lemma 5.7.2 involves reducing G to a smaller graph G'. Either G' is fractionally $(\Delta - \epsilon)$-colourable by minimality, in which case we finish easily, or G' contains a K_Δ

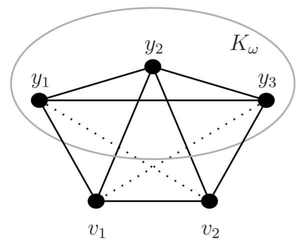

Figure 5.5: A bump.

or $C_5 \boxtimes K_2$ (when $\Delta = 5$), in which case we proceed on a case-by-case basis.

To simplify things, we first need to prove a couple of lemmas that exclude induced subgraphs of G.

Definition 1. *Suppose we have a set $Y = \{y_1, y_2, y_3\}$ of vertices in a maximum clique C, and two adjacent vertices v_1 and v_2 such that $N(v_1) \cap Y = \{y_1, y_2\}$ and $N(v_2) \cap Y = \{y_2, y_3\}$. Then we say that the set $X = C \cup \{v_1, v_2\}$ is a* bump *(see Figure 5.5).*

Lemma 5.7.6. *G does not contain a bump.*

Proof. Suppose to the contrary that G contains a bump X. To reach a contradiction we take a fractional $(\Delta - \epsilon)$-colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$-colouring of G as follows.

First, extend κ by colouring v_1 and y_3 with the same set of colours. This is possible because v_1 has at most $\Delta - 3$ neighbours in G', and y_3 has at most one neighbour in G', so $|\alpha(v_1) \cap \alpha(y_3)| \geq \Delta - \epsilon - (\Delta - 3) - 1 > 1$.

Next we extend κ by giving v_2 and y_1 common colour of total weight $\frac{1}{2}$, and leaving them only partially coloured. This is possible because at this point, v_2 has at most $\Delta - 1$ coloured neighbours, and y_1 has at most 3 coloured neighbours, but both are adjacent to v_1 and y_3. Therefore $|\kappa[N(y_1) \cup N(v_2)]| \leq \Delta - 2 + 1 = \Delta - 1$, and so $|\alpha(y_1) \cap \alpha(v_2)| \geq 1 - \epsilon \geq \frac{1}{2}$.

At this point observe that $|\kappa[Y]| = \frac{3}{2} \leq (\Delta - \epsilon) - 2 - (\Delta - 4)$, so we may now greedily extend κ by colouring the $\Delta - 4$ vertices in $C \setminus Y$, since each of these has at most two coloured neighbours in G'. All that remains is to complete the colouring of v_2, y_1, and y_2. First we finish colouring y_1; we can do this greedily because at this point $|\kappa[N(y_1)]| \leq \Delta - 2$, since y_2 is uncoloured and $\kappa[v_1] = \kappa[y_3]$. Next we greedily finish colouring v_2, which again we can do because at this point $|\kappa[N(v_2)]| \leq \Delta - 2$, since y_2 is uncoloured and $\kappa[v_1] = \kappa[y_3]$.

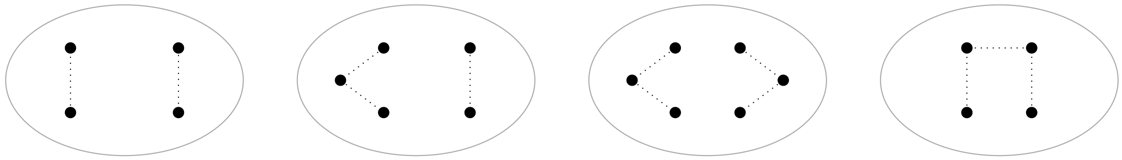

Figure 5.6: Configurations of edges missing from a K_Δ that are forbidden for, respectively, $\Delta \geq 5$ (Lemma 5.7.7), $\Delta \geq 6$ (Lemma 5.7.8), $\Delta \geq 7$ (Lemma 5.7.9), and $\Delta \geq 7$ (Lemma 5.7.10).

Finally we must extend to y_2, which we can do greedily: since $\kappa[v_1] = \kappa[y_3]$ and $|\kappa[v_2] \cap \kappa[y_1]| \geq \frac{1}{2}$, $|\kappa[N(y_2)]| \leq \Delta - \frac{3}{2}$, so $|\alpha(y_2)| \geq \frac{3}{2} - \epsilon \geq 1$. Thus G is fractionally $(\Delta - \epsilon)$-colourable, a contradiction. □

We already know, thanks to Lemma 5.7.1, that K_Δ minus an edge cannot appear in G. But given restrictions on Δ, we can forbid other subgraphs arising as K_Δ minus a small number of edges. We use variations of the approach for bumps: we extend a partial fractional colouring of the graph by leaving a set of vertices to the end, then finishing greedily, having already given their neighbourhoods lots of repeated colour.

Lemma 5.7.7. *G cannot contain K_Δ minus a matching of size two.*

Proof. Suppose to the contrary that G contains a subgraph X on Δ vertices, with vertices $v_1, v_2, v_3, v_4 \in V(X)$ such that the non-edges of $G[X]$ are exactly $\{v_1v_2, v_3v_4\}$. We first consider the case where $\Delta \geq 6$. We begin with a fractional $(\Delta - \epsilon)$-colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$-colouring of G as follows.

First, we extend κ by colouring v_1 and v_2 with the same set of colours. Each of v_1, v_2 has at most two coloured neighbours in G', and so $|\alpha(v_1) \cap \alpha(v_2)| \geq (\Delta - \epsilon) - 4 \geq 1$. Thus it is possible to choose $\kappa[v_1] = \kappa[v_2]$.

Next, we extend κ by colouring v_3 and v_4 in such a way that $\kappa[v_3] \cap \kappa[v_4] \geq \frac{1}{2}$. Each of v_3, v_4 has at most two coloured neighbours in G' as well as neighbours v_1 and v_2 which have the same set of colours, and so $|\alpha(v_3) \cap \alpha(v_4)| \geq (\Delta - \epsilon) - 5 \geq 1 - \epsilon \geq \epsilon \geq \frac{1}{2}$. Thus we may choose $\kappa[v_3]$ and $\kappa[v_4]$ as claimed. We now have $|\kappa[v_1, v_2, v_3, v_4]| \leq \frac{5}{2}$.

It remains to colour the $\Delta - 4$ vertices in $X \setminus \{v_1, v_2, v_3, v_4\}$. We can do this easily because for each such vertex, the total weight of colours appearing twice in its neighbourhood is at least $1 + \epsilon$.

Therefore as we colour greedily, the weight on the closed neighbourhood will never exceed $\Delta - \epsilon$. Thus G is fractionally $(\Delta - \epsilon)$-colourable, a contradiction.

Now we consider the case where $\Delta = 5$. Let u denote the neighbour of v_5 outside X; if u does not exist, we can add a pendant vertex to v_5 and call it u, for the sake of our argument. We begin with a fractional $(\Delta - \epsilon)$-colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$-colouring of G as follows, considering three subcases based on $f = |\kappa[N(v_3)] \cap \kappa[N(v_4)]|$.

If $f < \frac{1}{3}$, we give v_1 and v_2 common colour of weight $\frac{2}{3}$, leaving them only partially coloured. We then put weight $\frac{2}{3} - |\kappa[u] \cap \kappa[v_1]|$ of colour from $\kappa[u] \setminus \kappa[v_1]$ onto $\{v_3, v_4\}$ (putting none on both), which is possible because there is at least $\frac{2}{3}$ colour in $\kappa[u] \cap (\alpha(v_3) \cup \alpha(v_4))$. We now extend κ to completely colour v_1 and v_2, which is possible because at this point $|\kappa[\{v_3, v_4\}]| \leq \frac{2}{3}$. Next we extend κ to completely colour v_3 and v_4, which is possible because at this point v_5 is uncoloured and $|\kappa[v_1] \cap \kappa[v_2]| \geq \frac{2}{3}$. Finally we extend the colouring to include v_5, which is possible because $|\kappa[v_1] \cap \kappa[v_2]| \geq \frac{2}{3}$ and $|\kappa[u] \cap \kappa[\{v_1, v_2, v_3, v_4\}]| \geq \frac{2}{3}$. So we may assume $f \geq \frac{1}{3}$.

If $f < \frac{2}{3}$, we give v_1 and v_2 common colour of weight $\frac{2}{3}$, leaving them only partially coloured. We then give v_3 and v_4 common colour of weight $\frac{1}{3}$, so at this point the total colour appearing on $N(v_3) \cup N(v_4)$ is at most $4 - \frac{1}{3} + \frac{2}{3} \leq 5 - \epsilon - \frac{1}{3}$ (because $f \geq \frac{1}{3}$). We then give $\{v_3, v_4\}$ enough colour from $\kappa[u]$ so that $|\kappa[u] \cap \kappa[\{v_1, v_2, v_3, v_4\}]| \geq \frac{1}{3}$; this is possible because $f < \frac{2}{3}$, and so $(\alpha(v_3) \cup \alpha(v_4) \cup \kappa[v_1, v_2]) \geq 4$. We may extend to finish colouring v_1, v_2, v_3, v_4 greedily, since v_5 is uncoloured and both $\kappa[v_1] \cap \kappa[v_2]$ and $\kappa[v_3] \cap \kappa[v_4]$ have size at least $\frac{1}{3}$. Finally we can extend the colouring to v_5, since the weight of colours appearing at least twice on $N(v_5)$ is at least $\frac{4}{3} \geq 1 + \epsilon$. So we may assume $f \geq \frac{2}{3}$.

This final case is easiest: we give v_1 and v_2 common colour of weight $\frac{2}{3}$, then give v_3 and v_4 common colour of weight $\frac{2}{3}$, then extend to completely colour $\{v_1, v_2, v_3, v_4\}$ greedily, then extend to v_5 greedily. The details are as in the previous cases, but easier. Thus G is fractionally $(\Delta - \epsilon)$-colourable, a contradiction. □

Lemma 5.7.8. *If $\Delta \geq 6$, G cannot contain K_Δ minus the edges of vertex disjoint paths, one of length one and one of length two.*

Proof. Suppose to the contrary that $\Delta \geq 6$ and G contains a subgraph X on Δ vertices, with vertices $v_1, v_2, v_3, v_4, v_5 \in V(X)$ such that the non-edges of $G[X]$ are exactly $\{v_1v_2, v_1v_3, v_4v_5\}$. We begin with a fractional $(\Delta - \epsilon)$-colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$-colouring of G as follows.

First we give v_1 and v_2 weight $\frac{1}{2}$ of common colour, leaving them only partially coloured. This is possible because v_1 and v_2 have, in total, at most $5 \leq \Delta - \epsilon - \frac{1}{2}$ coloured neighbours in $G \setminus X$. Next we give v_4 and v_5 the same colour, which is possible because at this point the weight of colour on their neighbourhoods totals at most $2 + 2 + \frac{1}{2} \leq \Delta - \epsilon - 1$, since they are both adjacent to v_1 and v_2. Next we extend κ to complete the colouring of v_1, v_2, v_3, v_4, and v_5 greedily, which we can do since each of these vertices has at least $\frac{1}{2}$ weight of repeated colour in its neighbourhood, and at least one uncoloured neighbour in X. Finally we extend greedily to the remaining vertices of X, which we can do since each such vertex is adjacent to v_1, v_2, v_4, and v_5, and therefore has repeated colour of weight at least $\frac{3}{2}$ in its neighbourhood. Thus G is fractionally $(\Delta - \epsilon)$-colourable, a contradiction. $\square$

Lemma 5.7.9. *If $\Delta \geq 7$, G cannot contain K_Δ minus the edges of two vertex-disjoint paths of length two.*

Proof. Suppose to the contrary that $\Delta \geq 7$ and G contains a subgraph X on Δ vertices, with vertices $v_1, \ldots, v_6 \in V(X)$ such that the non-edges of $G[X]$ are exactly $\{v_1v_2, v_2v_3, v_4v_5, v_5v_6\}$. We begin with a fractional $(\Delta - \epsilon)$-colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$-colouring of G as follows.

First we give v_1 and v_2 the same colour. Next we give v_4 and v_5 weight $\frac{1}{2}$ of common colour. We then extend greedily to complete the colouring of v_3, v_4, v_5, and v_6, then extend greedily to complete the colouring of G. We can do this because, similar to Lemma 5.7.7, v_1 and v_2 together have at most 5 neighbours in $G \setminus X$, as do v_4 and v_5. $\square$

Lemma 5.7.10. *If $\Delta \geq 7$, G cannot contain K_Δ minus the edges of a three-edge path.*

Proof. Suppose to the contrary that $\Delta \geq 7$ and G contains a subgraph X on Δ vertices, with vertices $v_1, v_2, v_3, v_4 \in V(X)$ such that the non-edges of $G[X]$ are exactly $\{v_1v_2, v_2v_3, v_3v_4\}$. We begin with a fractional $(\Delta - \epsilon)$-colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$-colouring of G as follows.

We first extend κ by colouring v_1 and v_2 with the same set of colours. Since v_1 has at most two coloured neighbours in G' and v_2 has at most three coloured neighbours, we have $|\alpha(v_1) \cap \alpha(v_2)| \geq (\Delta - \epsilon - 5) \geq 2 - \epsilon \geq 1$, and so from this set we choose $\kappa[v_1] = \kappa[v_2]$.

We next extend κ by giving v_3 and v_4 weight $\frac{1}{2}$ of common colour, which is possible because v_3 and v_4 together have at most 5 neighbours in $G \setminus X$, and weight 1 of colour appearing in their neighbourhood in X. We may then extend greedily to complete the colouring of v_3 and v_4. Now since the weight of colours appearing twice in X is at least $\frac{3}{2}$, we may extend the colouring to the rest of X greedily. Thus G is fractionally $(\Delta - \epsilon)$-colourable, a contradiction. $\square$

We are now ready to prove Lemma 5.7.2.

Proof of Lemma 5.7.2. Suppose G contains a clique C of size $\Delta - 1$ and a vertex w outside C with at least two neighbours in C. Call the vertices in C $v_1, \ldots, v_\omega$, and suppose w is adjacent to v_1 and v_2. Let the neighbours of v_1 and v_2 outside $C \cup \{w\}$ be denoted y and z, if they exist. We may actually assume they exist, since adding them as pendant vertices does not affect our proof adversely.

We choose w, v_1, and v_2 such that if possible, w is in a K_ω, and subject to that, if possible, v_1 and v_2 do not have a common neighbour outside $C \cup \{w\}$, i.e. $y \neq z$. We construct one of two reduced graphs from G, depending on whether or not y and z are distinct.

Case 1: $y \neq z$.

Let p and p' be the neighbours of v_3 outside C. Subject to whether or not we can choose w to be in a K_ω and whether or not we can choose v_1 and v_2 such that $y \neq z$, we choose w, v_1, v_2, and v_3 such that w and v_3 are nonadjacent and $|\{p, p'\} \cap \{y, z\}|$ is minimum. Choose v_4 nonadjacent to w as well, noting that this is possible since by Lemma 5.7.1, w has at least two non-neighbours in C. Construct the graph G_1 from $G - C$ by making y adjacent to z and making w adjacent to p and p'. Clearly $\Delta(G_1) \leq \Delta$.

We claim that G_1 is not fractionally $(\Delta - \epsilon)$-colourable; if it is then we extend a $(\Delta - \epsilon)$-colouring κ of G_1 to a colouring of G as follows. First, we extend κ by giving v_3 the same colours as w. Since all the coloured neighbours of v_3 are adjacent to w in G_1, we have $\kappa[w] \subseteq \alpha(v_3)$, and so we may choose $\kappa[v_3] = \kappa[w]$. We now greedily extend to the vertices $v_4, \ldots, v_\omega$, which is possible because v_1 and v_2 remain uncoloured; it now remains to colour v_1 and v_2. Since $\kappa[v_3] = \kappa[w]$, it follows that $|\alpha(v_1)| \geq (\Delta - \epsilon) - (\Delta - 3) - 1 \geq 2 - \epsilon$ and $|\alpha(v_2)| \geq 2 - \epsilon$. Further, since $|\kappa[y] \cap \kappa[z]| = 0$ we have $|\kappa[N(v_1)] \cap \kappa[N(v_2)]| \leq \Delta - 3$, and so $|\alpha(v_1) \cup \alpha(v_2)| \geq 2$. Thus we may apply Lemma 5.2.2 to extend κ to v_1 and v_2. It follows that G is fractionally $(\Delta - \epsilon)$-colourable, a contradiction. This proves the claim.

Therefore by the minimality of G we may assume that either G_1 contains a Δ-clique, or $\Delta = 5$ and G_1 contains a copy of $C_5 \boxtimes K_2$.

We claim that if $\Delta = 5$, G_1 does not contain a copy X of $C_5 \boxtimes K_2$. Suppose to the contrary that adding the edges wp, wp', yz to G yields a copy of $C_5 \boxtimes K_2$. Since G does not contain two intersecting copies of K_4, X contains two disjoint edges that are not edges of G. It follows that $w, y, z \in V(X)$. Further, since $C_5 \boxtimes K_2$ is 5-regular, wp and wp' both belong to $E(X)$ and further no vertex in $V(X)$ has a neighbour in $G \setminus (X \cup \{v_1, v_2, v_3\})$. Therefore $\{v_1, v_2, v_3\}$ is a clique cutset of size three, contradicting the fact that every proper induced subgraph of G is fractionally $(\Delta - \epsilon)$-colourable. This proves the claim.

We may now move on to the more complicated task of proving that $\omega(G_1) = \omega$. Suppose G_1 contains a Δ-clique C'.

Our first claim is that $\{w, y, z\} \in C'$ and $yz \notin E(G)$. By Lemma 5.7.1, adding a single edge to G cannot create a Δ-clique. It follows that $w \in V(C')$. Suppose that $|\{y, z\} \cap C'| \leq 1$ or that $yz \in E(G)$. Again by Lemma 5.7.1, p, p' must be distinct and belong to C'. Now, in G, w has $\omega - 2$ neighbours in the ω-clique $C' - w$, and so w does not belong to an ω-clique by Lemma 5.7.1. On the other hand, v_3 has two neighbours in a ω-clique (namely p and p') and does belong to a maximum clique, contradicting our choice of w. This proves the first claim.

Our second claim is that $|\{p, p'\} \cap \{y, z\}| = 1$. Suppose $|\{p, p'\} \cap \{y, z\}| = 0$. Then the edges in $\{wp, wp', yz\} \setminus E(G)$ either consist of a single edge, a two-edge matching, or a 2-edge path disjoint from a third edge. By Lemmas 5.7.1, 5.7.7, and 5.7.8, we know that they consist of a 2-edge path disjoint from a third edge, and that $\Delta = 5$. In particular, it follows from the first claim that w is adjacent to both p and p'. Let p'' and p''' denote the neighbours of v_4 outside C. Since G does not contain a bump by Lemma 5.7.6, both y and z have only one neighbour in C. We may therefore exchange the roles of v_3 and v_4 without violating the disjointness of $\{p, p'\}, \{y, z\}$. By the minimality of G, the new resulting reduced graph G_1' (constructed as was G_1, but with v_3 and v_4 swapped) has a K_Δ. Since y is adjacent to w, v_1, p, p', p'', and p''', the sets $\{p, p'\}$ and $\{p'', p'''\}$ must intersect. Since $\{p, p', v_3, v_4\}$ cannot be a clique by Lemma 5.7.1, $\{p, p'\} \neq \{p'', p'''\}$.

Therefore we may assume $p' = p'''$ and that $|\{p, p', p''\}| = 3$. But then p' is adjacent to p, p'', y, z, v_3, and v_4, contradicting the fact that $\Delta = 5$. Therefore $|\{p, p'\} \cap \{y, z\}| \neq 0$.

Suppose $|\{p, p'\} \cap \{y, z\}| = 2$. We may assume $p = y$ and $p' = z$. Recall that we have chosen v_3 so as to minimize $|\{p, p'\} \cap \{y, z\}|$. Each of w, y, z has at most three neighbours in C, since $\{w, y, z\} \in C'$ by the first claim. Furthermore, if w belongs to a K_ω in G, then it has only two neighbours in C. If $\Delta = 5$, by Lemma 5.7.1, v_4 sees neither y nor z (since v_3 sees y and z), contradicting our choice of v_3. If $\Delta \geq 6$ and w is in a K_ω in G, then there is a vertex in $C \setminus N(w)$ that is adjacent to at most one of y, z and nonadjacent to w, contradicting our choice of v_3. If $\Delta \geq 6$ and w is not in a K_ω in G, then either there is a vertex in $C \setminus N(w)$ adjacent to at most one of y, z, contradicting our choice of v_3, or else every vertex in $C \setminus N(w)$ sees both of y, z. In this latter case we can relabel: relabel y to w', v_1 to v_1', v_3 to v_2', w to y', z to z', and v_4 to v_3'. Since v_4 was chosen to be nonadjacent to w, we have a labelling that contradicts the minimality of $|\{p, p'\} \cap \{y, z\}|$. This proves the second claim. We may now assume that $y = p$ and that $|\{y, z, p'\}| = 3$.

Our third claim is that $p' \in C'$. Suppose to the contrary that $p' \notin C'$. Then in G, w has $\omega - 1$ neighbours in $V(C')$. Thus w belongs to an ω-clique in $G - C$, and therefore has exactly two neighbours in C. Also, since $wz \in E(G)$ and G does not contain a bump by Lemma 5.7.6, v_2 is the only neighbour of z in C. Further, y belongs to an $(\omega - 1)$-clique in $G - C$ and has at most three neighbours in C, and at most two if $\Delta = 5$. Therefore, there is a vertex in C with no neighbour in $\{w, y, z\}$, contradicting our choice of v_3. This proves the third claim.

We now know that $y = p$ and $\{w, y, p', z\} \subseteq V(C')$. Since G does not contain a bump and since $wz \in E(G)$, we know that z has only one neighbour in C. Therefore by our choice of v_3 minimizing $|\{p, p'\} \cap \{y, z\}|$, every vertex in C is adjacent to w or y. Thus $\Delta = 6$ and each of w and y has three neighbours in C.

To complete the proof, we now fractionally colour G directly, beginning with a fractional $(\Delta - \epsilon)$-colouring κ of $G - C - \{w, y\}$. We first extend κ by colouring w and v_3 with the same set of colours. Since v_3 and w together have at most four coloured neighbours, we have $|\alpha(v_3) \cap \alpha(w)| \geq (6 - \epsilon) - 4 \geq 1$, and so we may choose $\kappa[v_3] = \kappa[w]$.

Next we extend κ by colouring y and v_2 so that $|\kappa[y] \cap \kappa[v_2]| \geq \frac{1}{2}$, which is possible because at this point, $|\kappa[N(y) \cup N(v_2)]| = 5$, since the only coloured vertices in $N(y) \cup N(v_2)$ are $C' - y$ and v_3 (which has the same colour as w). We now have $\kappa[\{v_2, v_3, w, y\}] \leq \frac{5}{2}$.

Next we extend κ by colouring v_4 and v_5. Since each of v_4, v_5 is adjacent to either w or y, we have $|\kappa[N(v_4)]| \leq \frac{7}{2}$ and $|\kappa[N(v_5)]| \leq \frac{7}{2}$. Thus $|\alpha(v_4)|, |\alpha(v_5)| \geq 2$ and so we may apply Lemma 5.2.2 to choose $\kappa[v_4]$ and $\kappa[v_5]$ greedily.

Finally we greedily extend κ to v_1. We have $\kappa[N(v_1)] \leq \frac{9}{2}$ since v_1 is adjacent to v_2, v_3, w, and y. Applying Lemma 5.2.2, we may choose $\kappa[v_1]$ from $\alpha(v_1)$. Thus G is fractionally $(\Delta - \epsilon)$-colourable, a contradiction.

This completes the proof of Case 1.

Case 2: $y = z$ and w is in a K_ω in G.

In this case, we know that we can choose w to be in a maximum clique, but we cannot make such a choice of w, v_1, v_2 for which $y \neq z$. Since w is in a maximum clique, it has only two neighbours in C. Therefore we may choose v_3 and v_4 to be nonadjacent to both w and y, since Lemma 5.7.1 implies that y has at least two non-neighbours in C. But we need further conditions on our vertex labelling. Denote by p, p' and q, q' the neighbours of v_3 and v_4 outside C, respectively. We choose a labelling of the vertices satisfying the following conditions:

*L*1 w is in a maximum clique. Subject to this condition,

*L*2 y is in a maximum clique if possible. Subject to this condition,

*L*3 v_3 and v_4 are not adjacent to w nor to y. Subject to satisfying the previous conditions,

*L*4 v_3 is chosen so that $|N(p) \cap N(p') \cap N(y)|$ is maximized.

Construct the graph G_2 from $G - C$ by making w adjacent to p and p' and making y adjacent to q and q'. Clearly $\Delta(G_1) \leq \Delta$.

We claim that G_2 is not fractionally $(\Delta - \epsilon)$-colourable; if it is then we extend a $(\Delta - \epsilon)$-colouring κ of G_2 to a colouring of G as follows. We begin by extending κ to colour v_3 with the same colour as w. Since v_3's only coloured neighbours are p and p', which are adjacent to w in G_2, we may choose $\kappa[v_3] = \kappa[w]$. We now extend κ to the remaining vertices in C. By the choice of $\kappa[v_3]$, we have $|\alpha(v_1)|, |\alpha(v_2)| \geq \Delta - \epsilon - 2$. Since each of the $\Delta - 4$ other uncoloured vertices has at most three coloured neighbours we find $|\alpha(v_i)| \geq \Delta - \epsilon - 3$ for $4 \leq i \leq \omega$. Third, the edges yq, yq' in G_2 ensure that $|\alpha(v_1) \cup \alpha(v_4)|, |\alpha(v_2) \cup \alpha(v_4)| \geq \Delta - \epsilon - 1$. Applying Lemma 5.2.2 to $C \setminus \{v_3\}$ (which has size $\Delta - 2$), we find a $(\Delta - \epsilon)$-colouring of G, a contradiction. This proves the claim.

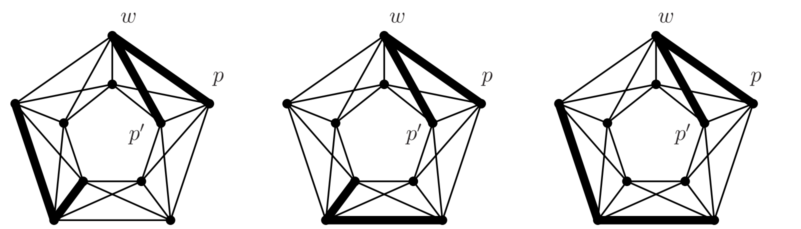

Figure 5.7: Three ways to form $C_5 \boxtimes K_2$ in Case 2.

Therefore by the minimality of G we may assume that either G_2 contains a Δ-clique, or $\Delta = 5$ and G_2 contains a copy of $C_5 \boxtimes K_2$. Let $F = E(G_2) \setminus E(G) \subseteq \{wp, wp', yq, yq'\}$. Let F_w and F_y denote the edges incident to w and y in G_2, respectively.

We claim that if $\Delta = 5$, G_2 does not contain a copy X of $C_5 \boxtimes K_2$. Suppose to the contrary that adding the edges wp, wp', yq, yq' to G creates a copy of $C_5 \boxtimes K_2$. Since G does not contain two intersecting copies of K_4, X contains at least two vertex-disjoint edges that are not edges of G. It follows that $w, y \in V(X)$. Further, since $C_5 \boxtimes K_2$ is 5-regular, $\{p, p', q, q'\} \subseteq V(X)$ and F contains all four edges wp, wp', yq, yq'. Since w belongs to a K_4 in G, p and p' must form the intersection of two K_4s in X. Since G does not contain a pair of intersecting K_4s, q and q' do not form the intersection of two K_4s in X, and moreover, y cannot be in $N(w) \cup N(p) \cup N(p')$ in X. Hence y does not belong to a 4-clique in G. See Figure 5.7, where y is the bottom left vertex. Observe that v_3 belongs to a maximum clique in G, and its neighbours p and p' belong to another maximum clique. Further, p and p' have a common neighbour in a third maximum clique. Since y is not in a maximum clique, this contradicts $L2$ in our choice of w and y, and proves the claim.

We now move on to the task of proving that $\omega(G_2) = \omega$. Suppose G_2 contains a Δ-clique C'.

Our first claim is that $|E(C') \cap F_y| \geq 1$ and $|E(C') \cap F_w| \geq 1$. We can see that $|E(C') \cap F_y| \geq 1$, otherwise $C' \setminus w$ is a maximum clique in G intersecting a maximum clique containing w, contradicting Lemma 5.7.1.

Suppose now that $|E(C') \cap F_w| = 0$. By the same argument, y cannot belong to a maximum clique in G. We know that C' must contain at least two edges in F, so $yq, yq' \in F \cap E(C')$. Therefore $|N(q) \cap N(q') \cap N(y)| \geq \omega - 2$ and these vertices, along with y form an $(\omega - 1)$-clique. Further $qq' \in E(G)$, and so q, q' belong to an ω-clique in G.

If $|\{p, p'\} \cap \{q, q'\}| = 1$, then we can relabel v_4 as w'; since v_4 is in a K_ω in G and has two neighbours in $C' \setminus \{y\}$, one but not both of which are adjacent to v_3, contradicting the fact that we are not in Case 1. If $|\{p,p'\} \cap \{q,q'\}| = 2$, this contradicts condition $L2$ in our choice of labelling, since G contains two vertices in the K_ω C having two neighbours in common in a disjoint K_ω $C' \setminus \{y\}$. Therefore $|\{p,p'\} \cap \{q,q'\}| = 0$.

Note that by $L3$ we have chosen v_3, v_4 nonadjacent to both w and y. In particular this means that $\{p,p'\}$ and $\{q,q',y\}$ are disjoint. By $L4$, we know $|N(p) \cap N(p') \cap N(y)| \geq |N(q) \cap N(q') \cap N(y)| \geq \omega - 2$. In particular this set must intersect $N(q) \cap N(q') \cap N(y)$. But since $\{p,p'\}$ and $\{q,q',y\}$ are disjoint, if $\{p,p'\} \cap C' = \emptyset$, there is a vertex of degree $\Delta + 1$, a contradiction. Therefore we may assume without loss of generality that $p \in C' \setminus \{q,q',y\}$. But then in G, p is adjacent to every other vertex in C', so its only other neighbour is v_3. Since y is nonadjacent to v_3, q, and q', $N(p) \cap N(p') \cap N(y) \subseteq C' \setminus \{q,q',y,p\}$, contradicting the fact that its size is at least $\omega - 2$. This proves the first claim.

Our second claim is that w and y belong to an ω-clique W in G. As a consequence, since this makes $\{w,y,v_1,v_2\}$ a clique, Lemma 5.7.1 tells us that $\Delta \geq 6$. To prove this, let W be the maximum clique in G containing w, and note that W is the closed neighbourhood of w in $G - C$. By the first claim, $w \in V(C')$ and $y \in V(C')$. By the choice of v_3, $y \notin \{p,p'\}$ and $w \notin \{q,q'\}$. It follows that $wy \in E(G)$, and so $y \in W$. This proves the second claim.

Our third claim is that the only edges between C and W are between $\{v_1,v_2\}$ and $\{w,y\}$. To see this assume otherwise, and denote the vertices of W $\{w,y,w_3,\ldots,w_\omega\}$. By the maximum degree, there must exist $3 \leq i,j \leq \omega$ such that v_i and w_j are adjacent.

To reach a contradiction we extend a fractional $(\Delta - \epsilon)$-colouring κ of $G - W - C$ as follows. First assign w and v_i the same colour, which is possible because together these vertices have at most weight 1 of colour on (the union of) their neighbourhoods. Then for some $i' \notin \{1,2,i\}$, give y and $v_{i'}$ colour $\frac{1}{2}$ in common, leaving them only partially coloured, noting that this is possible because at this point y and $v_{i'}$ have colour at most $1+2=3$ on their neighbourhoods (since w and v_i have the same colour). Next we greedily extend to all vertices of $W \setminus \{w,y,w_j\}$, noting that this is possible because all these vertices are adjacent to y and w_j, which together have only $\frac{1}{2}$ colour on them at this point. We then greedily extend to w_j, which is possible because w_j is adjacent to w, y, and v_i, which together have weight $\frac{3}{2}$ colour on them. Next we greedily extend to complete the colouring of all vertices of $(C \cup \{y\}) \setminus \{v_1,v_2\}$, which is clearly possible because v_1 and v_2 are still uncoloured. Finally we extend to v_1 and v_2, which is possible because both are complete to $\{w, y, v_i, v_{i'}\}$, a set of four vertices with at most $\frac{5}{2}$ colour on them. This contradicts the fact that G is not fractionally $(\Delta - \epsilon)$-colourable, and proves the third claim.

Our fourth claim is that $\{p, p'\} \cap \{q, q'\} \neq \emptyset$. By the second claim, neither w nor y has any neighbours outside of W in $G - C$. By the first claim, C' contains an edge in F_w and an edge in F_y; we may assume without loss of generality that $p \in V(C')$. By the third claim $\{p, p', q, q'\} \cap W = \emptyset$, so G contains no edges between $\{w, y\}$ and $\{p, p', q, q'\}$. Therefore since C' is a clique in G_2, yp must be in F, so $p \in \{q, q'\}$. This proves the fourth claim.

Without loss of generality, for the remainder of Case 2 we assume $p \in V(C')$ and $p = q$. Thus we can also assume that p is adjacent to w_3 and w_4 in W. By the third claim p does not belong to W.

We now complete the proof of Case 2. To do so we fractionally colour G by extending a $(\Delta - \epsilon)$-colouring κ of $G - W - C - \{p\}$ as follows. We begin to extend κ by assigning $\kappa[w] = \kappa[v_5]$, noting that v_5 may or may not be adjacent to p. This is possible since together these vertices have at most two coloured neighbours. Next we give v_1 and w_5 colour $\frac{1}{2}$ in common, leaving them partially uncoloured. This is possible since at this point $|\kappa[N(v_1) \cup N(w_5)]| \leq 3$. Next, we extend κ by giving w_4 and v_4 the same set of colours, noting that since both are adjacent to p, at this point at most $\frac{7}{2} \leq \Delta - \epsilon - 1$ colour appears on their neighbourhoods, so this is possible. Next, we give w_3 and v_3 common colour $\frac{1}{2}$, noting that both are adjacent to p. Since $\kappa[N(v_3) \cap C] = \kappa[(N(w_3) \cap W) \cup \{v_3\}]$ and $|\kappa[N(v_3) \cap C]| = \frac{5}{2}$, we have $|\alpha(v_3) \cap \alpha(w_3)| \geq (\Delta - \epsilon) - \frac{5}{2} - 2 \geq 1$. We now greedily extend κ to colour $W - \{w, y, w_3, w_4\}$, which is possible since y and w_3 together have weight $\frac{3}{2}$ not yet coloured. Next we give y and v_3 weight $\frac{1}{2}$ of colour in common and leave them partially uncoloured, which is possible because at this point $|\alpha(y)| \geq \frac{3}{2}$, and $|\kappa[\tilde{N}(v_3)] \setminus \kappa[\tilde{N}(y)]| \leq 1$. We can now greedily extend to $C - \{v_1, \ldots, v_5\}$, since v_1 and v_2 together have weight $\frac{3}{2}$ not yet coloured. Next we can extend to complete the colouring of w_3, since y and p together have weight $\frac{3}{2}$ not yet coloured. Next we can extend to complete the colouring of y, since v_1 and v_2 together have weight $\frac{3}{2}$ not yet coloured.

Finally we can complete the colouring by extending greedily to complete the colouring of v_1 and v_2, since each has weight at least $\frac{3}{2}$ of colour appearing twice on its neighbourhood. This completes the proof of Case 2.

This completes the proof of Case 2.

Case 3: $y = z$ and w is not in a K_ω in G.

In this case, by the choice of w, there exists no vertex in G belonging to a maximum clique that has two neighbours in a different maximum clique. Also, we know that every pair of vertices in C has either zero or two common neighbours outside of C, for otherwise with a better choice of w, v_1, v_2 we would be in Case 1. Thus $N(w) \cap V(C) = N(y) \cap V(C)$. By Lemma 5.7.1, $|V(C) \setminus N(w)| \geq 2$. Again denote by p, p' and q, q' the neighbours of v_3 and v_4 outside C, respectively. We choose v_3 and v_4 from $V(C) \setminus N(w)$ to maximize $|\{p, p', q, q'\}|$. Subject to this, v_3 and v_4 are chosen to maximize $|\{wp, wp', yq, yq'\} \cap E(G)|$. Note that $|\{p, p'\} \cap \{q, q'\}| \in \{0, 2\}$, that $\{w, y\} \cap \{p, p', q, q'\} = \emptyset$, and that in particular, y is nonadjacent to v_4.

Noting that $w, y \notin \{p, p', q, q'\}$, we construct the graph G_2 from $G - C$ as in Case 2 by making w adjacent to p and p' and making y adjacent to q and q'. As in Case 2, we may assume G_2 is not fractionally $(\Delta - \epsilon)$-colourable; if it is then we extend a $(\Delta - \epsilon)$-colouring κ of G_2 to a colouring of G. (Observe that the colouring argument given in Case 2 does not make use of the fact that w belongs to a maximum clique in that case.)

Therefore we may assume that either G_2 contains a Δ-clique, or $\Delta = 5$ and G_2 contains a copy of $C_5 \boxtimes K_2$. As in the previous case, let $F = E(G_2) \setminus E(G) \subseteq \{wp, wp', yq, yq'\}$. Let F_w and F_y denote the edges of F incident to w and y in G_2, respectively.

We claim that if $\Delta = 5$, G_2 does not contain a copy X of $C_5 \boxtimes K_2$. Suppose to the contrary that adding the edges wp, wp', yq, yq' to G creates a copy of $C_5 \boxtimes K_2$. Since G does not contain two intersecting copies of K_4, X contains two vertex-disjoint edges of F. It follows that $w, y \in V(X)$, and since $\Delta = 5$, Lemma 5.7.1 tells us that w and y are not adjacent. Further, since $C_5 \boxtimes K_2$ is 5-regular, $\{p, p', q, q'\} \subseteq V(X)$ and F contains all four edges wp, wp', yq, yq'. Since w does not belong to a K_4 in G, p and p' do not form the intersection of two K_4s in X. Likewise, neither do q and q'. Also, if $\{p, p'\} \cap \{q, q'\} \neq \emptyset$ then $|\{p, p'\} \cap \{q, q'\}| = 2$ (since we are not in Case 1), which is impossible because intersection of the neighbourhoods of two nonadjacent vertices in $C_5 \boxtimes K_2$ is the intersection of two K_4s, a contradiction. Therefore w, y, p, p', q, q' are six distinct vertices.

Since exchanging the roles of v_3 and v_4 cannot reduce $|F|$, G contains no edges from $\{wq, wq', yp, yp'\}$. It follows that $pp' \in E(G)$ and $qq' \in E(G)$. Therefore by symmetry, bearing in mind that w and y are nonadjacent in both G and G_2, the only possible case is shown in Figure 5.8. Note here that there is a different choice of w that would put us in Case 1, a contradiction.

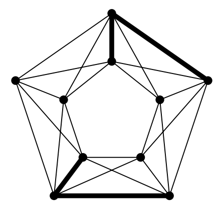

Figure 5.8: The only way to form $C_5 \boxtimes K_2$ in Case 3. If w is the top vertex, we may instead choose w as the vertex immediately below it to put us in Case 1.

We now proceed to prove that $\omega(G_2) < \Delta$. Suppose G_2 contains a Δ-clique C'.

Our first claim is that $|E(C') \cap F_w| \geq 1$ and $|E(C') \cap F_y| \geq 1$. Suppose that $|E(C') \cap F_w| = 0$. Then clearly $y \in V(C')$, and by Lemma 5.7.1, both edges yq, yq' belong to $E(C')$, and $qq' \in E(G)$. But then $C' - y$ is an ω-clique containing two neighbours of v_4, which also belongs to an ω-clique. This contradicts our choice of w. By a symmetric argument, $|E(C') \cap F_y| \geq 1$. This proves the first claim.

Our second claim is that $wy \in E(G)$ and $\Delta \geq 6$. By the first claim, w and y belong to $V(C')$. By the choice of v_3 and v_4, $w, y \notin \{p, p', q, q'\}$. Thus $wy \in E(G)$, and so w, y, v_1, v_2 form a K_4. If $\Delta = 5$ this contradicts Lemma 5.7.1. This proves the second claim.

Our third claim is that $|E(C') \cap F| \geq 3$. Suppose that $|E(C') \cap F| = 2$. By Lemma 5.7.7, the two edges in $E(C') \cap F$ do not form a matching, and so they form a two-edge path. By the first claim, one of the edges must be between w and y, contradicting the second claim. This proves the third claim.

Our fourth claim is that $|E(C') \cap F| = 4$. Suppose that $|E(C') \cap F| = 3$. By Lemma 5.7.8, at least two pairs of the edges in $E(C') \cap F$ intersect. Since $w, y \notin \{p, p', q, q'\}$ the edges $E(C') \cap F$ do not form a triangle, so they form a three-edge path. By Lemma 5.7.10 and the second claim, $\Delta = 6$.

Since $wy \in E(G)$ and by symmetry between w and y and between p and p', we may assume $p = q$ and the edges of the path are $p'w, wp, py$. Since $|\{p, p'\} \cap \{q, q'\}| \neq 1$, $p' = q'$ and $pp' \in E(G)$. By the choice of v_3, v_4 maximizing $|\{p, p', q, q'\}|$, v_5 must be complete to $\{p, p'\}$ or to $\{w, y\}$. But then v_5 belongs to two 5-cliques in G, contradicting Lemma 5.7.1. This proves the fourth claim.

We now know that $|E(C') \cap F| = 4$. Suppose that the edges in $E(C') \cap F$ form two vertex-disjoint two-edge paths. Then by Lemma 5.7.9, $\Delta = 6$. Now $|\{p, p', q, q'\}| = 4$ and so $wq, wq', yp, yp' \in$

$E(G)$. This contradicts the choice of v_3 and v_4, for reversing their roles would yield $|F| = 0$.

Since we are in Case 3, the edges in $E(C') \cap F$ therefore form a cycle of length four. It follows that $\{p, p'\} = \{q, q'\}$ and $wy, pp' \in E(G)$. By the choice of v_3 and v_4 maximizing $|\{p, p', q, q'\}|$, each of $v_5, \ldots, v_\omega$ is complete to either $\{w, y\}$ or $\{p, p'\}$. Therefore by Lemma 5.7.1, $\Delta \geq 7$. Since each of w, y, p, p' is adjacent to $\Delta - 3$ vertices of C' in G, each has at most three neighbours in C. Therefore $\Delta = 7$, and G is isomorphic to the graph $(C_5 \boxtimes K_3) - 2v$ pictured in Figure 5.2. Thus G is indeed fractionally $\frac{13}{2}$-colourable and thus fractionally $(\Delta - \epsilon)$-colourable, a contradiction.

This completes the proof of Case 3, and the proof of the lemma. $\square$

5.8 A superlocal version of Reed's conjecture

In this section we prove Theorem 5.2.6. We have already mentioned Reed's Conjecture and its fractional relaxation in Section 5.2.1.

Conjecture 5.8.1 (Reed [69]). *Every graph satisfies* $\chi \leq \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil$.

Inspired by structural observations, King conjectured that McDiarmid's Theorem 5.2.4 holds in the integer setting [57].

Conjecture 5.8.2 (King [57]). *Every graph G satisfies* $\chi(G) \leq \lceil \max_{u \in V(G)}(\omega(v) + d(v) + 1) \rceil = \lceil \max_{u \in V(G)} \rho(v)) \rceil$.

A typical example of a graph G for which $\gamma(G)$ is far from $\chi(G)$ is the star $K_{1,r}$. For such graphs we have $\Delta(G) + \omega(G) + 1 = \max_v d(v) + \omega(v) + 1$, so the bound offered by the local conjecture isn't any better. And yet a greedy colouring algorithm can very easily 2-colour a star. So can we get a better bound when vertices that are hard to colour (i.e. have high $\gamma_\ell(v)$) form a stable set? The answer, at least in the fractional setting and for certain graph classes, is yes. Our idea is that a graph should be easy to colour if no two vertices with high $\rho(v)$ are adjacent. This gives rise to the invariants $\gamma_{\ell\ell}$ and $\gamma'_{\ell\ell}$, which we define as follows:

$$\begin{aligned}
\text{For } uv \in E(G),\ \text{define } \gamma'_{\ell\ell}(uv) \text{ as } & \tfrac{1}{4}(d(u)+d(v)+\omega(u)+\omega(v)+2) \\
&= \tfrac{1}{2}(\rho(u)+\rho(v)). \\
\text{Define } \gamma'_{\ell\ell}(G) \text{ as } & \max_{uv\in E(V)} \gamma'_{\ell\ell}(uv). \\
\text{For } uv \in E(G),\ \text{define } \gamma_{\ell\ell}(uv) \text{ as } & \lceil \gamma'_{\ell\ell}(uv) \rceil. \\
\text{Define } \gamma_{\ell\ell}(G) \text{ as } & \lceil \gamma'_{\ell\ell}(G) \rceil.
\end{aligned}$$

In [34], we posed the natural conjecture regarding these invariants:

Conjecture 5.8.3. *Every graph G satisfies $\chi(G) \le \lceil \gamma_{\ell\ell}(G) \rceil$.*

One piece of evidence in support of this conjecture is the fact that the fractional relaxation holds; Theorem 5.2.6 can be restated in this notation as follows.

Theorem 5.8.4. *Every graph G satisfies $\chi_f(G) \le \gamma'_{\ell\ell}(G)$.*

In [34], we proved that Conjecture 5.8.3 holds for graphs with no stable set of size 3 and for line graphs and quasi-line graphs. The proofs closely follow the proofs of the Local Reed's Conjecture for the corresponding graph classes, which appear in [18] and [57].

The proofs of Theorems 5.2.3, 5.2.4, and 5.8.4 all rely on the same natural fractional colouring algorithm, originally due to Reed [65]: we add equal weight to every maximum stable set until a vertex is completely coloured, then we discard all completely coloured vertices and continue the process, respecting the fact that discarding vertices changes the set of maximum stable sets. Improving the bounds we get is merely a matter of refining the analysis.

In the following discussion, we will use a slightly simpler definition of a fractional colouring than we have been working with. Clearly, to show G is fractionally k-colourable, it is enough to find a nonnegative weighting w on the stable sets of G such that $\sum_S w(S) \le k$, and for every vertex v, $\sum_{S\ni v} w(S) = 1$. The proof of Theorem 5.2.4 relies on the following lemma, whose proof appears in §2.2 of [57].

Lemma 5.8.5. *Let S be a maximum stable set of G chosen uniformly at random. Then for any vertex v, $\mathbb{E}(|S \cap N(v)|) \ge 2 - (\omega(v)+1)\mathbf{Pr}(v \in S)$.*

Before proving Theorem 5.8.4 we need an easy generalization. For adjacent vertices u and v we define $N(u,v)$ as $(N(u) \cup N(v)) \setminus \{u,v\}$.

Lemma 5.8.6. *Let S be a maximum stable set of G chosen uniformly at random. Then for any adjacent vertices u and v,*

$$\mathbb{E}(|S \cap N(u,v)|) \geq 4 - (\omega(v)+2)\mathbf{Pr}(v \in S) - (\omega(u)+2)\mathbf{Pr}(u \in S) - \sum_{w \in N(v) \cap N(u)} \mathbf{Pr}(w \in S). \quad (5.22)$$

Proof. We know by Lemma 5.8.5 that

$$\begin{aligned} \mathbb{E}(|S \cap N(v)|) &\geq 2 - (\omega(v)+1)\mathbf{Pr}(v \in S) \text{ and} && (5.23) \\ \mathbb{E}(|S \cap N(u)|) &\geq 2 - (\omega(u)+1)\mathbf{Pr}(u \in S). && (5.24) \end{aligned}$$

By linearity of expectation we have

$$\mathbb{E}(|S \cap N(u,v)|) = \mathbb{E}(|S \cap N(u)|) + \mathbb{E}(|S \cap N(v)|) - \mathbb{E}(|S \cap \tilde{N}(u) \cap \tilde{N}(v)|). \quad (5.25)$$

Also by linearity of expectation, we have

$$\mathbb{E}(|S \cap \tilde{N}(u) \cap \tilde{N}(v)|) = \mathbf{Pr}(u \in S) + \mathbf{Pr}(v \in S) + \sum_{w \in N(v) \cap N(u)} \mathbf{Pr}(w \in S). \quad (5.26)$$

Substituting (5.23), (5.24), and (5.26) into (5.25) gives us (5.22). □

We are now ready to prove Theorem 5.8.4.

Proof of Theorem 5.8.4. We fractionally colour G using the following iterative method.

1. Set $w(S) = 0$ for every $S \in \mathcal{S}$. Set $G_0 = G$. Set $i = 0$.

   Set $T = 0$. T stands for total weight used.

   For each $v \in V$, set $wo_v = 0$ (wo stands for weight on).

2. If $V(G_i) = \emptyset$ or $T = \gamma'_{\ell\ell}(G)$ then stop.

3. For each vertex v of G_i, let $p_i(v)$ be the probability that v is in a uniformly random maximum stable set of G_i. Set $low = \min\{\frac{1-wo_v}{p_i(v)} | v \in V(G_i)\}$. Set $val_i = \min(low, \gamma'_{\ell\ell}(G) - T)$.

4. Let $\mathcal{S}_i$ be the set of maximum stable sets of G_i. For each stable set in $\mathcal{S}_i$, increase $w(S)$ by $\frac{val_i}{|\mathcal{S}_i|}$. For each vertex v of G_i, increase wo_v by $p_i(v)val_i$. Increase T by val_i.

5. Let G_{i+1} be the graph induced by those vertices v which satisfy $wo_v < 1$. Increment i and go to Step 2.

Our choice of val_i ensures two things: that T never exceeds $\gamma'_{\ell\ell}(G)$, and that if the ith iteration is not the last, then $V(G_{i+1})$ is properly contained in $V(G_i)$. Thus the algorithm must terminate.

We claim that at the end of the procedure, the $w(S)$ weights give a fractional $\gamma'_{\ell\ell}(G)$-colouring. It is easy to show by induction that at the end of each iteration and for every $v \in V$, $wo_v = \sum_{\{S \in \mathcal{S} | v \in S\}} w(S)$ and $T = \sum_{S \in \mathcal{S}} w(S)$. The definitions of low and val_i ensure that no wo_v is ever more than 1. We stop if $V(G_i) = \emptyset$ or $T = \gamma'_{\ell\ell}(G)$; in the first case we know that we have the desired fractional colouring. We must now show that the same is true in the second case. It suffices to show that in this case, each $wo_v = 1$.

So assume that for some v we have $wo_v < 1$ when we complete the process. For each vertex u and iteration i, denote by $a_i(u)$ the amount by which wo_u was augmented in iteration i, i.e. $a_i(u) = val_i p_i(u)$. There are two cases; we will show that each results in a contradiction.

Case 1: v has a neighbour u with $wo_u < 1$.

In this case $\{u, v\} \subseteq V(G_i)$ for every i. For every i, let S be a maximum stable set drawn at random from $\mathcal{S}_i$. Then by Lemma 5.8.6,

$$val_i \mathbb{E}(|S \cap N(u,v)|) = \sum_{x \in N(u,v)} a_i(x) \geq 4val_i - (\omega(v)+2)a_i(v) - (\omega(u)+2)a_i(u) - \sum_{w \in N(u) \cap N(v)} a_i(w)$$

Summing over all iterations,

$$
\begin{aligned}
\sum_{x\in N(u,v)} wo_x &\geq 4T - (\omega(v)+2)wo_v - (\omega(u)+2)wo_u - \sum_{w\in N(u)\cap N(v)} wo_w \\
&> \omega(u)+\omega(v)+d(u)+d(v)+2-(\omega(v)+2)-(\omega(u)+2)-|N(u)\cap N(v)| \\
&= d(u)+d(v)-|N(u)\cap N(v)|-2 = |N(u,v)|,
\end{aligned}
$$

a contradiction since $wo_x \leq 1$ for each $x \in N(u,v)$.

Case 2: Every neighbour u of v has $wo_u = 1$ at the end of the procedure.

For every neighbour u of v there exists some j such that $u \in V(G_j)$ but $u \notin V(G_{j+1})$. Choose u maximizing j; this implies that $N_{G_i}(v) = \emptyset$ for all $i > j$, and consequently $a_i(v) = val_i$ for each $i > j$. When $i \leq j$ we again have

$$
\sum_{x\in N(u,v)} a_i(x) \geq 4val_i - (\omega(v)+2)a_i(v) - (\omega(u)+2)a_i(u) - \sum_{w\in N(u)\cap N(v)} a_i(w)
$$

by Lemma 5.8.6. Summing over the iterations up to j we see

$$
\begin{aligned}
& \sum_{x\in N(u,v)} \sum_{i\leq j} a_i(x) \\
\geq\ & 4(T - \sum_{i>j} a_i(v)) - (\omega(v)+2)\sum_{i\leq j} a_i(v) - (\omega(u)+2)\sum_{i\leq j} a_i(u) - \sum_{w\in N(u)\cap N(v)} \sum_{i\leq j} a_i(w) \\
=\ & d(u)+d(v)+\omega(u)+\omega(v)+2-4\sum_{i>j} a_i(v) - \\
& (\omega(v)+2)\sum_{i\leq j} a_i(v) - (\omega(u)+2)\sum_{i\leq j} a_i(u) - \sum_{w\in N(u)\cap N(v)} \sum_{i\leq j} a_i(w) \\
\geq\ & d(u)+d(v)+\omega(u)+\omega(v)+2-(\omega(v)+2)\sum_{i} a_i(v) - (\omega(u)+2)wo_u - \sum_{w\in N(u)\cap N(v)} wo_w \\
>\ & d(u)+d(v)-|N(u)\cap N(v)|-2 = |N(u,v)|,
\end{aligned}
$$

where the third inequality follows since $\omega(v)+2 \geq 4$. This is a contradiction as $wo_x \leq 1$ for each $x \in N(u,v)$.

It follows that for every $v \in V(G)$, $wo_v = 1$. This completes the proof. □

Chapter 6

On the excluded grid theorem

6.1 Introduction

In the Graph Minors project, Robertson and Seymour introduced the notions of treewidth and tree decomposition of graphs as an important tool in the proof of Wagner's conjecture. [1] In particular, as a step toward proving Wagner's conjecture for the class of graphs not containing a fixed planar graph as a minor, they proved the so-called Excluded Grid Theorem [75].

Theorem 6.1.1 (Roberton, Seymour [75]). *For every k, there exists $f(k)$ such that every graph not containing a $k \times k$-grid as a minor has treewidth at most $f(k)$.*

We will give precise definitions in the next section. In Robertson and Seymour's original proof, they showed that one can take $f(k)$ to be a certain extremely large function of k containing iterated exponential towers. In a subsequent paper with Thomas, they improved on this and showed that $f(k)$ could be taken to be about 20^{2k^5} [73]. At the same time, they suggested that this relationship might be tightened to take $f(k) = O(k^2 \log k)$. They also gave examples of graphs with treewidth at least a multiple of $k^2 \log k$ not containing a $k \times k$-grid minor, so this would be best possible. These are still the best known lower bounds, but Demaine et al. conjecture that there exist graphs with treewidth $\Omega(k^3)$ and no $k \times k$-grid minor[25].

A polynomial upper bound on $f(k)$ of any kind remained elusive for many years, although the question received a fair bit of attention from other researchers. Demaine and Hagiaghayi showed

[1] Wagner's conjecture states that for every infinite sequence of graphs $G_1, G_2, \dots$ there exist $i < j$ such that G_i is isomorphic to a minor of G_j.

that for each fixed graph H, graphs excluding an H-minor and a $k \times k$-grid minor have treewidth at most $c_H k$, for some very large constant depending on H [24]. For general graphs, Diestel et al. gave a simpler proof of a bound similar to Robertson et al.'s in [27]. Leaf and Seymour improved Robertson et. al's exponential function to take $f(k) = 2^{8k^2 \log k}$[61]. Kawarabayashi and Kobayashi gave a different proof of a similar result [54]. In a landmark achievement, Chekuri and Chuzhoy gave the first polynomial upper bound for $f(k)$. They showed that one can take $f(k) = O(k^{99})$ in [13],[12], and later improved this to $f(k) = O(k^{20})$ in [21] [22].

A key tool in Chekuri and Chuzhoy's proof is the existence, in graphs with large treewidth, of many disjoint subgraphs with 'good' linkedness properties (we will give precise definitions soon). Our contribution in this chapter is a quantitative improvement of their result. Our proof also has the advantage of being self-contained, while the result in [12] is implicitly shown inside the proof of a complex algorithm. We use the same general scheme as [12] to obtain our main theorem, but in several places we use different techniques. In particular we avoid probabilistic arguments. Moreover in Section 6.6.2 we prove a theorem about partitioning a graph into parts with relatively more edges inside than leaving each part, which may be of independent interest. With some work (i.e. following arguments in [12]), Theorem 6.3.1 implies an upper bound $f(k) = O(k^{\delta})$, for some $\delta < 99$, but > 20. So this improves on Chekuri and Chuzhoy's first result, but not their most recent. The work presented in this chapter is joint with Paul Seymour and has not appeared elsewhere.

6.2 Definitions and notation

Before we can state our results precisely, we need to define some terms. A tree decomposition of a graph G is a pair (T, B) where T is a tree on vertices $v_1, \ldots, v_\ell$ and $B = \{B_1, \ldots, B_\ell\}$ is a multiset of subsets of $V(G)$, called *bags*, satisfying $\cup_i B_i = V(G)$. Further for each edge $uv \in E(G)$ there exists some B_i containing both u and v, and the set of bags containing a given vertex v correspond to the vertices of a subtree of T. The *width* of a tree decomposition is the size of its largest bag minus 1, and the *treewidth* of a graph is the minimum width of a tree decomposition. The treedwith of G is denoted $\mathrm{tw}(G)$.

A set of vertices $T \subseteq V(G)$ is *vertex-well-linked* if for any two equal-sized subsets T_1, T_2 of T (not necessarily disjoint) there are $|T_1|$ vertex-disoint paths, each joining a vertex in T_1 to a vertex in T_2. A set $T \subseteq V(G)$ is *edge-well-linked* if for any two equal-sized subsets T_1, T_2 of T (not necessarily disjoint) there are $|T_1|$ edge-disoint paths, each joining a vertex in T_1 to a vertex in T_2. If $0 < \alpha \leq 1$, we say a set T is *α-well-linked* if for every separation (A, B) in G we have $|A \cap B| \geq \alpha \min\{|A \cap T|, |B \cap T|\}$.

Reed proved that treewidth and the size of a largest vertex-well-linked set are tied by a constant factor [70].

Lemma 6.2.1. *Let G be a graph, and suppose k is the size of the largest vertex-well-linked set in G. Then $\frac{k}{4} - 1 \leq tw(G) \leq k - 1$.*

If $S \subseteq V(G)$ we denote by $\Gamma(S)$ those vertices in S with a neighbour in $V(G) \setminus S$. We will also need a simple lemma whose proof can be found in [12].

Lemma 6.2.2. *Let $x_1, \ldots, x_n$ be integers with $\sum_{i=1}^{k} x_i = N$ and $x_i \leq \frac{2N}{3}$ for each i. Then there exists a partition (A, B) of $[n]$ such that $\sum_{i \in A} x_i \geq \frac{N}{3}$ and $\sum_{i \in B} x_i \geq \frac{N}{3}$.*

Let G be a graph with maximum degree $\Delta(G)$ and assume that G has a set $\mathcal{T}$ of k vertex-well-linked vertices. We can now define the main object of our interest.

Definition 6.2.3. *Let h be an integer and $\alpha \leq 1$ be a positive number. We say $S \subseteq V(G)$ is an (α, h)-good router for $\mathcal{T}$ if*

(a) $S \cap \mathcal{T} = \emptyset$,

(b) $\Gamma(S)$ is α-well-linked in $G[S]$,

(c) there exist h vertex disjoint paths between S and $\mathcal{T}$.

The set $\mathcal{T}$ will generally be fixed in what follows, and so we usually just call S an (α, h)-good router.

6.3 The main result

We prove the following.

Theorem 6.3.1. *Let G be a graph and $T \subseteq V(G)$ be a set of k vertex-well-linked vertices. Suppose that r, h, α satisfy*

1. $2700(r+1)^2 h\alpha^{-1}\Delta(G) \le k$

2. $2\alpha\Delta(G)\log(512krh^2\alpha^{-2}\Delta(G)^2) < \frac{1}{8r}$

Then there exist r disjoint (α, h)-good routers in G.

In [12] they show implicitly that G has r disjoint (α, h)-good routers for certain values of $h = O(k^{\epsilon})$, $r = O(k^{\frac{\epsilon}{20}})$ and $\alpha^{-1} = \Omega(k^{\frac{\epsilon}{10}})$ for some $\epsilon < 1$ (they use $\epsilon \approx \frac{1}{98}$ but $\epsilon = \frac{10}{11}$ seems to work). Their proof may imply something more general, but their techniques require at least that $h\alpha^{-1} = O(k)$ and $\alpha^{-1} = \Omega(r^2)$ and $\Delta(G) = O(\log^3(k))$. Our result implies theirs, and gives more flexibility in the parameters r, h and α. In particular, it weakens the dependence of α on r- under the conditions of Theorem 6.3.1, one could take $\alpha^{-1} = O(r\Delta(G)\log k)$.

The rest of the chapter is dedicated to proving Theorem 6.3.1. It is organized as follows. In Section 6.4 we reduce the problem to an easier formulation to work with. Then, in Section 6.5 we state the two main lemmas we need, and deduce Theorem 6.3.1. These two lemmas correspond roughly to results that Chekuri and Chuzhoy use in [12] but our proofs are completely different. The last three sections contain proofs.

6.4 An alternate characterization of good routers

Let G be a graph, and fix a set $\mathcal{T}$ of vertex-well-linked vertices. As was observed in [12], the next lemma shows that Property (c) in the definition of a good router can be replaced by a lower bound on the size of $|\Gamma(S)|$ if we assume that G is minimal with the property that $\mathcal{T}$ is vertex-well-linked. For the rest of this chapter, we will make this minimality assumption on G. We also assume that $\mathcal{T}$ is a vertex-well-linked set of maximum size k, and so $\mathrm{tw}(G) \le 4k$ by Lemma 6.2.1.

Lemma 6.4.1. *Let G be a graph, $\mathcal{T}$ a set of k vertex-well-linked vertices. Suppose further that G is minimal, with respect to vertex-deletion, such that $\mathcal{T}$ is vertex-well-linked. If $S \subset V(G)$ satisfies Properties (a), (b) and*

(c') $|\Gamma(S)| \ge \frac{4h}{\alpha}$

then S is an (α, h)-good router for T.

Lemma 6.4.1 is an easy corollary of the next lemma, which essentially comes from [14], and Menger's theorem [64].

Lemma 6.4.2. *Let $H_1, H_2 \subseteq V(G)$ be disjoint subsets of vertices and suppose that H_1 is vertex-well-linked and H_2 is α-well-linked for some $0 < \alpha \leq 1$. Let G' be obtained from G by adding a vertex s complete to H_1 and a vertex t complete to H_2. Now suppose that the minimum $s-t$ vertex cut in G' has size $\gamma \leq k$, and that $\gamma \leq \frac{\alpha}{4}|H_2|$. Then there exists a vertex $v \in V(G)$ such that H_1 is still vertex-well-linked in $G \setminus \{v\}$.*

If G is minimal such that $\mathcal{T}$ is vertex-well-linked and S satisfies Properties (a), (b) and (c'), then by Menger's theorem and Lemma 6.4.2 there are at least $\frac{\alpha}{4}|\Gamma(S)|$ vertex-disjoint paths between S and $\mathcal{T}$. Thus Property (c) is implied by Property (c'), and so S is an (α, h)-good router.

A proof of Lemma 6.4.2 can be found in Section 6.8.

6.5 A roadmap of the proof

Recall, we want to find r disjoint (α, h)-good routers for some appropriately chosen parameters. As described in Section 6.4, we assume G is minimal such that $\mathcal{T}$ is vertex-well-linked. We also assume hereafter for simplicity that the vertices in $\mathcal{T}$ have degree 1. This is easy to achieve since replacing a vertex in $\mathcal{T}$ with a new degree-1 vertex adjacent to it doesn't affect the well-linkedness of $\mathcal{T}$.

There are two major steps to the proof of Theorem 6.3.1. We begin with a partition of the vertex set of $G \setminus \mathcal{T}$, such that each set X in the partition has $|\Gamma(X)| \leq \frac{4h}{\alpha}$ and $\Gamma(X)$ is α-well-linked in $G[X]$ and $G[X]$ is connected. That is, each part satisfies Properties (a) and (b) but not Property (c'). Assume the partition is chosen so that the graph H obtained from G by contracting each part into a single vertex and deleting loops has a minimum number of edges. By construction H has maximum degree $\Delta(H) \leq \frac{4h\Delta(G)}{\alpha}$ and H has treewidth $< k$, since H is a minor of G. The following lemma allows us to partition $V(H) \setminus \mathcal{T}$ into r parts where the number of edges in each part is relatively large compared to the number of edges leaving.

Lemma 6.5.1. *Let H be a connected graph with maximum degree $\Delta(H)$, treewidth $tw(H) < k$ and a set $\mathcal{T} \subset V(H)$ of k vertices of degree 1. Suppose that $\mathcal{T}$ is edge-well-linked. Suppose further that*

$|E(H \setminus \mathcal{T})| \geq 225(r+1)^2\Delta(H)$.

Then there exist nonempty subsets $\{X_1, \ldots, X_r\}$ *of* $V(H) \setminus \mathcal{T}$ *such that for each* $1 \leq i \leq r$

- $|\delta_H(X_i)| \leq 32kr\Delta(H)^2$
- $|E_H(X_i)| \geq \frac{|\delta_H(X_i)|}{8r}$

This lemma will be proved in Section 6.6.

The partition of H given by Lemma 6.5.1 corresponds naturally to a partition of G. The next step is to show that each of these parts, taken as a subgraph of G, contains an (α, h)-good router. To achieve this, we will repartition each of the r parts X_i into a collection of sets, such that each set X satisfies Properties (a) and (b), and such that the new partition has a smaller number of edges between parts than there were in $H[X_i]$. The minimality of $|E(H)|$ will then imply that one of the parts must satisfy Property (c'). The repartitioning will be given by the following lemma.

Lemma 6.5.2. *Let x be a number satisfying $4\alpha\Delta(G)\log(x) \leq \frac{1}{2}$. Let G be a graph with maximum degree $\Delta(G)$, and $X \subseteq G$ with $|\Gamma(X)| \leq x$. Then, there exists a partition $\mathcal{C}$ of X, where each part $C \in \mathcal{C}$ has $\Gamma(C)$ α-well-linked in C and such that the number of edges of G with ends in different parts of $\mathcal{C}$ is at most $2\alpha\Delta(G)|\Gamma(X)|\log(|\Gamma(X)|)$.*

Lemma 6.5.2 will be proved in Section 6.7, but for now we already have the tools necessary to prove our main result, Theorem 6.3.1.

Proof of Theorem 6.3.1. Let $\mathcal{X}$ be a partition of $V(G) \setminus \mathcal{T}$ such that each part $X \in \mathcal{X}$ satisfies Properties (a) and (b) but not Property (c'). Such a partition exists, since each vertex taken as a singleton set satisfies (a) and (b) but not (c'). Let H be the graph obtained from G by contracting each part in X to a single vertex and deleting loops, and assume that the partition $\mathcal{X}$ is chosen so that $|E(H)|$ is minimum. Now, by Property (c') the graph H has maximum degree $\Delta(H) \leq 4h\alpha^{-1}\Delta(G)$. Also, since H is a minor of G, we have $\mathrm{tw}(H) \leq \mathrm{tw}(G) < k$. Since $\mathcal{T}$ is vertex-well-linked in G, it is edge-well-linked in H.

We claim that $|E(H \setminus \mathcal{T})| \geq \frac{k}{3}$. To see this, we contend that there is a partition of $\mathcal{T}$ into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$ so that $|\mathcal{T}_1| \geq \frac{k}{3}$ and $|\mathcal{T}_2| \geq k3$ and so that if $t, s \in \mathcal{T}$ have a common neighbour in $V(H) \setminus \mathcal{T}$, then they both belong to the same part. Then, there are $\frac{k}{3}$ edge-disjoint paths from $\mathcal{T}_1$ to $\mathcal{T}_2$ by the edge-well-linkedness of $\mathcal{T}$. By the choice of the partition, each of these must use an edge in $E(H \setminus \mathcal{T})$ and the claim follows. To obtain such a partition, it suffices to observe that each vertex in H has degree at most $4h\alpha^{-1}\Delta(G) \leq \frac{k}{3}$ and to apply Lemma 6.2.2.

We deduce that $|E(H \setminus \mathcal{T})| \geq \frac{k}{3} \geq 900(r+1)^2 h\alpha^{-1}\Delta(G) \geq 225(r+1)^2\Delta(H)$. Then, according to Lemma 6.5.1, we can find a partition of $V(H)\setminus\mathcal{T}$ into r non-empty parts $\{X_1, \ldots, X_r\}$ satisfying for each $1 \leq i \leq r$

- $|\delta_H(X_i)| \leq 32kr\Delta(H)^2$
- $|E_H(X_i)| \geq \frac{|\delta_H(X_i)|}{8r}$

Let $Y_1, \ldots, Y_r$ be the subsets of $V(G)$ obtained by uncontracting the vertices in each set X_i. Any subset of Y_i satisfies Property (a) easily, since the X_i are disjoint from $\mathcal{T}$.

Now, let $x = 32kr\Delta(H)^2$. Since $\Delta(H) \leq 4h\alpha^{-1}\Delta(G)$ it follows from our assumptions that $4\alpha\Delta(G)\log(x) \leq \frac{1}{2}$. It then follows from Lemma 6.5.2 that there exists a partition of each Y_i into parts satisfying Property (b) with at most

$$
\begin{aligned}
&2\alpha\Delta(G)|\Gamma(Y_i)|\log(|\Gamma(Y_i)|)\\
&\leq 2\alpha\Delta(G)|\delta(Y_i)|\log(|\delta(Y_i)|)\\
&\leq 2\alpha\Delta(G)\log(x)|\delta(Y_i)|\\
&\leq 2\alpha\Delta(G)\log(512krh^2\alpha^{-2}\Delta(G)^2)|\delta(Y_i)|\\
&< \frac{|\delta(Y_i)|}{8r}
\end{aligned}
$$

edges going between different parts. If none of the parts in this partition satisfies Property (c') then this contradicts the minimality of $|E(H)|$. So Y_i has a subset satisfying Properties (a), (b) and (c'). This gives r disjoint (α, h)-good routers, as needed.

□

6.6 Proof of Lemma 6.5.1

In this section, we give the proof of Lemma 6.5.1. In order to do so, we need to treat the cases where the graph has a large number of vertices separately from the case where the number of vertices is bounded. We need different machinery for each of these two cases. In Section 6.6.1 we present tools to deal with the case when the number of vertices is large. Then in Section 6.6.2 we develop what we will need to deal with the bounded case. We then complete the proof in Section 6.6.3.

6.6.1 Dual branch-decompositions

To deal with the case where the graph has a large enough number of vertices, we establish here the existence of a certain decomposition of a graph along bounded size edge cuts. Most of these results are essentially proved in Section 3 of Graph Minors X [76], but we give the theorems explicitly for completeness.

A *ternary tree* is a tree where every vertex has degree either 1 or 3. For a tree T we denote by $L(T)$ its set of leaves. The main result in this section is the following.

Lemma 6.6.1. *Suppose G has treewidth $< k$ and maximum degree Δ. Then there exists a ternary tree T and a bijective mapping between $V(G)$ and $L(T)$ with the following property: Each edge $e \in E(T)$ partitions $L(T)$ into two sets in the natural way, say X_e and $V(G) \setminus X_e$. For each e there are at most $k\Delta$ edges between X_e and its complement.*

We will give a sketch of the proof, but for that we need a little background. First, some definitions. Let us define, for each $X \subseteq V(G)$, a function $\kappa_0(X) = |\{uv \in E(G); |\{u, v\} \cap X| = 1\}|$. Then let $\kappa(X) = \kappa_0(X) - k\Delta$, and observe that $\kappa(X) = \kappa(V(G) \setminus X)$ and κ is submodular. Hence κ meets the requirements to be a *connectivity function* as defined in [76]. The *efficient* subsets of $V(G)$ are those which correspond to one side of edge cuts with at most $k\Delta$ edges. A *bias* is a set $\mathcal{B}$ of efficient subsets satisfying

- if $X \subseteq V(G)$ is efficient then $\mathcal{B}$ contains one of $X, V(G) \setminus X$
- if $X, Y, Z \subseteq V(G)$ then $X \cup Y \cup Z \neq V(G)$.

We say that a bias $\mathcal{B}$ extends a set $\mathcal{A}$ of efficient sets if $\mathcal{A} \subseteq \mathcal{B}$.

An *incidence* is a pair (v, e) where $v \in V(T)$ and e is an edge of T incident with v. An *exact tree-labelling over* $\mathcal{A}$ is a pair (T, α) where T is a ternary tree and α is a map from the incidences of T to the efficient subsets of $V(G)$ such that

1. for each $e = uv \in E(T)$, $\alpha(u, e) = V(G) \setminus \alpha(v, e)$

2. for each incidence (v,e) in T where v is a leaf, either $\alpha(v,e) = V(G)$ or $\alpha(v,e) \cup X = V(G)$ for some $X \in \mathcal{A}$

3. if $v \in V(T)$ has degree three, incident with edges e, f, g, say, then $\alpha(v,e) \cup \alpha(v,f) \cup \alpha(v,g) = V(G)$, and these three sets are mutually disjoint.

In [76] it is shown that for any set $\mathcal{A}$ of efficient subsets, there is a bias extending $\mathcal{A}$ if and only if there is no exact tree-labelling over $\mathcal{A}$.

A *tangle* of order k is a collection $\mathcal{T}$ of separations of order $< k$ in G such that

i. for every separation (A,B) of order $< k$ either (A,B) or (B,A) belongs to $\mathcal{T}$

ii. if $(A_1,B_1),(A_2,B_2),(A_3,B_3)$ belong to $\mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$

iii. if $(A,B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Assertions i.,ii. and iii. are called the tangle axioms. The *tangle number* of G, denoted $\text{tn}(G)$ is the maximum order of a tangle in G. We have (see for example [46])

$$\text{tn}(G) \leq \text{tw}(G) + 1 \leq \tfrac{3}{2}\text{tn}(G).$$

Now we need a lemma that is similar to Lemma 4.3 (1) in [76]. Let $\mathcal{A} = \{\{v\}; v \in V(G)\}$.

Lemma 6.6.2. *There is a bias extending $\mathcal{A}$ if and only if G has a tangle of order $k+1$.*

Proof of Lemma 6.6.2. If $\mathcal{T}$ is a tangle in G of order $k+1$ then let $\mathcal{B} = \{V(A); (A,B) \in \mathcal{T}\}$. Since $|A \cap B| \leq k$ we have $|E(A, V(G) \setminus A)| \leq k\Delta$ so $\mathcal{B}$ is a bias and it extends $\mathcal{A}$ by the third tangle axiom.

Conversely, let $\mathcal{B}$ be a bias extending $\mathcal{A}$ and let $\mathcal{T}$ be the set of all separations (A,B) of order $\leq k$ with $V(A) \in \mathcal{B}$. We claim that $\mathcal{T}$ is a tangle of order $k+1$. Well, if (A,B) is a separation of order $\leq k$ then both $V(A)$ and $V(B)$ are efficient, so one of them belongs to $\mathcal{B}$, say $V(A)$. Then $(A,B) \in \mathcal{T}$. So the first axiom holds, and clearly the second and third do as well. This completes the proof of Lemma 6.6.2. $\square$

Proof of Lemma 6.6.1. Let $\mathcal{A} = \{\{v\}; v \in V(G)\}$. The above discussion implies that the following are equivalent.

1. There is an exact tree-labelling over $\mathcal{A}$
2. There is no bias extending $\mathcal{A}$
3. There is no tangle of order $k+1$ in G
4. $tn(G) \leq k$

Consequently if $\mathrm{tw}(G) \leq k-1$ then there is an exact tree-labelling (T, α) over $\mathcal{A}$.

We now need to give the bijection between $L(T)$ and $V(G)$. From the definition of an exact tree-labelling, for each incidence (v, e) of T where v is a leaf, we have $\alpha(v, e) = V(G)$ or $\alpha(v, e) = V(G) \backslash x$ for some $x \in V(G)$.

Claim 2. *We may assume T has no leaf incidences (v, e) with $\alpha(v, e) = V(G)$.*

Suppose there is such an incidence (v, e). Let u be the other end of e. We have $\alpha(u, e) = \emptyset \notin \mathcal{A}$. In particular u is not a leaf. Assume u is incident with edges $e_1 = uv_1$ and $e_2 = uv_2$. Let T' be obtained from T as follows. Let $V(T') = V(T) \setminus \{u, v\}$ and $E(T') = E(T) \setminus \{e, e_1, e_2\} \cup \{f = v_1v_2\}$. We now define an exact tree-labelling over $\mathcal{A}$ using T' instead of T. For each incidence (w, h) of T', define $\alpha'(w, h) = \alpha(w, h)$ if (w, h) is an incidence of T. Define $\alpha'(v_1, f) = \alpha(v_1, e_1)$ and $\alpha'(v_2, f) = \alpha(v_2, e_2)$. It is easy to check that (T', α') is an exact tree-labelling over $\mathcal{A}$. The claim follows by repeatedly removing leaf incidences in this manner.

Now, for each leaf incidence (v, e) of T, define $\gamma(v) = V(G) \setminus \alpha(v, e)$; note that $\gamma(v)$ is always a single vertex. We claim that γ is the desired bijection between $L(T)$ and $V(G)$. Since each $\alpha(v, e)$ corresponds to an efficient subset, i.e. an edge cut with at most $k\Delta$ edges, it is enough to prove the following.

Claim 3. *Let $e = uv \in E(T)$ and let T_e^u and T_e^v be the components of $T \setminus e$ containing u and v, respectively. Then $\{\gamma(w) : w \in L(T) \cap T_u\} = \alpha(v, e)$ and $\{\gamma(w) : w \in L(T) \cap T_v\} = \alpha(u, e)$.*

Proof. If e is a leaf edge, then the claim is clearly true. Suppose then that both u and v have degree 3, and suppose that u is incident with edges $e, e_1 = uv_1, e_2 = uv_2$. Assume the claim is true for e_1 and e_2. We have $L(T) \cap T_e^v = L(T) \cap T_{e_1}^u \cap T_{e_2}^u$ and $L(T) \cap T_e^u = L(T) \cap (T_{e_1}^{v_1} \cup T_{e_2}^{v_2})$. From the definition of a tree-labelling, we have $\alpha(u, e) = V(G) \setminus (\alpha(u, e_1) \cup \alpha(u, e_2)) = \alpha(v_1, e_1) \cap \alpha(v_2, e_2)$.

Also $\alpha(v,e) = V(G) \setminus \alpha(u,e) = \alpha(u,e_1) \cup \alpha(u,e_2)$. So the claim is true for e as well. Inductively the claim holds for all edges. $\square$

This completes the proof of Lemma 6.6.1. $\square$

The decomposition given by Lemma 6.6.1 will allow us to obtain the disjoint subsets of a graph called for in Lemma 6.5.1 by partitioning the ternary tree appropriately.

Lemma 6.6.3. *Let T be a ternary tree and suppose $r < L(T)$. Then there exists a partition of T into subtrees $T_1, \ldots, T_r$, each containing at least $\frac{L(T)}{2r-1}$ leaves, and $\delta_T(T_i) \leq r-1$ for each $1 \leq i \leq r$. Further, there exists a set of $\lfloor \frac{r}{2} \rfloor$ disjoint subtrees $T_1, \ldots, T_{\lfloor \frac{r}{2} \rfloor}$, each containing at least $\frac{L(T)}{2r-1}$ leaves, and $\delta_T(T_i) \leq 3$ for each i.*

Proof. Let $\mathcal{F}$ denote the family of subtrees of T which contain at least $\frac{L(T)}{2r-1}$ leaves in $L(T)$ and which have at most $r-1$ edges leaving them. We begin by finding r disjoint subtrees in $\mathcal{F}$ (but not necessarily with union spanning the entire tree). To do this, we use the Helly property for a family of subtrees of a tree. Namely, either there exist r disjoint subtrees in $\mathcal{F}$ or there exists a set of $r-1$ vertices in T meeting every member of $\mathcal{F}$. If we find the r disjoint trees then we're done; else we find $r-1$ vertices that hit all subtrees in the family. We can assume that none of these vertices belong to $L(T)$ (because in this case $\frac{L(T)}{2r-1} > 1$ and so any member of $\mathcal{F}$ containing a leaf also contains its parent). Now, removing these $r-1$ vertices, we are left with at most $2r-1$ connected components. Each of them is a tree with at most $r-1$ edges leaving to the rest of T. One of them must have $\frac{L(T)}{2r-1}$ leaves, contradicting the fact that we hit all such subtrees. So there must exist r disjoint trees each with at least $\frac{L(T)}{2r-1}$ leaves, we want to grow these subtrees so that they partition T. We can add vertices to the subtrees greedily until they partition the vertices of T. This proves the first assertion.

Then consider the 'touching graph' of the subtrees, i.e. make a vertex for each subtree and join two if there is an edge between them in T. This touching graph is a simple tree, and it has r vertices, and therefore $r-1$ edges, and hence average degree ≤ 2. It follows that at least half of the subtrees in our set have at most 3 edges sticking out. This proves the second assertion and completes the proof of Lemma 6.6.3.

$\square$

6.6.2 A partition into sets containing many edges relative to the number leaving

In this section, we show the following lemma which almost gives the subsets that we are looking for in Lemma 6.5.1, but without the guarantee of an upper bound on the number of edges leaving each set. We will therefore apply it in the case when the graph has a bounded number of vertices (and hence a bounded number of edges).

Lemma 6.6.4. *Let G be a graph with maximum degree Δ and let $r > 0$. Suppose that $|E(G)| \geq 225r^2\Delta(G)$. Then there exists a partition of $V(G)$ into r parts $\{X_1, \dots, X_r\}$, with the property that for each $1 \leq i \leq r$, $4(r-1)|E(X_i)| \geq |\delta(X_i)|$.*

Proof. Choose a partition $\{X_1, \dots, X_r\}$ of $V(G)$ so that the product $\prod_{1 \leq i \leq r} |E(X_i)|$ is maximized. Consider two parts X_i and X_j, say, and let $v \in X_i$ with x_i neighbours in X_i and x_j neighbours in X_j. From the maximality of the chosen partition we have

$$(|E(X_j)| + x_j)(|E(X_i)| - x_i) \leq |E(X_i)||E(X_j)|$$

or simply $|E(X_j)|x_i \geq |E(X_i)|x_j - x_i x_j$. Summing this inequality over all vertices in X_i it follows that

$$|E(X_j)|(2|E(X_i)|) \geq |E(X_i)||E(X_i, X_j)| - 2|E(X_i)|\Delta.$$

Simplifying and summing over all $j \neq i$, we obtain

$$\sum_{i \neq j} |E(X_j)| + (r-1)\Delta \geq \tfrac{1}{2}|\delta(X_i)|.$$

If $|E(X_i)| \geq \Delta$, then the required inequality $4(r-1)|E(X_i)| \geq |\delta(X_i)|$ holds. Thus it is enough to show that for each i we have $|E(X_i)| \geq \Delta$.

Suppose to the contrary that $|E(X_1)| < \Delta$, say. We claim that there exists some $i \neq 1$ with $|E(X_i)| \geq 225\Delta$. For if not, then the total number of edges in G is less than $(\Delta + (r-1)225\Delta)(1+(r-1)) + r\Delta \leq 225r^2\Delta$, contrary to our hypothesis. Without loss of generality assume $|E(X_2)| \geq 225\Delta$. Then $|E(X_1)||E(X_2)| \leq \frac{1}{225}|E(X_2)|^2$.

By Lemma 6.6.5 (see below) there exists a partition of $X_1 \cup X_2$ into two parts Y_1, Y_2 such that $|E(Y_1)|, |E(Y_2)| \geq \frac{1}{15}(|E(X_1)| + |E(X_2)| + |E(X_1, X_2)|)$. But then $|E(Y_1)||E(Y_2)| > \frac{1}{225}|E(X_2)|^2$, contradicting the maximality of $\prod_{1\leq i\leq r}|E(X_i)|$.

$\square$

Lemma 6.6.5. *Let G be a graph with maximum degree at most Δ (possibly with loops) and suppose $|E(G)| \geq \frac{5}{2}\Delta$. Then there exists a partition $\{X_1, X_2\}$ of $V(G)$ such that $|E(X_1)|, |E(X_2)| \geq \frac{1}{15}|E(G)|$.*

Proof. We begin by obtaining an auxiliary graph G' from G, where G' has exactly four vertices and every vertex has degree at least half the maximum degree (note G' may have maximum degree greater than Δ).

Since $|E(G)| \geq \frac{5}{2}\Delta$, the graph G must have at least 5 vertices. If G has two vertices the sum of whose degrees is less than Δ, then identify these two vertices, keeping loops. Abusing notation slightly, let G denote this new graph. Repeat this step until there no longer exists such a pair of vertices in G. Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the degree sequence in G. Then $d_1 + d_2 > \Delta$ and $d_3, \ldots, d_n \geq \frac{1}{2}(d_1 + d_2)$. Thus, let G' be obtained from G by identifying the two vertices of smallest degree. Observe that in G' every vertex has degree at least half that. Identifying the two vertices in G' with smallest degree preserves this property; so until G' has exactly four vertices, repeat this step. Any partition of $V(G')$ satisfying the conclusion of the lemma gives a corresponding partition of $V(G)$ which also satisfies it, by unidentifying vertices.

Now, write Δ' and E' for the maximum degree and number of edges in G', respectively, and observe that $\Delta' \leq E' \leq 2\Delta'$. Let v_1, v_2, v_3, v_4 be the four vertices of G' and for $i \neq j$ let e_{ij} be the number of edges between v_i and v_j. Finally let ℓ_i denote the number of loops incident with v_i. For each i, we have $2\ell_i + \sum_{j\neq i} e_{ij} \geq \frac{1}{4}E'$. It follows that for each i, there exists $j \neq i$ such that $\ell_i + e_{ij} \geq \frac{1}{12}E'$. If we can find two disjoint pairs i, j with this property then this gives the desired partition. Assume for a contradiction then, that the only pairs satisfying the previous inequality are $1,2$ and $1,3$ and $1,4$ and without loss of generality suppose $e_{12} \leq e_{13} \leq e_{14}$. We can assume further that $e_{24} + \ell_2 + \ell_4 < \frac{1}{15}E'$ and $e_{23} + \ell_2 + \ell_3 < \frac{1}{15}E'$ and $e_{34} + \ell_3 + \ell_4 < \frac{1}{15}E'$. By assumption $2\ell_1 + e_{12} + e_{13} + e_{14} \leq 2(2\ell_2 + e_{12} + e_{23} + e_{24})$; we deduce that $2\ell_1 + e_{13} + e_{14} < e_{12} + \frac{4}{15}E'$. It follows that $e_{14} + \ell_1 < \frac{4}{15}E'$ and therefore that $e_{12} + \ell_1, e_{13} + \ell_1 < \frac{4}{15}E'$ as well, a contradiction.

$\square$

6.6.3 The proof

We are now ready to prove Lemma 6.5.1. Let us first recall the statement.

Lemma. *Let H be a connected graph with maximum degree $\Delta(H)$, treewidth $tw(H) < k$ and a set $\mathcal{T} \subset V(H)$ of k vertices of degree 1. Suppose that $\mathcal{T}$ is edge-well-linked. Suppose further that $|E(H \setminus \mathcal{T})| \geq 225(r+1)^2\Delta(H)$.*

Then there exist nonempty subsets $\{X_1, \dots, X_r\}$ of $V(H) \setminus \mathcal{T}$ such that for each $1 \leq i \leq r$

- $|\delta_H(X_i)| \leq 32kr\Delta(H)^2$
- $|E_H(X_i)| \geq \frac{|\delta_H(X_i)|}{8r}$

Proof of Lemma 6.5.1. We consider two cases, depending on the size of $|V(H)|$.

Case 1: $|V(H \setminus \mathcal{T})| \geq 64kr\Delta(H)$.

In this case we can actually obtain a stronger conclusion than the lemma statement, namely

- $|\delta_H(X_i)| \leq 5k\Delta(H)$
- $|E_H(X_i)| \geq \frac{|\delta_H(X_i)|}{2}$.

By Lemma 6.6.1, there exists a ternary tree T and a bijection between $V(H) \setminus \mathcal{T}$ and $L(T)$ such that for each edge of T, the corresponding cut in H contains at most $k\Delta$ edges. According to Lemma 6.6.3 there exist r disjoint subtrees of T, say $T_1, \dots, T_r$, each containing at least $\frac{|V(H \setminus \mathcal{T})|}{4r-1}$ leaves of T and such that for each i we have $|\delta_T(T_i)| \leq 3$. It follows that there exist r subsets of $V(H \setminus \mathcal{T})$, say $X_1, \dots, X_r$ with $|X_i| \geq \frac{|V(H \setminus \mathcal{T})|}{4r-1}$ and $|\delta_H(X_i)| \leq 3k\Delta(H) + k \leq 5k\Delta(H)$ (where the additional k term comes from the terminals $\mathcal{T}$). Now, since H is connected each vertex in X_i either belongs to $\Gamma(X_i)$ or has a neighbour in X_i, and so $|E_H(X_i)| \geq \frac{1}{2}(\frac{|V(H \setminus \mathcal{T})|}{4r-1} - |\delta_H(X_i)|)$. Since $|V(H \setminus \mathcal{T})| \geq 64kr\Delta(H)$, we have $|E_H(X_i)| \geq |\delta_H(X_i)|(\frac{r}{r-\frac{1}{4}} - \frac{1}{2}) \geq \frac{|\delta_H(X_i)|}{2}$, as required.

To be sure each set X_i is nonempty we need $\frac{|V(H \setminus \mathcal{T})|}{4r-1}$ positive, but this is true since $8kr\Delta(H) > 1$.

Case 2: $|V(H \setminus \mathcal{T})| \leq 64kr\Delta(H)$.

In this case we have $|E(H \setminus \mathcal{T})| \leq 32kr\Delta(H)^2$. By hypothesis, we also have $|E(H \setminus \mathcal{T})| \geq 225(r+1)^2\Delta(H)$. Applying Lemma 6.6.4, there exist $r+1$ disjoint subsets $X_1 \dots X_{r+1} \subseteq V(H \setminus \mathcal{T})$ with the property that $|E(X_i)| \geq \frac{1}{4r}|\delta(X_i)|$ for each $1 \leq i \leq r+1$.

Now, viewing each X_i as a subset of $|V(H)|$, we can assume, by discarding one set if necessary, that each of $X_1, \ldots, X_r$ has at most $\frac{k}{2}$ neighbours in $\mathcal{T}$. Because $\mathcal{T}$ is edge-well-linked, edges incident with vertices in $\mathcal{T}$ make up at most half of the edges in $\delta(X_i)$. Thus each X_i satisfies $|\delta(X_i)| \leq 32kr\Delta(H)^2$ and $|E(X_i)| \geq \frac{1}{8r}|\delta(X_i)|$.

□

6.7 Proof of Lemma 6.5.2

Let us restate Lemma 6.5.2 for convenience.

Lemma. *Let x be a number satisfying $4\alpha\Delta(G)\log(x) \leq \frac{1}{2}$. Let G be a graph with maximum degree $\Delta(G)$, and $X \subseteq G$ with $|\Gamma(X)| \leq x$. Then, there exists a partition $\mathcal{C}$ of X, where each part $C \in \mathcal{C}$ has $\Gamma(C)$ α-well-linked in C and such that the number of edges of G with ends in different parts of $\mathcal{C}$ is at most $2\alpha\Delta(G)|\Gamma(X)|\log(|\Gamma(X)|)$.*

Proof of Lemma 6.5.2. We begin with $X \subseteq G$; the set X has a subset $\Gamma(X)$ of vertices which have neighbours in $V(G) \setminus X$. For convenience, we will consider X along with a 'half-edge' incident with each vertex in $\Gamma(X)$. We'll denote the set of half-edges by $\delta(X)$. If C is a subset in a partition of X we denote by $\Gamma(C)$ those vertices that either have a neighbour in a different part, or belong to $\Gamma(X)$.

We begin with the partition $\mathcal{C} = \{X\}$ and, by hypothesis $|\Gamma(X)| \leq x$. If $\Gamma(X)$ is α-well-linked in X then this is the desired decompostion of X with 0 edges.

So X is not α-well-linked, and so there exists a separation (A, B) of X with $|A \cap B| < \alpha \min\{|A \cap \Gamma(X)|, |B \cap \Gamma(X)|\}|$. Assume that $|A \cap \Gamma(X)| \leq |B \cap \Gamma(X)|$. Then there are at most $\alpha|A \cap \Gamma(X)|\Delta(G)$ edges between A and $X \setminus A$. We have $\alpha\Delta(G) \leq \frac{1}{6}$ so $|\Gamma(A)| \leq \frac{2}{3}|\Gamma(X)|$. We now have a partition of X into A and $X \setminus A$; if both are α-well-linked we stop, otherwise we further decompose pieces along small cuts. We continue refining the partition iteratively until we either find a large α-well-linked cluster or we get a decomposition of X into small α-well-linked clusters. In the first situation, we are happy. In the second situation, let us count the number of edges we have between parts.

We use a potential function argument. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a partition of X. For each edge $e \in E(X) \cup \delta(X)$ we will assign a potential function $\phi(e, \mathcal{C})$.

First, for each $z > 0$ define $\rho(z) = 4\alpha\Delta(G)\log(z)$, and we need that $\rho(x) < \frac{1}{2}$. Let

$$\phi(e, \mathcal{C}) := \begin{cases} \phi(e) = 1 + \rho(z) + \rho(z') & \text{if for some } i, j, e \in E(C_i, C_j), \text{ and } |\Gamma(C_i)| = z, |\Gamma(C_j)| = z', \\ \phi(e) = 1 + \rho(z) & \text{if } e \in \delta(X) \text{ and some end of } e \in C_i \text{ with } |\Gamma(C_i)| = z, \\ 0 & \text{otherwise, i.e. both ends belong to the same part.} \end{cases}$$

Observe that $\phi(e) \leq 2$, for every edge e. So the sum of all potentials of edges in $E(X) \cup \delta(X)$ is at most twice the number of such edges.

At the start the total potential is just $|\Gamma(X)|(1 + 4\alpha\Delta(G)\log(|\Gamma(X)|))|$. Each time we refine the partition we claim that the potential decreases. Consider partitioning a part $C \subseteq X$ into two new sets (C_1, C_2). Assume that $|\Gamma(C) \cap C_1| \leq |\Gamma(C) \cap C_2|$. We have $|\Gamma(C_1)|, |\Gamma(C_2)| \leq |\Gamma(C)|$, since $\alpha\Delta(G) < 1$. For each edge $e \in \delta(C) \cap \delta(C_2)$ the value of $\phi(e)$ decreases by $\rho(|\Gamma(C)|) - \rho(|\Gamma(C_2)|)$. For each edge $e \in \delta(C) \cap \delta(C_1)$, since $|\Gamma(C_1)| \leq \frac{2}{3}|\Gamma(C)|$ the potential decreases by at least $4\alpha\Delta(G)(\log(|\Gamma(C)|) - \log(\frac{2}{3}|\Gamma(C)|)) \geq 4\alpha\Delta(G)\log(1.5)$, so we have a total decrease of at least $4\log(1.5)\alpha\Delta(G)|C_1 \cap \Gamma(C)|$. Finally, those edges $e \in E(C_1, C_2)$, which previously had potential 0, now have a positive $\phi(e) = 1 + \rho(|\Gamma(C_1)| + \rho(\Gamma(C_2))) \leq 2$. So in total the increase in potential from these edges is at most $2\alpha\Delta(G)|C_1 \cap \Gamma(C)|$ and overall, the potential is decreased.

So, when we have refined the partition of X to a point where all parts are α-well-linked, we are left with a total potential of at most $|\Gamma(X)|(1 + 4\alpha\Delta(G)\log(|\Gamma(X)|))|$. Each half-edge in $\delta(X)$ contributes at least 1 to the total, and so we must have created at most $2\alpha\Delta(G)|\Gamma(X)|\log(|\Gamma(X)|)$ new edges, as needed. □

6.8 Proof of Lemma 6.4.2

Here we give the proof of Lemma 6.4.2. The proof is an adaptation of a similar result in [14] but we provide it for completeness.

Proof of Lemma 6.4.2 (adapted from [14]). We know there exists a separation (A, B) with $H_1 \subseteq A$ and $|H_2 \cap (B \setminus A)| \geq \frac{1}{2}H_2$ and $|A \cap B| = \gamma$. The second condition is true since $\gamma \leq \frac{1}{2}|H_2|$. Choose such a separation (A, B) that also minimizes $|B|$. (Note we don't necessarily require $H_2 \subseteq B$ here.) We claim there's a vertex in $B \setminus A$ that we can delete. If not, then pick a vertex $v \in B \setminus A$ and there's some separation (S, T) with $|S \cap H_1| = \ell$ and $|S \cap T| = \ell$ and $v \in S \cap T$.

Now, denote by $i = |H_2 \cap (B \setminus A) \cap (S \setminus T)|$ and $j = |H_2 \cap (B \setminus A) \cap (T \setminus S)|$ and $h = |(H_2 \cap S \cap T) \setminus A|$.

By submodularity of vertex cuts we have

$$|A \cap B| + |S \cap T| \geq |(S \cap A) \cap (B \cup T)| + |(B \cap T) \cap (S \cup A)|.$$

Also, we have

$$|A \cap B| + |S \cap T| \geq |(A \cap T) \cap (B \cup S)| + |(B \cap S) \cap (A \cup T)|.$$

Recall that $|A \cap B| + |S \cap T| = \ell + \gamma$.

We know (since $H_1 \subseteq A$) that $|(S \cap A) \cap (B \cup T)| \geq \ell$ and also $|(A \cap T) \cap (B \cup S)| \geq \ell$. So we can assume that $|(B \cap S) \cap (A \cup T)| \leq \gamma$ and $|(B \cap T) \cap (S \cup A)| \leq \gamma$ otherwise we are done.

We claim that $(T \setminus S) \cap B$ and $(S \setminus T) \cap B$ are both nonempty. For, suppose $(T \setminus S) \cap B = \emptyset$. Then $A \cap B \subseteq S$, and since $v \in B \setminus A$ there are at least $\gamma + 1$ vertices in $(A \cap B) \cup \{v\} \subseteq (B \cap S) \cap (A \cup T)$. The analogous argument for $(S \setminus T) \cap B$ is similar. Therefore the minimality of B implies that both $i < \frac{1}{2}|H_2|$ and $j < \frac{1}{2}|H_2|$.

From the α-well-linkedness of H_2 we must have $\alpha|H_2 \cap A| \leq \gamma$ and so $(i+j+h) \geq |H_2| - \frac{\gamma}{\alpha}$. We can assume by symmetry that $i < j$ and so $|(B \cap T) \cap (S \cup A)| \geq \alpha(i+h) \geq \alpha(|H_2| - \frac{\gamma}{\alpha} - j) > \alpha(\frac{1}{2}|H_2| - \frac{\gamma}{\alpha})$. Combined with the fact that $\gamma \geq |(B \cap T) \cap (S \cup A)|$ we conclude that $\gamma > \frac{\alpha}{4}|H_2|$, a contradiction. $\square$

Chapter 7

An algorithm for p-centres in δ-hyperbolic graphs

7.1 Introduction

In this final chapter, we change gears and present an algorithmic result with applications in large graph analytics. This is joint work with William Sean Kennedy and Iraj Saniee and has been submitted for publication; a manuscript can be found on the arXiv [33].

The p-center algorithm is a discrete variant of one of the most frequently used clustering algorithms, the so-called k-means clustering. The goal of the p-center algorithm is to identify on a given graph a pre-specified number p of vertices (called centers), such that the maximum distance of any graph vertex to its nearest center is minimized. For any given p, the algorithm naturally partitions a graph into p clusters induced by the position of its p-centers. Clusters induced by the p-centers are not necessarily balanced in size as they are determined by the metric properties of the graph. Thus p-center clustering is more appropriate for distance-based partitioning or classification than other frameworks, such as community detection. Unfortunately, as a clustering algorithm the complexity of the p-center algorithm is generally prohibitive, $O(n^p)$ for an n-vertex graph, making it inapplicable to even moderate size graphs.

Proved nearly four decades ago, Shier's minimax result for trees and metric trees leads to an exact algorithm with quasilinear time complexity (in the number of vertices and edges of the graph) for determination of an optimal set of p-centers by repeatedly finding diametrical pairs of vertices and removing a ball from the graph containing one end of the current diametrical pair [80]. Hochbaum and Shmoys [49] gave a (multiplicative) 2-approximation algorithm for determining p-centres in graphs satisfying the triangle inequality with running time $O(m \log_2 m)$. Subsequently, Dyer and Frieze [30] improved this to a 2-approximation algorithm with running time $O(np)$. These algorithms are, in a sense, best possible as Hsu and Nemhauser [50] show that determining an α-approximate solution to p-centers is NP-hard whenever $\alpha < 2$.

In an insightful paper, Chepoi and Estellon essentially applied Shier's technique from [80] to graphs with small hyperbolic constant, δ [16]. These are graphs whose metric structure differs from the metric structure of a tree by a fixed constant (as we will explain in Section 7.2 and, in particular, Section 7.2.2 and Figure 7.1. For more details see [44, 8, 16]). The algorithmic version of this scheme gives rise to what is essentially an $O(n^3)$ time approximation for p-center on an n-vertex graph with hyperbolic constant δ appearing both as a prefactor in the complexity expression and also in the degree of approximation in terms of an additive constant to the radius of the optimal p-center partition. Of course the cubic time complexity $O(n^3)$ is still impractical for graphs of hundreds of thousands to millions of vertices as would be even a quadratic-time approximation.

Since there is evidence that real-life networks extracted from social media, co-authorship and collaboration, friendship and many other settings have small hyperbolic constants [56], it would be desirable to know if the cubic complexity is tight or can be further reduced, at least by negotiating on the degree of the approximation. In this chapter we will see that by giving up to 3δ in the (additive) approximation, one can achieve a quasilinear time p-center approximation. As such, this scheme is the first p-center approximation applicable to large graphs, particularly when p is relatively small, for example in the range $10 - 10^4$ and n is large, for example, $10^5 - 10^9$ vertices.

In the following sections we describe how the cubic complexity of [16] to quasilinear reduction is achieved without adding more than 3δ to the radius of the optimal p-center clusters. In Section 7.2 we outline necessary definitions, in particular, for geodesic metric spaces (Section 7.2.1) and hyperbolicity (Section 7.2.2). We then turn to a more formal discussion of p-centers, p-packings, and the dual problems which take center stage in our discussion (Section 7.3). In Section 7.3.1 we focus on algorithms for solving and approximating these problems on δ-hyperbolic graphs. The formal statements of our main results are also found in Section 7.3.1. Section 7.4 contains the proofs of the main results.

7.2 Definitions and notation

In this chapter we consider only simple, undirected graphs. Let $G = (V, E)$. To each edge uv, we associate a line segment of length 1, so that we may refer to any point on uv at distance t from u and $1 - t$ from v ($0 \leq t \leq 1$). This (uncountably infinite) set of points of G is denoted $A(G)$. We will use the notation $n = |V(G)|$ and $m = |E(G)|$. The distance $d(u, v)$ between any two points u and v in $A(G)$ is the length of a shortest path between them in G. When u and v are vertices, we write $[u, v]$ to refer to a shortest (also called *geodesic*) path. Note that shortest paths need not be unique. For a geodesic path $P = [u, v]$ and $i \in [0, d(u, v)]$, the point $P[i]$ is the one at distance i from u on P.

7.2.1 Geodesic metric spaces and graphs

Let (X, d) be a metric space. If x, y are points in X, a *geodesic segment* $[x, y]$, when it exists, is a continuous curve parametrized by the line segment $[a, b]$ of length $d = d(x, y)$. That is, a map $\rho : [0, d] \to X$ with $\rho(0) = x, \rho(d) = y$ and $d(\rho(s), \rho(t)) = |s - t|$ for each $s, t \in [0, d]$. A metric space is *geodesic* if for every pair of points there exists a geodesic segment joining them. Note that geodesic segments need not be unique, e.g. a diagonal pair of points on a cycle.

Any graph as we have defined above can be viewed as a geodesic metric space $(A(G), d)$. Such a metric space is called *graphic* and it will be convenient in what follows to think of graphs in this way. In a graphic metric space, a geodesic $[x, y]$ is simply a shortest path from x to y regardless of x and y being in $V(G)$ or in $A(G)$.

Let $S \subseteq X$ be compact. The *diameter* $\mathrm{diam}(S)$ of S is the maximum length of a geodesic between two vertices in S. For $u \in S$, $F_S(u)$ is the set of points in S whose distance from u is maximum. Two points $u, v \in S$ are *diametrical* if $d(u, v) = \mathrm{diam}(S)$. They are *locally diametrical* if $u \in F_S(v)$ and $v \in F_S(u)$. It follows that $d(u, v) \leq diam(S)$ for $v \in F_S(u)$ and $d(v, u) \leq diam(S)$ for $u \in F_S(v)$.

If v is a point of $A(G)$ and $r \in \mathbb{R}$, we write $B_r(v)$ for the closed ball of radius r about v, i.e. all points at distance at most r from v. For a geodesic path $P = [u, v]$ and for the length

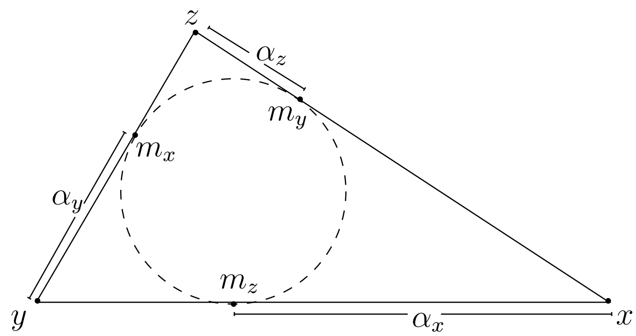

Figure 7.1: A geodesic triangle $\Delta(x, y, z)$ with internal points m_x, m_y and m_z and internal distances α_x, α_y and α_z labelled.

$0 \leq \theta < d(u, v)$, the point $i = [u, v][\theta] \in A(G)$ is at distance θ from u on P. When there is no ambiguity, we identify the point $i = P[\theta]$ with the length θ. Clearly the two points $[u, v][i]$ and $[v, u][i]$ do not generally coincide.

7.2.2 Hyperbolicity

The concept of hyperbolicity of a metric space was introduced by Rips and Gromov in [44]. There are several essentially equivalent definitions but in this chapter we will mainly use the *δ-thin-triangle* characterization.[1] For points x, y, z in X, we write $\Delta(x, y, z)$ to denote a *geodesic triangle* formed by x, y, z; that is the union of three geodesics $[x, y], [y, z], [x, z]$ (usually the choice of geodesics won't matter).

Given a geodesic triangle $\Delta \equiv \Delta(x, y, z)$, let π be half the perimeter, $\pi = \frac{1}{2}(d(x, y) + d(y, z) + d(x, z))$ and define $\alpha_x = \pi - d(y, z)$ and similarly $\alpha_y = \pi - d(x, z)$ and $\alpha_z = \pi - d(x, y)$. Thus $\alpha_x + \alpha_y = d(x, y)$ and so on. One can imagine a triangle drawn in the Euclidean plane with side lengths $d(x, y), d(x, z)$ and $d(y, z)$. Its inscribed circle would touch the triangle sides $[x, y], [y, z]$ and $[z, x]$ at points m_z, m_x and m_y respectively. From elementary geometry, $[x, y][\alpha_x] = [y, x][\alpha_y] = m_z$ and $[y, z][\alpha_y] = [z, y][\alpha_z] = m_x$ and $[z, x][\alpha_z] = [x, z][\alpha_x] = m_y$, as illustrated in Figure 7.1.

The points m_x, m_y, m_z are called the *internal points* and $\alpha_x, \alpha_y, \alpha_z$ the *internal distances* corresponding to x, y, z respectively in Δ. The *insize* of the triangle Δ is the maximum of $\max_{\theta \in [0, \alpha_x]} d([x, y][\theta], [x, z][\theta])$, $\max_{\theta \in [0, \alpha_y]} d([y, x][\theta], [y, z][\theta])$, and $\max_{\theta \in [0, \alpha_z]} d([z, x][\theta], [z, y][\theta])$.

[1]For a comprehensive treatment of δ-hyperbolicity see [8].

Definition 2. *Let (X, d) be a geodesic metric space, and $\delta \geq 0$. Let δ be minimum such that the insize of every geodesic triangle is at most δ. We say that X is δ-hyperbolic (equivalently, the hyperbolicity of X is δ).*

If G is a graph whose associated graphic metric space is δ-hyperbolic then we say G is δ-hyperbolic. The reader may verify that every tree is 0-hyperbolic. Hyperbolicity is sometimes defined in terms of a four-point condition.

Lemma 7.2.1 (4-point condition, see Proposition 1.22 in [8])**.** *Let (X, d) be a δ-hyperbolic metric space. There is a constant $\delta_{4-point} \leq \delta$ such that for any 4 points $x, y, z, w \in X$, their ordered set of sums of opposite sides, without loss of generality $d(x, y) + d(w, z) \geq d(x, z) + d(y, w) \geq d(x, w) + d(y, z)$, satisfy $d(x, y) + d(w, z) - d(x, z) - d(y, w) \leq 2\delta_{4-point}$.*

The fact that in a δ-hyperbolic metric space $\delta_{4-point}$ is always less than or equal to δ follows directly from the proof of Proposition 1.22 on page 411.

7.3 p-centers and p-packings

Let (X, d) be a geodesic metric space and S be a compact subset of X. Throughout this chapter we rely on two intimately related notions, p-centers and p-packings.

Definition 7.3.1 (p-centers)**.** A set $C \subset X$ *r-dominates* S if for every point $s \in S$ there exists a point $c \in C$ with $d(s, c) \leq r$. The *p-radius* of S, denoted by $r_p(S)$, is the minimum r such that there exists a set of at most p points $C_p(S)$ that r-dominates S. The points in $C_p(S)$ are called p-centers of S.

Definition 7.3.2 (p-packings)**.** A set $D \subseteq S$ is an *r-dispersion* in S if each pair of points $s, s' \in D$, $s \neq s'$, $d(s, s') \geq r$. The *p-diameter* of S, denoted by $d_p(S)$, is the maximum r such that there exists a set of at least p points $D_p(S)$ that is an r-dispersion in S. The points in $D_p(S)$ are called a *p-packing.*

Consider a set of p points C which r-dominate S. By definition, for any choice of $p + 1$ points D, each $d \in D$ is within r of some $c \in C$, and by the pigeonhole principle, at least two, say a_1 and

a_2, are within r of the same $c \in C$. Hence,

$$d(a_1, a_2) \leq d(a_1, c) + d(a_2, c) \leq 2r.$$

So, $\min_{i \neq j} d(a_i, a_j) \leq 2r$. Since this holds for all choices of C and D, we have the following observation which first appeared in [80].

Observation 7.3.3. $r_p(S) \geq \frac{1}{2} d_{p+1}(S)$.

It turns out that these two invariants are equal whenever S has a tree-metric. Indeed, Shier showed the following.

Theorem 7.3.4 (Shier [80])**.** *Let T be a tree. Then $r_p(T) = \frac{1}{2} d_{p+1}(T)$.*

As discussed in Section 7.2.2, δ-hyperbolic spaces are treelike, by which we mean that they possess a metric structure that differs from a tree metric by δ. Therefore, it is logical to attempt to extend Shier's result on p-center covering and packing to such structures. Chepoi and Estellon [16] do exactly this by giving an elegant extension of Shier's theorem to δ-hyperbolic spaces.

Theorem 7.3.5 (Chepoi and Estellon [16])**.** *Let X be a δ-hyperbolic metric space and S a finite subset of X. Then*

$$r_p(S) \leq \frac{1}{2} d_{p+1}(S) + \delta$$

This relationship between $r_p(S)$ and $d_{p+1}(S)$ is a key element in algorithms for approximating p-centers and p-packing.

7.3.1 Algorithms for p-centers and p-packings

The p-packing problem, sometimes referred to as the p-dispersion problem, has received some attention in the literature. For example it is known to be NP-hard [38]. Highly relevant to our work is the heuristic that iteratively adds each of the p points by maximizing the points' distance from previously chosen points (see for example [39, 67]). This heuristic is shown to be a 2-approximation algorithm by Ravi, Rosenkrantz and Tayi [67]. For more information, we refer the interested reader to [40] that has an empirical comparison of ten p-dispersion heuristics.

To our knowledge, the previous best algorithm in terms of an additive error not exceeding δ for the p-radius follows from the Chepoi-Estellon bound (Theorem 7.3.5). Indeed, the proof in [16] leads to a polynomial algorithm to solve p-centres in graphs with an additive error of δ on the p-radius.[2] Specifically, in time $O((n^3 \log n + n^2 m) \log(\mathrm{diam}(G)))$ the authors in [16] determine a set U of p points such that U $(r_p + \delta)$-dominates $V(G)$. Their algorithm involves finding diametrical pairs of vertices in subsets of $V(G)$ $O(n \log(\mathrm{diam}(G)))$ times. Johnson's algorithm [52] finds the diameter in time $O(n^2 \log n + nm)$; hence the running time in Chepoi-Estellon [16] follows.

As pointed out in the introduction, in this work we leverage the fact that instead of finding diametrical pairs, one can just use *locally* diametrical pairs (introduced in Section 7.2.1) with significant reduction in computational time with only a small penalty in the p-radius. Our main result is the following.

Theorem 7.3.6. *Let G be a δ-hyperbolic graph, $p \geq 3$ an integer and $r_p(G)$ the optimal radius of the p-center for G. There exists an algorithm to find a set of p points that $(r_p + 3\delta)$-dominates G. Further, the algorithm runs in time $O(n \log n + (m+n)((2p+1)(\lceil 4 + 3\delta + 2\delta \log_2 n \rceil) + (p+1))) = O(p(\delta + 1)(m+n) \log n)$.*

Though the Chepoi-Estellon algorithm [16] achieves a better approximation (an additive factor of δ instead of our 3δ), its running time is $O((n^3 \log n + n^2 m) \log(\mathrm{diam}(G)))$. We first show below how to improve their running time by a factor of n (Lemma 7.3.9), but this approach still remains infeasible for large graphs. When $p \in \{1, 2\}$ we can achieve the same Chepoi-Estellon p-radius bound but in quasilinear time.

Theorem 7.3.7. *Let (X, d) be a δ-hyperbolic metric space, S a finite subset of X and $p \in \{1, 2\}$. There exists an algorithm to determine a set of p points that $(r_p + \delta)$-dominate S. Further, the algorithm runs in time $O((2\delta + 1)t_X)$, where t_X is the time required to find the set of points at maximum distance from a given point in X. In particular in a δ-hyperbolic graph the running time is $O((2\delta + 1)(m + n))$.*

For $p = 1$, the previous best algorithm we know of is due to Chepoi et al. [15]: the approximation

[2] The cited result also gives rise to an algorithm for general δ-hyperbolic spaces whose running time depends on the time to compute $F_S(x)$ for $x \in X$ and $S \subseteq X$. Because our interest is primarily in graphs, we direct the reader to [16] for details.

error is $\leq 5\delta$, and the computation requires just two breadth-first searches. In contrast, we require $2\delta + 1$ breadth-first searches to achieve the smaller additive factor of δ.

The remainder of this section is organized as follows. We start by showing how to improve the time complexity of the Chepoi-Estellon algorithm by only approximately finding diametrical pairs of vertices, that is via finding locally diametrical pairs. In the proofs of our main results, we will repeatedly apply this idea, showing that it is sufficient to solve the easier and computationally more efficient approximate version of this expensive sub-problem. We then move on to proofs of Theorems 7.3.7 and 7.3.6 in Sections 7.4 and 7.4.1, respectively.

Recall from Section 7.2.1 that a pair of vertices $\{u, v\}$ is *locally diametrical* if there is no vertex w such that $d(u, w) > d(u, v)$ or $d(v, w) > d(v, u)$. Clearly a diametrical pair is locally diametrical but the converse is not true in general (e.g., a cycle with handles). It turns out to be sufficient to find locally diametrical pairs in the main lemma of [16]. Indeed, the following lemma is simply Lemma 1 from [16], but with the requirement that u and v be diametrical replaced with the weaker property of being locally diametrical.

Lemma 7.3.8. *Let X be a δ-hyperbolic metric space and $S \subseteq X$ be a compact set and $r \in \mathbb{R}$. Suppose that u and v are locally diametrical in S and let $[u, v]$ be a geodesic. Let $c = [u, v][r]$. Then $B_{2r}(u) \cap S \subseteq B_{r+\delta}(c) \cap S$.*

The proof of Lemma 1 in [16] works essentially unchanged to prove Lemma 7.3.8 by replacing diametrical pairs with locally diametrical pairs. Since we will use a refined version of the same argument that is needed for Lemma 7.3.8 in the proof of Theorem 7.3.6, we skip the proof of Lemma 7.3.8. We prove below (Lemma 7.3.10) that we can find a locally diametrical pair with at most $2\delta + 1$ breadth-first searches. Hence, we achieve the following significant reduction in the run time of the Chepoi-Estellon algorithm.

Lemma 7.3.9. *Let G be a δ-hyperbolic graph and p an integer. There exists an algorithm to find a set of p points that $(r_p + \delta)$-dominates $V(G)$ that runs in time $O(n^2 \log(\mathrm{diam}(G))(2\delta + 1))$.*

It remains to show how to efficiently determine locally diametrical pairs.

Lemma 7.3.10. *Given a δ-hyperbolic graph G and $S \subseteq V(G)$. There is an algorithm that finds a locally diametrical pair of vertices by performing at most $2\delta + 1$ breadth-first searches; that is, the running time is $O((2\delta + 1)(m + n))$.*

Proof. Choose a vertex $u \in S$ arbitrarily and find a vertex $v_1 \in F_S(u)$ by BFS. Then, find $v_2 \in F_S(v_1)$. Next, find a vertex $v_3 \in F_S(v_2)$. If $d(v_2, v_3) = d(v_1, v_2)$, then let $v = v_1$ and $w = v_2$ and we have found a locally diametrical pair. Otherwise $d(v_2, v_3) > d(v_1, v_2)$ and continue the process until v_k, v_{k+1} are found such that $d(v_k, v_{k+1}) = d(v_k, F_S(v_k))$ and $d(v_k, v_{k+1}) = d(v_{k+1}, F_S(v_{k+1}))$. This must happen for at most $k \leq diam(S) - d(v_1, v_2)$. But by Proposition 3 in [15] $d(v_1, v_2) \geq diam(S) - 2\delta_{4-point} \geq diam(S) - 2\delta$ so k cannot exceed 2δ. This means no more than $(2\delta + 1)$ BFS steps or no more than $O(2\delta+1)(m+n)$ steps are needed for finding a locally diametrical pair starting from $u \in S$. Then algorithm returns the locally diametrical pair (v_k, v_{k+1}).

□

7.4 Approximating p-centers

In general, in searching for p-centers, first we approximately solve the dual problem, that is, we find D, a $(p+1)$-packing, with $|D| \geq p+1$ such that

$$\max\{\ r \mid d(s,s') \geq r,\ \forall s \neq s' \in D\} \leq d_{p+1}(V).$$

This together with Observation 7.3.3 yields

$$\frac{1}{2}\max\{\ r \mid d(s,s') \geq r,\ \forall s \neq s' \in D\} \leq r_p(V). \tag{7.1}$$

Given these $(p+1)$-points we find a set of p-points C such that setting $\lambda = \frac{1}{2}\max\{\ r \mid d(s,s') \geq r,\ \forall s \neq s' \in D\}$,

1. C λ-dominates the points in D, and
2. for each $a \in D$ there exists some $a' \in D$ and $c \in C$ such that c is on a geodesic between a and a'.

We prove later that these two properties together with δ-hyperbolicity allow us to show that for a carefully-selected set D, the p points in C $(\lambda + 3\delta)$-dominate V, that is,

$$\min\{\ r \mid for\ each\ x \in V,\ \exists c \in C\ with\ d(x,c) \leq r\} \leq \lambda + 3\delta. \tag{7.2}$$

Substituting the value of λ in (7.2) and applying (7.1) yields,

$$\begin{aligned} & \min\{\ r \mid \textit{for each } x \in V,\ \exists c \in C \textit{ with } d(x,c) \leq r\} \\ \leq\ & \frac{1}{2}\max\{\ r \mid d(s,s') \geq r,\ \forall s \neq s' \in D\} + 3\delta \\ \leq\ & r_p(V) + 3\delta. \end{aligned}$$

It follows that C $(r_p(V) + 3\delta)$-dominates V as desired.

We now apply this approach to find a 1-center of a graph.

Theorem 7.4.1. *Let G be a δ-hyperbolic graph. There exists an algorithm to find a point c that $(r_1 + \delta)$-dominates $V(G)$. The algorithm requires time $O((2\delta + 1)(m + n))$.*

Proof. Let x, y be a locally diametrical pair of vertices and let $[x, y]$ be a geodesic segment. As described above, set $\lambda = \frac{d(x,y)}{2}$ and choose $c = [x, y][\lambda]$. Clearly, $C = \{c\}$ satisfies Properties 1 and 2 above. We now show that $C = \{c\}$ $(\lambda + \delta)$-dominates V.

Let z be any point in V and consider the geodesic triangle $\Delta(x, y, z)$ as depicted and labeled in Figure 7.2. Without loss of generality, assume that $d(y, z) \leq d(x, z)$. Since (x, y) is locally diametrical, then

$$d(y, z) \leq d(x, z) \leq d(x, y)$$

which implies that

$$\alpha_z \leq \alpha_y \leq \alpha_x.$$

(This means that in the figure c lies to the right of m_z, as shown.) Then

$$d(z, c) \leq \alpha_z + \delta + d(c, m_z) \leq \alpha_z + \delta + \lambda - \alpha_y \leq \delta + \lambda.$$

As the claim holds for any z, c $(\lambda + \delta)$-dominates $V(G)$, and therefore, since $\lambda = \frac{1}{2}d(x, y) \leq \frac{1}{2}d_2(V) \leq r_1(V)$, the latter inequality by Observation 7.3.3, and thus c $(r_1 + \delta)$-dominates $V(G)$, as desired. To complete the proof, we note that by Lemma 7.3.10, x, y and $[x, y][\lambda]$ can be found

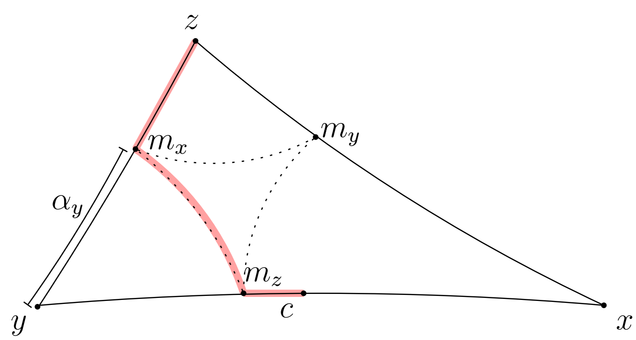

Figure 7.2: A geodesic triangle $\Delta(x, y, z)$ with points m_x, m_y, m_z and c labelled as in the proof of Theorem 7.4.1. Dashed lines indicate a distance $\leq \delta$ and the red line indicates the upper estimate for $d(z, c)$.

in time $O((2\delta + 1)(m + n))$. $\square$

We note that in the course of the above proof we demonstrated the following fact that we shall reuse.

Observation 7.4.2. *Let z be any vertex in $V(G)$, (x, y) a locally diametrical pair of vertices, $c \in A(G)$ the mid-point of $[x, y]$ and $\lambda = \frac{d(x,y)}{2}$. Then $d(z, c) \leq \lambda + \delta$.*

In extending these proof techniques to the general case for $p > 1$, we run into the following two difficulties, each costing us an additional δ in our approximation error. First, Property 2 only guarantees that p of the $\binom{p+1}{2}$ pairs of points in D have a geodesics connecting them containing some point $c_i \in C$. This will force us use two geodesic triangles to bound the distance from some points in V to their closest center in C. Second, in achieving the quasilinear runtime, we are only able to find a $(\lambda + 2\delta)$-approximation for the $(p+1)$-packing problem. We omit further details until Section 7.4.1.

To finish off this section, we prove that when $p = 2$ we can find a 2-center solution which $(r_2 + \delta)$-dominates G. Like Theorem 7.4.1, this is stronger than our general result (Theorem 7.3.6) and the proof does not use the machinery outlined at the beginning of Section 7.4 that relies on Properties 1 and 2. Theorems 7.4.1 and 7.4.3 may be special cases of a general and stronger result than our main result, so we include it.

Theorem 7.4.3. *Let G be a δ-hyperbolic graph. There exists an algorithm to find points c_1, c_2 that $(r_2 + \delta)$-dominate $V(G)$. The algorithm requires time $O((2\delta + 1)(m + n))$.*

Proof. Let x, y be a locally diametrical pair of vertices and let $[x, y]$ be a geodesic segment. Choose z so that $\min\{d(z, x), d(z, y)\}$ is maximized (requires two BFS). We let our 3-packing be $D = \{x, y, z\}$. Assume without loss of generality that $d(x, y) \geq d(x, z) \geq d(y, z)$, and so, $\lambda = \frac{1}{2}\max\{\ r \mid d(s, s') > r,\ \forall s \neq s' \in D\} = \frac{1}{2}d(y, z)$.

We choose $c_1 = [x, y][\lambda]$ and $c_2 = [y, x][\lambda]$. We claim that $C = \{c_1, c_2\}$ satisfy Equation 7.2, with $t = 1$, and so, C $(r_2 + \delta)$-dominates G.

To prove the claim, let $\Delta_1 = \Delta(x, y, z)$ be a geodesic triangle. Let w be any point of G and let $\Delta_2 = \Delta(x, y, w)$ be a geodesic triangle so that Δ_1 and Δ_2 share the geodesic $[x, y]$. We will show that $\min\{d(w, c_1), d(w, c_2)\} \leq \lambda + \delta$. Take $\alpha_x, \alpha_y, \alpha_w$ and m_x, m_y, m_w to denote the internal distances and points in Δ_2. Without loss of generality assume $d(w, x) \leq d(w, y)$ which implies that $d(w, x) \leq d(y, z) = 2\lambda$ and $\alpha_x \leq \alpha_y$. We distinguish two cases, as illustrated in Figure 7.3.

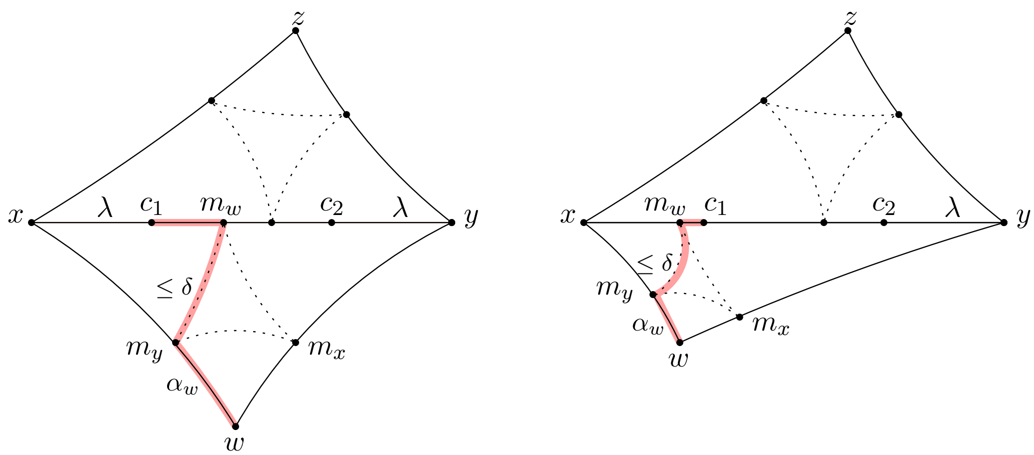

Figure 7.3: Figure for Cases 1 and 2 in the proof of Theorem 7.4.3. The red lines indicate the upper estimate for $d(w, c_1)$. Dashed lines indicate a distance $\leq \delta$.

Case 1: $\lambda < \alpha_x^2 < d(x, y) - \lambda$

From the choice of z, it follows that either $d(w, x) \leq d(y, z) = 2\lambda$ or $d(w, y) \leq 2\lambda$. Assume without loss of generality that $d(w, x) = d(w, m_y^2) + d(m_w^2, x) \leq 2\lambda$. Therefore, $d(w, c_1) \leq d(w, m_y^2) + d(m_y^2, m_w^2) + d(m_w^2, c_1) \leq d(w, m_y^2) + \delta + d(m_w^2, x) - \lambda \leq \lambda + \delta$.

Case 2: $\alpha_x^2 \leq \lambda$

In this case m_w^2 lies between x and c_1 on the geodesic segment $[x, y]$. By the local maximality of x and y, we have $d(y, w) = \alpha_y^2 + \alpha_w^2 \leq \alpha_y^2 + \alpha_x^2 = d(x, y)$ and so $d(w, m_y^2) = \alpha_w^2 \leq \alpha_x^2 = d(x, m_w^2)$.

Then $d(w, c_1) \leq d(w, m_y^2) + d(m_y^2, m_w^2) + d(m_w^2, c_1) \leq d(x, c_1) + \delta = \lambda + \delta$.

To complete the proof, we need only show that c_1, c_2 can be found in $O((2\delta+1)(m+n))$ time. By Lemma 7.3.10, x and y can be found in time $O((2\delta+1)(m+n))$ and the vertex z can be found by doing a breadth-first search rooted at x and one rooted at y. Given $D = \{x, y, z\}$, the vertices c_1 and c_2 can then be found by storing the last breadth-first search used in finding x and y and $\lambda = \frac{1}{2}\min\{d(x, x), d(y, z)\}$. The runtime now follows. $\square$

7.4.1 The general algorithm

Our algorithm and proof follow the same three basic steps, though each step is more involved. As a reminder these three steps are 1) approximately solving the dual problem, or finding a $(p+1)$-packing, 2) deriving p-points from this dual solution that satisfy Properties 1 and 2, and 3) bounding the approximation guarantee by showing Equation 7.2.

It turns out the difficult part of these three steps is Step 1. For this step, we need to extend the notion of a 'locally diametrical pair' to a 'locally diametrical set' in such a way that i) it provides us with both the tools we need to satisfy Properties 1 and 2 and ii) it can be determined efficiently. We find a set of $(p+1)$ vertices $D = \{v_0, v_1, ..., v_p\}$ with

$$\lambda(D) := \tfrac{1}{2}\max\{\ r \mid d(s, s') \geq r,\ \forall v_i \neq v_j \in D\}$$

such that the following three properties hold

(a) (Vertex relabeling) $d(v_0, v_i) = 2\lambda(D)$ for some $v_i \in D$,

(b) (Extending locally diametrical pairs to locally diametrical sets) For each $v_i \in D$ with $d(v_i, v_j) = 2\lambda(D)$ for some v_j, there exists no $w \in V(G)$ with $d(w, v_k) > 2\lambda(D), \forall v_k \in D \setminus \{v_i\}$, and

(c) (δ-hyperbolic version of locally diametrical sets) for each $i \geq 1$, there exists no vertex $v \in V(G)$ with $d(v_0, v) > d(v_0, v_i) + 2\delta$ and $d(v_i, v) \leq 2\lambda(D)$ and $d(v, v_j) > 2\lambda(D)$ for each $j \neq i$.

These three requirements provide us with what is needed to determine a set of $(p+1)$ vertices satisfying Properties 1 and 2. Specifically, we prove

Lemma 7.4.4. *Let G be a δ-hyperbolic graph and $\Lambda_n = \lceil 4 + 3\delta + 2\delta \log_2 n \rceil$. There exists an algorithm to find a set D of $p+1$ vertices satisfying (a), (b) and (c). The algorithm runs in time $O(n \log n + (m+n)((2p+1)\Lambda_n + (p+1)))$.*

Given a set of $p+1$ vertices satisfying Properties (a), (b) and (c) it is straightforward to find $C = \{c_1, ..., c_p\}$ satisfying Properties 1 and 2. For each $1 \leq i \leq p$, let c_i be the vertex at distance λ from v_i on the shortest path from v_i to v_0, i.e. $c_i = [v_i, v_0][\lambda]$.

Lemma 7.4.5. *Let G be a δ-hyperbolic graph. Suppose that $D = \{v_0, v_1, ..., v_p\}$ satisfy (a), (b) and (c). Then the set of p points $C = \{c_i \mid c_i = [v_i, v_0][\lambda]\}$ $(\lambda + 3\delta)$-dominate G.*

As described above (beginning of Section 4), such C $(r_p(V) + 3\delta)$-dominates V as desired. So, given the Lemmas 7.4.4 and 7.4.5, the proof of Theorem 7.3.6 follows once establishing the runtime, which we do now. First, determining the set D takes $O(n \log n + (m+n)((2p+1)\Lambda_n + (p+1)))$. Given D, the set of vertices $\{c_i, 1 \leq i \leq p\}$ can clearly be constructed by performing a breadth-first search rooted at v_0. Theorem 7.3.6 now follows.

In the next two sections we establish Lemmas 7.4.4 and 7.4.5. Lemma 7.4.4 is the more interesting of the two proofs, and takes us deeper into the analysis of locally diametrical sets. The proof of Lemma 7.4.5 is a sophistication of the ideas in Theorems 7.4.1 and 7.4.3. We begin with that lemma.

7.4.2 Proof of Lemma 7.4.5

We show that every vertex of G is at distance at most $\lambda + 3\delta$ from some centre c_i. Let $w \in V(G)$ and suppose that w is at distance greater than $\lambda + 3\delta$ from each centre. Property (b) implies $d(w, v_i) \leq 2\lambda$ for some i. We prove below the following claim.

Claim 4. *$d(w, v_j) > 2\lambda$ for each $j \neq i$.*

Using the claim, we can prove Lemma 7.4.5. Consider the geodesic triangle $\Delta(v_i, v_0, w)$, and recall that c_i belongs to the geodesic $[v_i, v_0]$. There are two cases to handle.

First, suppose that $d(v_i, m_w) \geq \lambda$. Then a w-c_i-path can be constructed by concatenating the geodesics from $[w, m_{v_0}], [m_{v_0}, m_w]$ and $[m_w, c_i]$, and so, since $d(w, v_i) \leq 2\lambda$

$$\begin{aligned} d(w, c_i) &\leq d(w, m_{v_0}) + d(m_{v_0}, m_w) + d(m_w, c_i) \\ &\leq d(w, m_{v_0}) + \delta + d(m_w, v_i) - \lambda \\ &\leq \lambda + \delta, \end{aligned}$$

a contradiction.

Otherwise, if $d(v_i, m_w) < \lambda$, then

$$\begin{aligned} \lambda + 3\delta < d(w, c_i) &\leq d(w, m_{v_0}) + d(m_{v_0}, m_w) + d(m_w, c_i) \\ &\leq d(w, m_{v_0}) + \delta + d(m_w, c_i). \end{aligned}$$

Since $d(v_i, c_i) = d(v_i, m_w) + d(m_w, c_i) = \lambda$, we deduce that $d(w, m_{v_0}) > d(v_i, m_w) + 2\delta$. It follows that $d(v_0, w) > d(v_0, v_i) + 2\delta$, which along with Claim 4, contradicts Property (c).

It follows that w is within $\lambda + 3\delta$ from at least one centre. We need only prove the claim.

Proof of Claim 4. Suppose that w is at distance at most 2λ from both v_i and v_j. Let c'_i and c'_j be the vertices at distance λ from i and j respectively on the geodesic $[v_i, v_j]$. We will show that at least one of $d(c_i, c'_i)$ and $d(c_j, c'_j)$ is at most δ. Consider the geodesic triangle $\Delta(v_i, v_j, v_0) = [v_i, v_j] \cup [v_i, v_0] \cup [v_j, v_0]$ and let $m_{v_i}, m_{v_j}, m_{v_0}$ be as described above. Assume for contradiction that both $d(c_i, c'_i)$ and $d(c_j, c'_j)$ are greater than δ. It follows that $d(v_i, m_{v_j}) < \lambda$ and $d(v_j, m_{v_i}) < \lambda$. But then $d(v_i, v_j) = d(v_i, m_{v_0}) + d(m_{v_0}, v_j) = d(v_i, m_{v_j}) + d(v_j, m_{v_i}) < 2\lambda$, a contradiction. Assume then, without loss of generality, that $d(c_i, c'_i) \leq \delta$.

Now consider the geodesic triangle $\Delta(v_i, v_j, w)$ and let m_w be defined as usual. First, suppose that $d(v_i, m_w) \geq d(v_i, c'_i)$. Then

$$d(w, c'_i) \leq d(w, m_{v_j}) + \delta + d(v_i, m_w) - d(v_i, c'_i) \leq d(w, v_i) + \delta - \lambda \leq \lambda + \delta.$$

Now, suppose that $d(v_i, m_w) < d(v_i, c'_i)$. Then

$$d(w, c'_i) \leq d(w, m_{v_i}) + \delta + d(v_j, m_w) - d(v_j, c'_i) \leq d(w, v_j) + \delta - \lambda \leq \lambda + \delta.$$

In either case, $d(w, c_i') \leq \lambda + \delta$, and so $d(w, c_i) \leq \lambda + 2\delta$, a contradiction. $\square$

7.4.3 Proof of Lemma 7.4.4

A proof sketch is as follows. We first show that we can a find $(p+1)$-packing that is within $O(\delta \log_2 n)$ of an optimal solution. To do so, we find a tree T which approximately preserves distances on our input graph G. It turns out that exactly solving the $(p+1)$-packings on trees can be done efficiently, though in contrast to before, we solve the p-centres first and use this to construct a dual solution in G. The fact that T is a good approximating tree allows us to bound how close our $(p+1)$-packing is to an optimal solution and in turn helps us achieve the quasilinear running time. Finally, given this initial $(p+1)$-packing, we iteratively improve the solution whenever possible until we achieve Properties (a), (b), (c). Clearly, (a) can hold for all solutions after relabelling, so the only difficulty is in insuring both (b) and (c) hold.

We will use the following theorem, which we will deduce from known results at the end of this section, to find our initial $(p+1)$-packing. Let $\Lambda_n = \lceil 4 + 3\delta + 2\delta \log_2 n \rceil$.

Theorem 7.4.6. *There exists an algorithm to find a set $\mathcal{P}$ of $p+1$ vertices satisfying $d(u,v) \geq \kappa, \forall u \neq v \in \mathcal{P}$, for some κ with $d_{p+1}(G) - \kappa \leq \Lambda_n$. The algorithm runs in time $O(n \log n)$.*

Given the set $\mathcal{P}$ of $(p+1)$-points from Theorem 7.4.6, we now describe an efficient iterative algorithm which finds $(p+1)$-points satisfying Properties (a), (b) and (c). Our argument bounds the number of iterations using the following potential function.

Definition 3. *Let G be a graph and let $\mathcal{P} \subseteq V$ be a set of p vertices and suppose that κ is the largest value such that $d(u,v) \geq \kappa$ for all $u \neq v \in \mathcal{P}$. Let $\eta(\mathcal{P})$ denote the number of vertices in $\mathcal{P}$ which are exactly at distance κ from at least one other vertex in $\mathcal{P}$. We define the potential of $\mathcal{P}$ as $\phi(P) := p(\kappa+1) - \eta(\mathcal{P})$.*

Algorithm 4 together with Subroutines 2 and 3 describe the algorithm. We first prove that if Algorithm 4 terminates then it is correct, that is, $\mathcal{P}'$ satisfies (a), (b) and (c). The algorithm terminates if the potential $\phi(\mathcal{P})$ has not increased after successive executions of Subroutines 3 and 2. As Subroutine 2 executes last, the returned $\mathcal{P}$ satisfies (a) and (b) as satisfying (b) is the stopping condition and, as mentioned above, (a) always holds after a relabelling. Since $\phi(\mathcal{P})$ is

Algorithm 1: Finding an initial set $\mathcal{P}$ of vertices

Input: Graph $G = (V, E)$ and integer p.
Output: A set $\mathcal{P}$ of $p+1$ vertices satisfying $d(u, v) \geq \kappa, \forall u \neq v \in \mathcal{P}$, for some κ with $d_{p+1}(G) - \kappa \leq \Lambda_n$.
Let $T = (V, F)$ be the tree determined by Theorem 7.4.7.
Let λ be the p-radius of the p-centers of T determined by Theorem 7.4.8.
Let $\mathcal{P}$ be a set of maximum size s.t. $d(u, v) \geq 2\lambda$ for each $u \neq v \in \mathcal{P}$ (Theorem 7.4.9).
Let $\mathcal{P}'$ be a set of $p+1$ unique vertices chosen arbitrarily from $\mathcal{P}$.
return $\mathcal{P} = \mathcal{P}'$

Subroutine 2: Satisfying Properties (a) and (b).

Input: A set $\mathcal{P}$ satisfying Property (a) for $\lambda(\mathcal{P})$.
Output: A set $\mathcal{P}'$ satisfying Property (a) and (b) for $\lambda(\mathcal{P}') \geq \lambda(\mathcal{P})$.
We say that a vertex $u \in \mathcal{P}$ is *improvable* to $w \notin \mathcal{P}$ if there exists $v \in \mathcal{P}$ with $d(u, v) = 2\lambda(\mathcal{P})$ and $d(w, x) > 2\lambda(\mathcal{P}), \forall x \in \mathcal{P} \setminus \{u\}$.

```
repeat
    for i from 0 to p do
        if v_i is improvable to some v then
            replace v_i in P with the improved vertex (P = (P \ v_i) ∪ {v}).
        else
            do nothing.
    end
until no vertex is found to be improvable.
return P' = P.
```

Subroutine 3: Satisfying Properties (a), (b) and (c).

Input: A set $\mathcal{P}$ satisfying Property (a) and (b) for $\lambda(\mathcal{P})$.
For this step we label the vertices of $\mathcal{P}$ in a specific way. Let v_0 and v_p be vertices in $\mathcal{P}$ with $d(v_0, v_p) = 2\lambda(\mathcal{P})$. Then label the remaining vertices of $\mathcal{P}$ as $\{v_0, v_1, \ldots, v_p\}$ so that $d(v_0, v_i) \geq d(v_0, v_j)$ for each $i > j$.
In this context, we say that a vertex v_i $(1 \leq i \leq p)$ is improvable to $v_i' \notin \mathcal{P}$ if $d(v_0, v_i') > d(v_0, v_i)$ and $d(v_i, v_i') \leq 2\lambda(\mathcal{P})$ and $d(v_i', v_j) > 2\lambda(\mathcal{P})$ for each $j \neq i$.

```
for i from 1 to p do
    if v_i is improvable then
        replace v_i in P with the vertex v_i' furthest from v_0 that satisfies d(v_i, v_i') ≤ 2λ(P) and
        d(v_i', v_j) > 2λ(P) for each j ≠ i.
end
return P.
```

unchanged by Subroutine 2, $\mathcal{P}$ is unchanged as well. For the purpose of analysis, we will adopt the following notation. Let $\{v_0, \ldots, v_p\}$ be the labelling specified in the description of Subroutine 3. Then, for each $v_i (i \geq 1)$, if it was improved, let v_i' be the vertex v_i was replaced by. Otherwise write $v_i' = v_i$. So, now consider $\mathcal{P} = \{v_0, v_1', \ldots, v_p'\}$, that is output by Subroutine 3. For contradiction,

Algorithm 4: Finding an optimal and optimized set of vertices

Input: A set $\mathcal{P}$ satisfying Property (a) for $\lambda(\mathcal{P})$.
Output: A set $\mathcal{P}'$ satisfying Property (a), (b) and (c) for $\lambda(\mathcal{P}') \geq \lambda(\mathcal{P})$.
Let $\mathcal{P}$ be the returned set of Subroutine 2 with input $\mathcal{P}$.
repeat
  Let $\mathcal{P}'$ be the returned set of Subroutine 3 with input $\mathcal{P}$.
  Let $\mathcal{P}$ be the returned set of Subroutine 2 with input $\mathcal{P}'$.
until $\phi(\mathcal{P}') = \phi(\mathcal{P})$
return $\mathcal{P}' = \mathcal{P}$

suppose that $\mathcal{P}$ does not satisfy Property (c). Then, there exist an index i and a vertex v_i'' with $d(v_0, v_i'') > d(v_0, v_i') + 2\delta$ and $d(v_i', v_i'') \leq 2\lambda(\mathcal{P})$ and $d(v_i'', v_j') \geq 2\lambda(\mathcal{P}), \forall j \neq i$. Further, by the choice of v_i', there must exist some index $j > i$ with $d(v_i'', v_j) < 2\lambda(\mathcal{P})$. We will apply Lemma 7.2.1 to reach a contradiction, using the vertices v_0, v_i', v_i'', v_j, as illustrated in Figure 7.4. By the choice of labelling, we have $2\lambda(\mathcal{P}) \leq d(v_0, v_j) \leq d(v_0, v_i) \leq d(v_0, v_i') < d(v_0, v_i'') - 2\delta$. There are three distance sums to consider. We claim that $d(v_0, v_i'') + d(v_i', v_j) > \max\{d(v_0, v_i') + d(v_i'', v_j), d(v_0, v_j) + d(v_i', v_i'')\} + 2\delta$. This is clear because both $d(v_i'', v_j) \leq 2\lambda(\mathcal{P})$ and $d(v_i', v_i'') \leq 2\lambda(\mathcal{P})$ while $d(v_i', v_j) \geq 2\lambda(\mathcal{P})$. By Lemma 7.2.1, this contradicts the δ-hyperbolicity of G. It follows that $\mathcal{P}$ also satisfies (c).

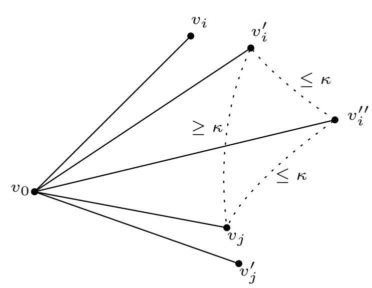

Figure 7.4: Proof that $\mathcal{P}$ satisfies Property (c)

It remains to prove that the algorithm terminates and to bound the runtime. To see that Algorithm 4 terminates, we first note that whenever a vertex in $\mathcal{P}$ is improved in Subroutine 2, the distance to its closest neighbour strictly increases. Therefore, after at most $p+1$ rounds of the repeat until loop Subroutine 2, $\lambda(\mathcal{P})$ strictly increases. Further, each round (except the last one) in which the potential doesn't change is proceeded by an iteration of Subroutine 3. By Theorem 7.4.6, for the initial $(p+1)$-points $\mathcal{P}^\star$, $d_{p+1}(G) - 2\lambda(\mathcal{P}^\star) \leq \Lambda_n$. Hence, $\phi(\mathcal{P}^\star) \geq (p+1)(2\lambda(\mathcal{P}^\star)+1) - (p+1) = (p+1)2\lambda(\mathcal{P}^\star) \geq (p+1)(d_{p+1}(G) - \Lambda_n)$. Further, any set of $p+1$ vertices has dispersion at most $d_{p+1}(G)$ and therefore has potential at most $(p+1)(d_{p+1}(G)+1)$. We conclude that the repeat until loop of Algorithm 4 can be executed at most $(p+1)\Lambda_n$ rounds in total.

We now examine the complexity of the algorithm. To obtain the initial set $\mathcal{P}$ as in Theorem 7.4.6 takes time $O(n \log n)$. Given a set $\mathcal{P}$, we can determine and record the set of distances $\{d(v, v_i) : v \in V(G), 0 \leq i \leq p\}$ by performing a breadth-first search rooted at each vertex $v_i \in \mathcal{P}$. From these distances, it can easily be checked in linear ($O(n)$) time whether a vertex is improvable. To complete the first round the first time we perform Subroutine 2, we must perform $p+1$ breadth-first searches. Each time a vertex is improved (in either Subroutine 2 or Subroutine 3), we need an additional one. From the discussion above it follows that at most $p+1+((p+1)+p)\Lambda_n$ breadth-first searches need be done. The algorithm therefore runs in time $O(n \log n+(m+n)((2p+1)\Lambda_n+(p+1)))$.

We now deduce Theorem 7.4.6 and its corresponding Algorithm 1. In finding our initial $(p+1)$-packing, we use the following definitions and results. For a graph G and constant k, we say that a tree T with vertex set $V(G)$ is a k-approximating tree if $|d_G(u,v) - d_T(u,v)| \leq k$ for every pair of vertices $u, v \in V$. Chepoi et al. showed in [15] that δ-hyperbolic graphs have good approximating trees that can be computed in linear ($O(m)$) time.

Theorem 7.4.7 ([15])**.** *Let $G = (V, E)$ be a δ-hyperbolic graph, and let $\Lambda_n = \lceil 4 + 3\delta + 2\delta \log_2 n \rceil$. There exists a Λ_n-approximating tree $T = (V, F)$ of G. Furthermore T can be computed from G in time $O(m)$.*

Fredrickson [43] showed that p-centres can be solved in linear time on trees.

Theorem 7.4.8 ([43])**.** *Let T be a tree and p an integer. There exists an algorithm to solve p-centres exactly on T in time $O(n)$.*

Shier proved in [80] (see Theorem 7.3.4) that in trees, the p-radius is always half the $p+1$-diameter. In [11] Chandrasekaran and Daughety gave an algorithm to find the p-diameter (and an optimal packing of size p) in a tree in time $O(n^2 \log n)$. Their technique involves a binary search for λ_p through repeated application of a subroutine which, when given a half-integer λ, produces a maximum number of points which are pairwise at distance $> 2\lambda$. More precisely,

Theorem 7.4.9 ([11])**.** *Let $T = (V, E)$ be a tree and let 2λ be an integer. There exists an algorithm which, in time $O(n \log n)$, produces a set $W \subseteq V$ of maximum size such that $d(u, v) \geq 2\lambda$ for each $u \neq v \in V$.*

Combining Theorem 7.4.9 with Theorem 7.3.4 and Theorem 7.4.8 we obtain an $O(n \log n)$ algorithm to find an optimal packing of size p in a tree.

Now, suppose that G is δ-hyperbolic and T is a Λ_n-approximating tree for G. By definition of an approximating tree, for every $u, v \in V$ we have $|d_T(u,v) - d_G(u,v)| \leq \Lambda_n$. It follows that $|d_{p+1}(T) - d_{p+1}(G)| \leq \Lambda_n$. Thus we obtain Algorithm 1 and Theorem 7.4.6 that yields our initial $(p+1)$-packing.

At this point, the reader may be wondering why we go to the trouble of Algorithm 1 to obtain our initial $(p+1)$-packing. Indeed, one could start with any packing at the beginning of Algorithm 4, and repeat rounds of Subroutines 2 and 3 until a packing satisfying Properties (a), (b) and (c) is found. However, as we have just seen, the number of times we may need to repeat the rounds is upper bounded by the difference between the dispersion of our initial set and the optimal dispersion d_{p+1}. When the initial set is chosen using Algorithm 1, this difference is $O(\delta \log n)$, whereas trying to save time choosing the initial set (say, by choosing it arbitrarily) may result in an additional linear factor in the complexity bound. Applying the greedy 2-approximation algorithm of Ravi, Rosenkrantz and Tayi mentioned in Section 7.3.1 adds a factor of d_{p+1}, which may also be linear. However, the practitioner may wish to experiment.

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Appendix: The unavoidable set of reducible configurations